RATIONAL WAVELETS ON THE REAL LINE

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ABSTRACT

Suppose $\{\phi_k\}_{k=0}^n$ is an orthonormal basis for the function space $\mathcal{L}_n$ of polynomials or rational functions of degree n with prescribed poles. Suppose $n = 2^s$ and set $\mathcal{V}_s = \mathcal{L}_n$. Then $k_n(z,w) = \sum_{k=0}^n \phi_k(z)\overline{\phi_k(w)}$, is a reproducing kernel for $\mathcal{V}_s$. For fixed w, such reproducing kernels are known to be functions localized in the neighborhood of $z = w$. Moreover, by an appropriate choice of the parameters $\{\xi_{nk}\}_{k=0}^n$, the functions $\{\varphi_{n,k}(z) = k_n(z,\xi_{nk})\}_{k=0}^n$ will be an orthogonal basis for $\mathcal{V}_s$. The orthogonal complement $\mathcal{W}_s = \mathcal{V}_{s+1} \ominus \mathcal{V}_s$ is spanned by the functions $\{\psi_{n,k}(z) = l_n(z,\eta_{nk})\}_{k=0}^{n-1}$ for an appropriate choice of the parameters $\{\eta_{nk}\}_{k=0}^{n-1}$ where $l_n = k_{n+1} - k_n$ is the reproducing kernel for $\mathcal{W}_s$. These observations form the basic ingredients for a wavelet type of analysis for orthogonal rational functions on the real line with respect to an arbitrary probability measure.

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1 Introduction

Consider a Hilbert space H of complex functions analytic in $X \subset \mathbb{C}$, equipped with an inner product $\langle \cdot, \cdot \rangle$, then it is a reproducing kernel Hilbert space if there exists a function k_w such that $k_w \in H$ for all $w \in X$ and $\langle f, k_w \rangle = f(w)$ for all $w \in X$ and for all $f \in H$. This $k_w(z) \equiv k(z,w)$, which is in fact unique, is called the reproducing kernel for H.

In such a reproducing kernel Hilbert space, it is well known that the solution of the problem

$$\inf_{f \in H} \{\|f\| : f(w) = 1\}$$

for some $w \in X$ is given by $f(z) = k(z,w)/k(w,w)$.

This property is the key to this paper, because it characterizes the reproducing kernel as a function which is localized near $z = w$. Indeed, without the constraint $f(w) = 1$, the solution would be $f \equiv 0$. With the constraint, the function is forced to take the value 1 at $z = w$ but to minimize the norm, it should be as close to zero as possible outside $z = w$. So what really happens is that the solution will be an approximation of this impulse by a function from H. If we solve this problem for subspaces of H with increasing dimension, then the reproducing kernels for these subspaces will approximate the Dirac impulse better and better, hence will be "narrower" near the peak at $z = w$ and will oscillate more by a Gibbs-like phenomenon. Thus the better we localize the function in the z-domain, the worse it will be localized in the frequency domain. This is a manifestation of the Heisenberg uncertainty principle.

This shows that two main ingredients of a wavelet analysis are present: localized functions (and it will turn out that they can indeed be used to generate basis functions) and a multiresolution idea where a function can be represented at increasing resolution levels.

This is the idea which will be elaborated in this paper. It is inspired by the paper of Fischer and Prestin [14] where a similar construction was used for orthogonal polynomials on the real line.

As a motivation to consider rational wavelets instead of polynomial wavelets, we refer to the examples where it will be shown that with the introduction of the poles, one can influence the shape of the wavelets considerably. Using a rational instead of a polynomial basis may result in a faster convergence of the approximation. If we have a rough idea about the location of the singularities of the function, then, using basis functions that have poles at those places, will result in a much faster convergence. For a similar reason, rational wavelets are also used in robust identification of linear systems (see [21]). Likewise, in many other problems in systems and control it is much better to have a low degree rational approximant rather than a high degree polynomial approximant. Also there, it is much better to use rational basis functions, rather than polynomial ones.

2 The function spaces

In this paper, we consider the Hilbert space of rational functions whose poles are in a prescribed set $\mathfrak{P} = \{1/\alpha_k : k = 1, 2, \ldots\}$ with all $\alpha_k \in \mathbb{R}$. Thus if Π_n denotes the set of polynomials of degree at most n, and if we set $\pi_0 = 1$ and $\pi_n(z) = \prod_{k=1}^n (1 - \alpha_k z)$ for $n = 1, 2, \ldots$, then we consider spaces of rational functions

$$\mathcal{L}_n = \left\{ \frac{p_n(z)}{\pi_n(z)} : p_n \in \Pi_n \right\}.$$

We assume that the functions are real valued on the extended real line $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and that the inner product is given by

$$\langle f, g \rangle = \int_{\hat{\mathbb{R}}} f(x) g(x) \mathrm{d}\mu(x),$$

with $\mathrm{d}\mu$ some positive measure on $\hat{\mathbb{R}}$ such that $\int_{\hat{\mathbb{R}}} \mathrm{d}\mu(t) = 1$.

It is obvious that if we set all $\alpha_k = 0$, then $\mathcal{L}_n = \Pi_n$, and the polynomial case is recovered.

Note that unlike the polynomial case, in general, a function in $\mathcal{L}_n$ can be finite at ∞, so that μ can have a mass point at ∞.

The orthonormal rational functions, i.e., the functions $\{\phi_0, \phi_1, \ldots\}$ such that $\phi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$, $\phi_n \perp \mathcal{L}_{n-1}$ and $\|\phi_n\| = 1$ were studied by M.M. Djrbashian [7, 8, 9, 10, 11, 12], (see also [20]), later by A. Bultheel, K. Pan, Xin Li, and in a long list of papers by Bultheel, González-Vera, Hendriksen and Njåstad. Most of the known results are brought together in the monograph [4]. These rational functions are also related to polynomials orthogonal with respect to varying measures as studied in several papers by López-Lagomasino and others e.g. [16, 17].

3 Reproducing kernels and orthogonal rational functions

The idea is to replace the basis of orthogonal rational functions $\{\phi_k\}$ by a basis consisting of reproducing kernels, because the latter have better localization properties. So let us first recall some general properties of reproducing kernels. Assume that we have a general separable Hilbert space $\mathcal{L}$ at our disposal. The following properties concerning the reproducing kernels are then well known (see for example [1, 19, 13]).

Theorem 3.1 *We consider a general complex separable Hilbert space $\mathcal{L}$ and an $(m+1)$-dimensional subspace $\mathcal{K} \subset \mathcal{L}$. We denote the reproducing kernel for $\mathcal{K}$ by $k(z,w)$. Then*

1. *For any orthonormal basis $\{\phi_0, \ldots, \phi_m\}$ of $\mathcal{K}$, the reproducing kernel for $\mathcal{K}$ is given by*

$$k(z,w) = \sum_{k=0}^{m} \phi_k(z)\overline{\phi_k(w)}.$$

It reproduces any function in $\mathcal{K}$ as follows

$$\langle f, k(\cdot, w)\rangle = f(w), \quad \forall f \in \mathcal{K}.$$

2. *For any set of distinct points $\{w_0, w_1, \ldots, w_m\}$ among the points of analyticity for $\mathcal{K}$,*

$$\langle k(z,w_j), k(z,w_i)\rangle = k(w_i, w_j)$$

and the matrix

$$M = [k(w_i, w_j)]_{i,j=0}^{m}$$

is positive semi-definite.

3. *The orthogonal projection of $f \in \mathcal{L}$ onto $\mathcal{K}$ is given by*

$$P_{\mathcal{K}} f = \langle f(z), k(z,w)\rangle, \quad \forall f \in \mathcal{L}.$$

4. *For any point w where it makes sense,*

$$\inf_{f\in\mathcal{K}}\{\|f\|^2 : f(w) = 1\} = [k(w,w)]^{-1}$$

and the minimizer is $f(z) = k(z,w)/k(w,w)$.

Let us now return to our spaces of rational functions that we introduced above. Then the following lemma is easy to verify.

Lemma 3.2 *Let $\{\phi_0, \ldots \phi_n\}$ be the orthonormal basis for the $(n+1)$-dimensional space of rational functions $\mathcal{L}_n$ as introduced above, then for any set of distinct points $\mathbf{x} = \{x_0, x_1, \ldots, x_n\}$ which are points of analyticity for functions in $\mathcal{L}_n$, the matrix*

$$\Phi_n(\mathbf{x}) = \begin{bmatrix} \phi_0(x_0) & \cdots & \phi_0(x_n) \\ \vdots & & \vdots \\ \phi_n(x_0) & \cdots & \phi_n(x_n) \end{bmatrix} \tag{3.1}$$

is invertible.

Proof. If it were not invertible, then there would exist a nonzero vector $\mathbf{c} = [c_0, \ldots, c_n]$ such that $\mathbf{c}\Phi_n(\mathbf{x}) = 0$. In other words, the function $\phi(z) = \sum_{k=0}^{n} c_k\phi_k(z) \in \mathcal{L}_n$ would vanish at $n+1$ distinct points $\{x_k\}_{k=0}^n$. Because ϕ is rational of degree at most n, it would be identically zero and this would imply that the functions $\{\phi_k\}_{k=0}^n$ were linearly dependent, which is impossible. □

This lemma shows the unisolvence (Haar condition) of the system $\{\phi_k\}$. See Davis [6, chap. II, Sect. 2.4]. This lemma entails immediately

Corollary 3.3 *If $k_n(z, w)$ is the reproducing kernel for $\mathcal{L}_n$ and the $\mathbf{x} = \{x_k\}_{k=0}^n$ are $n+1$ distinct points of analyticity for $\mathcal{L}_n$, then the functions $\{\varphi_{nj}(z) = k_n(z, x_j)\}_{j=0}^n$ form a basis for $\mathcal{L}_n$.*

Proof. Since, with the matrix $\Phi_n(\mathbf{x})$ as in the previous lemma:

$$\begin{bmatrix} k_n(z, x_0) \\ \vdots \\ k_n(z, x_n) \end{bmatrix} = \Phi_n^H(\mathbf{x}) \begin{bmatrix} \phi_0(z) \\ \vdots \\ \phi_n(z) \end{bmatrix}, \tag{3.2}$$

the statement follows because $\{\phi_k\}_{k=0}^n$ is a basis and $\Phi_n(\mathbf{x})$ is invertible. □

4 Multiresolution

Now we want a multiscale representation of a function. If we use the orthonormal basis ϕ_n and consider the Fourier expansion

$$f(z) \sim c_0\phi_0 + c_1\phi_1(z) + c_2\phi_2(z) + \cdots, \quad c_k = \langle f, \phi_k \rangle,$$

then as we add more and more terms, we shall add more and more details to the low resolution approximation. We shall consider the spaces $\mathcal{L}_{2^s}$ for $s = 0, 1, \ldots$ and set $\mathcal{V}_s = \mathcal{L}_{2^s}$. The orthogonal complement of $\mathcal{V}_s$ in $\mathcal{V}_{s+1}$ is denoted by $\mathcal{W}_s$. Thus if we assume from now on that n always has the meaning of 2^s, then we can define the nested spaces $\mathcal{V}_s$ and the orthogonal complements $\mathcal{W}_s$ in $\mathcal{V}_{s+1}$ as

$$\mathcal{V}_{-1} = \mathcal{L}_0, \quad \mathcal{W}_{-1} = \mathcal{V}_0 \ominus \mathcal{V}_{-1}, \quad \mathcal{V}_s = \mathcal{L}_n \quad \text{and} \quad \mathcal{W}_s = \mathcal{V}_{s+1} \ominus \mathcal{V}_s, \quad s = 0, 1, \ldots$$

We call the orthogonal projection of a function onto $\mathcal{V}_s$ the representation of that function at scale or resolution level s. For $s = -1$, for example, this is just a constant c_0, which is the weighted (by μ) average of the function.

Obviously, $\mathcal{V}_s = \text{span}\{\phi_k\}_{k=0}^n$ and $\mathcal{W}_s = \text{span}\{\phi_k\}_{k=n+1}^{2n}$ can both be generated by our orthonormal basis functions. These basis functions are in general not localized both in the z and in the frequency domain and as we mentioned before, we want to switch to a basis of reproducing kernels. However, since the reproducing kernels $k_n(z,w)$ are expressed in terms of the ϕ_k, we first discuss the basis of orthogonal rational functions (ORF) in some detail.

5 The ORF basis

Recall that we required the points α_k to be all in $\mathbb{R}$, so that the poles $1/\alpha_k$ are all on $\hat{\mathbb{R}}\setminus\{0\}$. It will also be assumed the the measure has infinite support and that an infinite sequence of orthogonal functions, as defined below, will exists.

Define the elementary factors

$$\zeta_k(z) = \frac{z}{1-\alpha_k z}, \quad k = 0, 1, \ldots$$

and set $B_0 = 1$ and $B_n = \zeta_1 \cdots \zeta_n$ for $n = 1, 2, \ldots$ and $\mathcal{L}_n = \text{span}\{B_0, B_1, \ldots, B_n\}$. The substar is defined by $f_*(z) = \overline{f(\overline{z})}$. Note that the basis functions B_k satisfy $B_{k*} = B_k$ so that $f \in \mathcal{L}_n \Rightarrow f_* \in \mathcal{L}_n$. Let the orthonormal functions ϕ_k be generated by orthonormalizing the basis $B_0, B_1, \ldots$, then it is easily checked that the coefficients of the ϕ_k with respect to the basis B_k are real. We remark that if all the $\alpha_k = 0$, then $B_k(z) = z^k$ and hence, $\mathcal{L}_n = \Pi_n$, the set of polynomials of degree at most n. We say that ϕ_n is regular (or that n is a regular index) if $p_n(1/\alpha_{n-1}) \neq 0$ where p_n is the numerator polynomial of ϕ_n, i.e., $\phi_n = p_n/\pi_n$ with $\pi_n(z) = \prod_{k=1}^n (1-\alpha_k z)$. We say that the system $\{\phi_k\}$ is regular if ϕ_k is regular for all $k = 0, 1, \ldots$. For consistency reasons, we shall say that a polynomial $p_n \in \Pi_n$ will have a zero at infinity whenever it has defective degree, i.e., $p_n \in \Pi_{n-1}$. Hence, in the polynomial case where all α_k are zero, the system of orthogonal polynomials will automatically be regular, because the orthogonal polynomials will never have defective degree.

If the system is regular, then the following generalization of the three-term recurrence relation for orthogonal polynomials exists. We refer to [4] for a proof.

Theorem 5.1 *Suppose that the orthonormal system $\{\phi_k\}$ is regular, then there holds a recurrence relation of the following form*

$$\phi_n(z) = \left(A_n\zeta_n(z) + B_n\frac{\zeta_n(z)}{\zeta_{n-1}(z)}\right)\phi_{n-1}(z) + C_n\frac{\zeta_n(z)}{\zeta_{n-2}(z)}\phi_{n-2}(z), \quad n = 2, 3, \ldots \tag{5.1}$$

with initial conditions

$$\phi_0(z) = 1, \quad \phi_1(z) = (A_1\zeta_1(z) + B_1)\phi_0(z).$$

Moreover A_n and C_n are all real and nonzero, and $A_n = -C_nA_{n-1}$, $n = 2, 3, \ldots$

Conversely, if the functions ϕ_k are given by such a relation, then they will be orthonormal with respect to some positive measure μ on $\hat{\mathbb{R}}$.

This theorem can also be found in several pieces and in slightly varying forms for example in [2, 3] where analogs are given for the unit circle.

The relation $A_n = -C_nA_{n-1}$ monitors the norms. This means that ϕ_k, $k = 2, 3, \ldots$ will be normalized to norm 1 if we choose ϕ_0 and ϕ_1 to be normalized. Recall that $\phi_0 = 1$ is normalized because we assumed that $\int \mathrm{d}\mu = 1$. So, if all the numbers C_n are given, then all the A_n will be uniquely defined once that A_1 is fixed. This value of A_1 is related to the orthonormality of ϕ_1. From the viewpoint of the Favard theorem, given all the C_n and the B_n, we can choose A_1 nonzero and then all the remaining A_n will be fixed. Thus all the orthogonal functions are fixed, and therefore also the orthogonality measure is fixed to a large extend. In this sense, A_1 will impose a certain condition on the orthogonality measure. The coefficient A_1 is a generalized standard deviation and somehow it controls the width of the weight function as is explained in Appendix A, Lemma A.1.

Another useful property we shall need later is that $A_n = 0$ iff ϕ_n is not regular. This is also proved in Appendix A, Lemma A.2.

We also have a Christoffel-Darboux type formula, which can be formulated as follows.

Theorem 5.2 (Christoffel-Darboux) *Let $\{\phi_k\}_{k=0}^n$ with $\phi_k \in \mathcal{L}_k \setminus \mathcal{L}_{k-1}$ be an orthonormal basis for $\mathcal{L}_n$, then the reproducing kernels $k_n(z,w) = \sum_{k=0}^n \phi_k(z)\overline{\phi_k(w)}$ for $\mathcal{L}_n$ satisfy*

$$A_{n+1}\frac{z-\overline{w}}{z\overline{w}}k_n(z,w) = \left[\overline{\left(\frac{\phi_n(w)}{\zeta_n(w)}\right)}\left(\frac{\phi_{n+1}(z)}{\zeta_{n+1}(z)}\right) - \left(\frac{\phi_n(z)}{\zeta_n(z)}\right)\overline{\left(\frac{\phi_{n+1}(w)}{\zeta_{n+1}(w)}\right)}\right].$$

For a proof we again refer to [4], where also the following generalization of a well known property of the polynomials can be found.

Theorem 5.3 *Let ϕ_n be the nth orthogonal rational function, then the zeros of the numerators of ϕ_n are simple and are in $\hat{\mathbb{R}}$. If ϕ_n is regular, then the numerators of ϕ_n and ϕ_{n-1} have no common zeros.*

Note that the presence of poles can disrupt the classical property for polynomials which says that the zeros of the orthogonal polynomials are real and interlace with the zeros of the neighboring orthogonal polynomials. In the rational case, the numerators of the orthogonal functions have simple zeros which are all in $\hat{\mathbb{R}}$, but it is not a priory sure that none of the zeros of the numerators coincides with with one of the numbers in $\hat{\mathfrak{P}} = \mathfrak{P} \cup \{\infty\} = \{1/\alpha_0, 1/\alpha_1, \ldots\}$ (we define $\alpha_0 = 0$). Because of this problem, one should consider zeros of quasi-orthogonal functions to construct quadrature formulas because such quasi-orthogonal functions can be guaranteed to have the appropriate number of real and simple zeros.

Quasi-orthogonal functions are defined by

$$Q_n(z,\tau_n) = \phi_n(z) + \tau_n\frac{\zeta_n(z)}{\zeta_{n-1}(z)}\phi_{n-1}(z) \in \mathcal{L}_n, \quad \tau_n \in \hat{\mathbb{R}}.$$

For $\tau_n = \infty$, this should be read as $Q_n(z,\tau_n) = [\zeta_n(z)/\zeta_{n-1}(z)]\phi_{n-1}(z)$. The numerator of a quasi-orthogonal function $Q_n(z,\tau_n)$, will have n simple zeros in $\mathbb{R} \setminus \mathfrak{P}_n$ with $\mathfrak{P}_n = \{1/\alpha_1, \ldots, 1/\alpha_n\}$, except for at most n values of $\tau_n \in \mathbb{R}$. Let us denote by $\mathfrak{E}_n$ the set of these exceptional points for τ_n. We call τ_n regular if $\tau_n \in \mathbb{R} \setminus \mathfrak{E}_n$. We call $Q_n(z,\tau_n)$ regular if ϕ_n is regular and τ_n is regular. Note that if $Q_n(z,\tau_n)$ is regular, then $Q_n(z,\tau_n)$ has n simple real zeros, which are not in $\{1/\alpha_1, \ldots, 1/\alpha_n\}$. Also observe that $\tau_n = \infty$ can never be a regular value because $z = 1/\alpha_{n-1}$ will always be a zero of the numerator of $Q_n(z,\infty)$ by construction. In fact the exceptional points $\mathfrak{E}_{n+1}$ are among the points of the form

$$\mathfrak{E}_{n+1} \subseteq \left\{-\frac{\phi_{n+1}(w)/\zeta_{n+1}(w)}{\phi_n(w)/\zeta_n(w)}, \quad w \in \mathfrak{P}_{n+1}\right\}.$$

With $\mathfrak{P}_{n+1} = \{1/\alpha_1, \ldots, 1/\alpha_{n+1}\}$, we have the following theorem.

Theorem 5.4 *If $Q_{n+1}(z,\tau_{n+1})$ is regular, then it has $n+1$ simple and real zeros in $\mathbb{R}\setminus\mathfrak{P}_{n+1}$. Let us denote them by $\{\xi_{n,k}\}_{k=0}^{n}$. Moreover, defining $\lambda_{nk}=[k_n(\xi_{nk},\xi_{nk})]^{-1}$, with $k_n(z,w)$ the reproducing kernel for $\mathcal{L}_n$, then equality holds in*

$$\langle f,g\rangle=\sum_{k=0}^{n}\lambda_{nk}f(\xi_{nk})g(\xi_{nk}),\quad \forall f,g\in\mathcal{L}_n.$$

Conversely, if the above equality holds for all $f,g\in\mathcal{L}_n$, then the nodes ξ_{nk} are the distinct zeros of some regular quasi-orthogonal rational function.

Again, the proof of this theorem can be found in [4].

We know that in the polynomial case $\tau_{n+1}=0$ is a regular value and the $n+1$ zeros of ϕ_{n+1} are simple and real. Moreover we have the Gaussian quadrature formula

$$\int_{\hat{\mathbb{R}}} f(x)\mathrm{d}\mu(x)=\sum_{k=0}^{n}\lambda_{nk}f(\xi_{nk}),\quad \forall f\in\Pi_{2n+1}.$$

Also this can be generalized to the rational case: if $\tau_{n+1}=0$ is a regular value, then the zeros of ϕ_{n+1} can be used in a quadrature formula which will be exact in a slightly larger space $\mathcal{L}_{n+1}\cdot\mathcal{L}_n$, and not just in $\mathcal{L}_n\cdot\mathcal{L}_n$.

Concerning the zeros $\{\xi_{nk}\}$ of these regular quasi-orthogonal functions, i.e., the nodes of these quadrature formulas, we show that we can choose one, for example ξ_{n0}, arbitrarily in $\mathbb{R}\setminus\mathfrak{P}_{n+1}$ and the other zeros are then given by the zeros of $k_n(z,\xi_{n0})$.

Theorem 5.5 *Let $k_n(z,w)$ be the reproducing kernel for $\mathcal{L}_n$ and assume that ϕ_{n+1} is regular. Choose w arbitrary in $\mathbb{R}\setminus\mathfrak{P}_{n+1}$ and define*

$$\tau_{n+1}=-\frac{\phi_{n+1}(w)/\zeta_{n+1}(w)}{\phi_n(w)/\zeta_n(w)}. \tag{5.2}$$

Then, if τ_{n+1} is finite (i.e. w is not a zero of the numerator of ϕ_n), then defining $\xi_{n0}=w$ and $\{\xi_{nk}\}_{k=1}^{n}$ as the n zeros of $k_n(z,w)$, we have that the numbers $\{\xi_{nk}\}_{k=0}^{n}$ are the zeros of the regular quasi-orthogonal function $Q_{n+1}(z,\tau_{n+1})$.

Conversely, if $\{\xi_{nk}\}_{k=0}^{n}$ are the zeros of some regular quasi-orthogonal function $Q_{n+1}(z,\tau_{n+1})$, then there exists a number $w\in\mathbb{R}\setminus\mathfrak{P}_{n+1}$, such that $\xi_{n0}=w$ and τ_{n+1} is given by (5.2), while $\{\xi_{nk}\}_{k=1}^{n}$ are the n zeros of $k_n(z,w)$.

Proof. Note that $w\notin\mathfrak{P}_{n+1}$ implies $\tau_{n+1}\notin\mathfrak{E}_{n+1}$ unless $\tau_{n+1}=\infty$. In that case w should be a zero of the numerator of ϕ_n, but since it is not one of the α_k, it has to be a zero of ϕ_n. In that case, it follows from the Christoffel-Darboux formula that, as a function of z, $k_n(z,w)\in\mathcal{L}_{n-1}$, and thus, it can never have $n+1$ zeros. However, if τ_{n+1} is finite, and thus regular, then it follows easily from the Christoffel-Darboux relation that the n zeros of $k_n(z,w)$ coincide with n zeros of $Q_{n+1}(z,\tau_{n+1})$. The remaining zero of $Q_n(z,\tau_{n+1})$ is obviously $z=w$.

The converse statement follows along the same lines. By Lemma A.5 in Appendix A, there are $n+1$ values of w for which the right-hand side in (5.2) will give the value of any τ_{n+1}. Because $Q_{n+1}(z,\tau_{n+1})$ is regular, w has to be in $\mathbb{R}\setminus\mathfrak{P}_{n+1}$ and because $\tau_{n+1}\neq\infty$, w will not be a zero of ϕ_n. Now the Christoffel-Darboux relation can be applied again to give the result. □

6 The ORK basis

We discussed the ORF basis for $\mathcal{L}_n$, but this has in general not the property of being a local basis. For a wavelet analysis, one would rather have a basis of kernel functions, which, if possible, should be chosen orthogonal.

By Corollary 3.3, we know that $\{k_n(z, x_j)\}_{j=0}^n$ forms a basis for $\mathcal{L}_n$ for almost any set of distinct points $\mathbf{x} = \{x_j\}_{j=0}^n$. The question is whether it is possible to choose the points in $\mathbf{x}$ such that this basis is orthogonal. In that case we would have a basis of orthogonal rational kernels (ORK).

It turns out that if we choose $x_j = \xi_{nj}$, $j = 0, \ldots, n$ to be the zeros of a regular quasi-orthogonal function $Q_{n+1}(z, \tau_{n+1})$, then the basis $\{\varphi_{nj}(z) = k_n(z, \xi_{nj})\}_{j=0}^n$ is orthogonal.

Theorem 6.1 *Let $k_n(z, w)$ be the reproducing kernel for $\mathcal{L}_n$ and let $\boldsymbol{\xi}_n = \{\xi_{nj}\}_{j=0}^n$ be the zeros of a regular quasi-orthogonal function $Q_{n+1}(z, \tau_{n+1})$. Then the basis for $\mathcal{L}_n$ defined by*

$$\varphi_{nj}(z) = k_n(z, \xi_{nj}), \quad j = 0, 1, \ldots, n$$

is orthogonal.

Proof. Let $\Phi_n = \Phi_n(\boldsymbol{\xi}_n)$ be defined by (3.1) then it follows from (3.2) and from

$$\langle \varphi_{ni}, \varphi_{nj} \rangle = \langle k_n(z, \xi_{ni}), k_n(z, \xi_{nj}) \rangle = k_n(\xi_{nj}, \xi_{ni}), \quad i, j = 0, \ldots, n$$

that $\{\varphi_{nj}\}_{j=0}^n$ will be an orthogonal basis if and only if

$$k_n(\xi_{nj}, \xi_{ni}) = \delta_{ij} k_n(\xi_{ni}, \xi_{ni}), \quad i, j = 0, \ldots, n.$$

In terms of the matrix Φ_n this reads

$$\Phi_n^H \Phi_n = \Lambda_n^{-1} \quad \text{or equivalently} \quad \Phi_n \Lambda_n \Phi_n^H = I$$

with $\Lambda_n = \operatorname{diag}(\lambda_{n0}, \ldots, \lambda_{nn})$ a diagonal matrix with $\lambda_{ni} = 1/k_n(\xi_{ni}, \xi_{ni}) \neq 0$. Writing out the relation $\Phi_n \Lambda_n \Phi_n^H = I$ elementwise gives

$$\sum_{k=0}^{n} \lambda_{nk} \phi_i(\xi_{nk}) \overline{\phi_j(\xi_{nk})} = \delta_{ij}, \quad i, j = 0, \ldots, n.$$

Because also $\langle \phi_i, \phi_j \rangle = \delta_{ij}$ for $i, j = 0, \ldots, n$, this means that the above quadrature formula is exact for the inner product of all basis functions in $\mathcal{L}_n$, hence for the inner product of any two functions in $\mathcal{L}_n$. By Theorem 5.4, this means that the nodes $\{\xi_{nk}\}$ should be the zeros of a regular quasi-orthogonal function $Q_{n+1}(z, \tau_{n+1})$. $\square$

7 The WRK basis

Now suppose that we know the function at resolution level $2n$, i.e., we know $f_{2n} \in \mathcal{L}_{2n}$. The problem is to decompose the function into two orthogonal functions

$$f_{2n} = f_n + g_n, \quad f_n \in \mathcal{L}_n, \quad g_n \in \mathcal{K}_n,$$

where $\mathcal{K}_n = \mathcal{L}_{2n} \ominus \mathcal{L}_n$ is the wavelet space. This is a trivial problem if we know the decomposition with respect to the ORF basis: if $f_{2n} = \sum_{k=0}^{2n} c_k\phi_k$, then $f_n = \sum_{k=0}^{n} c_k\phi_k$ while $g_n = \sum_{k=n+1}^{2n} c_k\phi_k$. However, for reasons that have been explained, we prefer not to use the ORF basis, but we use the ORK basis instead. We know how to express the function f_n in terms of an ORK basis. The remaining problem is to write g_n in terms of a basis which is generated from the reproducing kernel $l_n(z,w)$ for $\mathcal{K}_n = \mathcal{L}_{2n} \ominus \mathcal{L}_n$. Obviously the reproducing kernel for $\mathcal{K}_n$ is

$$l_n(z,w) = k_{2n}(z,w) - k_n(z,w) = \sum_{k=n+1}^{2n} \phi_k(z)\overline{\phi_k(w)}.$$

The main question in this respect is: Can we find n real numbers $\{\eta_{nj}\}_{j=0}^{n-1}$ such that the functions $\{\psi_{nj}(z) = l_n(z,\eta_{nj})\}_{j=0}^{n-1}$ form a basis for $\mathcal{K}_n$ and if possible, can it be made orthogonal?

This problem for the basis of $\mathcal{K}_n$ is not as trivial as it was in Corollary 3.3 for the basis of $\mathcal{L}_n$. There is indeed no guarantee that for an arbitrary set of distinct points $\mathbf{y}_n = \{y_k\}_{k=0}^{n-1}$ (on $\mathbb{R}$ or not) the matrix

$$\Psi_n(\mathbf{y}_n) = \begin{bmatrix} \phi_{n+1}(y_0) & \cdots & \phi_{n+1}(y_{n-1}) \\ \vdots & & \vdots \\ \phi_{2n}(y_0) & \cdots & \phi_{2n}(y_{n-1}) \end{bmatrix} \qquad (7.1)$$

would be invertible. However, using the following lemma, it is possible to prove that there always exists a set of points $\{y_j\}_{j=0}^{n-1}$ in $\mathbb{R} \setminus \mathfrak{P}_{2n}$ which make this matrix invertible.

Lemma 7.1 *Assume that $Q \in \mathbb{C}^{(2n+1)\times(2n+1)}$ is a square matrix such that $Q^H Q = D$ with D invertible and diagonal. Assume moreover that this Q is partitioned as*

$$Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}$$

with $Q_1 \in \mathbb{C}^{(n+1)\times(n+1)}$ and hence $Q_4 \in \mathbb{C}^{n\times n}$. If Q_1 is invertible and if $Q_2 \in \mathbb{C}^{n+1\times n}$ is of full rank n, then Q_4 is invertible.

Proof. Let D be partitioned into the two parts $D_1 \in \mathbb{R}^{(n+1)\times(n+1)}$ and $D_2 \in \mathbb{R}^{n\times n}$. We know from $Q^H Q = D$ that

$$Q_2^H Q_1 + Q_4^H Q_3 = 0$$

so that rank $(Q_2^H Q_1) =$ rank $(Q_4^H Q_3)$. Because Q_1 is invertible and rank $(Q_2) = n$, we find that rank $(Q_4^H Q_3) = n$. Now if Q_4 were singular, then there would exist a nonzero vector $\mathbf{c} \in \mathbb{C}^{1\times n}$ such that $\mathbf{c}Q_4^H = 0$, hence also $\mathbf{c}Q_4^H Q_3 = 0$. In other words, rank $(Q_4^H Q_3) < n$, which is a contradiction. □

So we have

Theorem 7.2 *Consider the zeros $\boldsymbol{\xi}_{2n} = \{\xi_{2n,k} : k = 0,1,\ldots,2n\}$ of the regular quasi-orthogonal function $Q_{2n+1}(z,\tau_{2n+1})$ for some $\tau_{2n+1} \in \mathbb{R} \setminus \mathfrak{E}_{2n+1}$. If we select $\mathbf{y}_n = \{y_k : k = 0,\ldots,n-1\}$ to be any n out of the $2n+1$ zeros in $\boldsymbol{\xi}_{2n}$, then the matrix $\Psi_n(\mathbf{y}_n)$ as defined in (7.1) will be invertible and hence $\{\psi_{nj}(z) = k_{2n}(z,y_j) - k_n(z,y_j)\}_{j=0}^{n-1}$ will form a basis for the wavelet space $\mathcal{L}_{2n} \ominus \mathcal{L}_n$.*

Proof. Since the ordering of the zeros $\xi_{2n,k}$ is completely arbitrary, we can always assume that we select the y_k to be the last elements of $\boldsymbol{\xi}_{2n} = \{\xi_{2n,j} : j = 0, \ldots, 2n\}$. Now consider the matrix $\Phi_{2n} = \Phi_{2n}(\boldsymbol{\xi}_{2n})$ of (3.1) where the evaluation is in the points of $\boldsymbol{\xi}_{2n}$, then $\Psi_n(\mathbf{y}_n)$ appears as the $n \times n$ right lower part of the matrix Φ_{2n}. By Lemma 3.2, it follows that taking the block of the first $n+1$ columns in Φ_{2n}, any selection of k different rows from it will result in a matrix of rank $\min\{k, n+1\}$. Thus the conditions of the previous lemma are satisfied and the theorem follows. □

In the polynomial case, another possibility exists to construct a WRK basis by choosing $\mathbf{y}_n = \{y_k\}_{k=0}^{n-1}$ to be the zeros of the polynomial ϕ_n. This was proved in [14]. We give an alternative proof.

Theorem 7.3 *Assume that* $\mathbf{y}_n = \{y_k\}_{k=0}^{n-1}$ *are the zeros of the orthogonal polynomial* ϕ_n*, then* $\{l_n(z, y_k)\}_{k=0}^{n-1}$ *forms a basis for* $\mathcal{K}_n = \mathcal{L}_{2n} \ominus \mathcal{L}_n$*.*

Proof. If it were not a basis, then the matrix $\Psi_n(\mathbf{y}_n)$ would be singular, and thus there would exist a nonzero vector $\mathbf{c} = [c_1, \ldots, c_n]$ such that $\Psi_n(\mathbf{y}_n)\mathbf{c}^T = 0$. Consider the function $\phi(z) = \sum_{k=1}^{n} c_k \phi_{n+k}$. This function vanishes in the zeros of ϕ_n. Thus it has to be of the form $\phi = \phi_n p_r$ where p_r is a polynomial of degree r with $0 \le r \le n$. On the other hand $\langle \phi, \phi_k \rangle = 0$ for any $k = 0, \ldots, n$. In particular $\langle \phi, \phi_{n-r} \rangle = \langle p_r \phi_{n-r}, \phi_n \rangle = 0$. This is however impossible because $p_r \phi_{n-r}$ is of strict degree n and can thus not be orthogonal to ϕ_n. Thus we have a contradiction and the $l_n(z, y_k)$ have to be independent. □

Example 7.1 Let us give an example. The Hermite polynomials are given by the recurrence relation (5.1) with all $\alpha_k = 0$ and initial conditions $\phi_0 = 1$, $\phi_1(z) = \sqrt{2}z$, and

$$A_n = \sqrt{\frac{2}{n}}, \quad B_n = 0, \quad C_n = -\sqrt{\frac{n-1}{n}}, \quad n = 2, 3, \ldots.$$

These are orthogonal with respect to the weight $w(x) = \pi^{-1/2} e^{-x^2}$, $x \in \mathbb{R}$, thus in our notation

$$\mathrm{d}\mu(x) = \frac{e^{-x^2}\mathrm{d}x}{\sqrt{\pi}}.$$

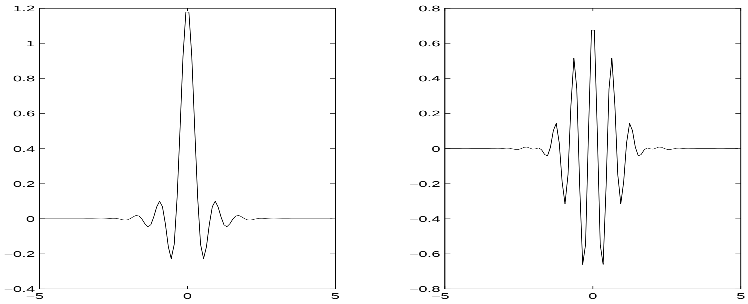
The scaling functions and wavelet functions multiplied by the weight are plotted in Figure 1. The wavelet plot in Figure 1(B) is of course the same as in Figure 2.7(d) of [14]. See also Example 12.1 for the effect of introducing poles.

Example 7.2 For the Chebyshev polynomials of the first kind we have plotted Figure 2. Of course this is directly related to the trigonometric bases studied in [5, 22].

This settles the question of how to find a basis of reproducing kernels for $\mathcal{K}_n$. The question of how one has to choose these points to make this basis orthogonal is yet an open problem. Even in the polynomial case, it is not known how this could be obtained.

FIG. 1: The scaling and wavelet functions $k_{32}(z, w)$ (left) and $l_{32}(z, w)$ (right) multiplied by the weight function in the case of Hermite polynomials. In Figure A: $w = 0$, in Figure B: $w = 1$.

A: $k_{32}(z, 0)$ and $l_{32}(z, 0)$



B: $k_{32}(z, 1)$ and $l_{32}(z, 1)$

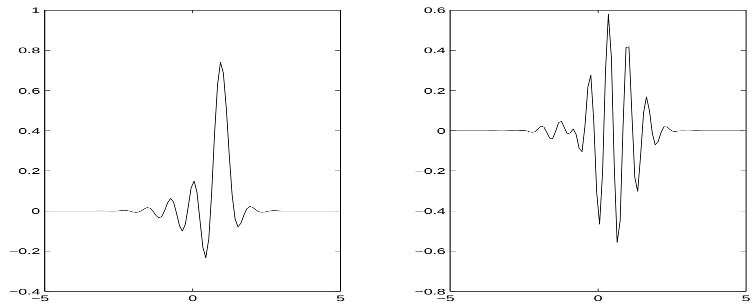
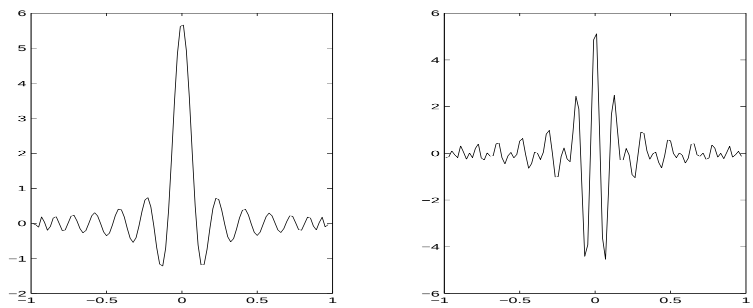


FIG. 2: The scaling and wavelet functions $k_{32}(z, 0)$ (left) and $l_{32}(z, 0)$ (right) in the case of Chebyshev polynomials.



8 Interpolation and biorthogonal bases

Although an orthogonal basis is easy for computations, it is almost as easy to work with biorthogonal basis functions. So, in many wavelet applications authors have designed biorthogonal basis functions, which leave more freedom in the design of the wavelet. By relaxing the orthogonality condition, one can give the wavelet more smoothness properties in combination with an essentially local support. In the case of reproducing kernels, the biorthogonal basis functions are directly related to interpolating functions. We will give a treatment that is completely parallel to the polynomial case.

Recall that the poles for $\mathcal{L}_n$ were taken from the set $\mathfrak{P}_n = \{1/\alpha_1, \ldots, 1/\alpha_n\} \subset \hat{\mathbb{R}} \setminus \{0\}$.

Let us define the Lagrange polynomials in Π_n for the interpolation points $\mathbf{x}_n = \{x_k\}_{k=0}^n$ (all distinct and on $\mathbb{R} \setminus \mathfrak{P}_n$) by

$$l_{nk}(z) = \frac{\prod_{j=0,j\neq k}^n (z - x_j)}{\prod_{j=0,j\neq k}^n (x_k - x_j)}, \quad k = 0, \ldots, n$$

and define

$$L_{nk}(z) = l_{nk}(z)\frac{\pi_n(x_k)}{\pi_n(z)}, \quad k = 0, \ldots, n$$

with

$$\pi_n(z) = \prod_{j=0}^n (1 - \alpha_j z).$$

Then obviously $L_{nk}(x_j) = \delta_{kj}$, $k, j = 0, \ldots, n$ while $L_{nk} \in \mathcal{L}_n$. We call these L_{nk} the fundamental Lagrange interpolating functions (FLIF) of $\mathcal{L}_n$ for the points $\mathbf{x}_n = \{x_k\}_{k=0}^n$. It immediately follows that for any function $f \in \mathcal{L}_n$ we may write

$$f(z) = \sum_{k=0}^n f(x_k) L_{nk}(z).$$

Defining the discrete inner product in $\mathcal{L}_n$

$$\langle f, g\rangle_{\mathbf{x}_n} = \sum_{k=0}^n f(x_k) g(x_k),$$

it is directly seen that $\langle L_{nk}, L_{nj}\rangle_{\mathbf{x}_n} = \delta_{kj}$. Thus the FLIF are an orthonormal basis for $\mathcal{L}_n$ with respect to this inner product. So, by the general theory of reproducing kernels, it follows that the solution of the problem

$$\min\{\|f\|_{\mathbf{x}_n}^2 : f \in \mathcal{L}_n; f(x_k) = 1\}$$

is given by $\sum_{j=0}^n L_{nj}(z)\overline{L_{nj}(x_k)} = L_{nk}(z)$. Moreover, if we set $\varphi_{nk}(z) = k_n(z, x_k)$ and $\tilde{\varphi}_{nk}(z) = L_{nk}(z)$, then because of the reproducing property of φ_{nk}

$$\langle \tilde{\varphi}_{nk}, \varphi_{nj}\rangle = \delta_{kj}.$$

In other words, $\{\varphi_{nk}\}_{k=0}^n$ and $\{\tilde{\varphi}_{nk}\}_{k=0}^n$ are biorthogonal bases for $\mathcal{L}_n$.

Note that we can characterize $\tilde{\varphi}_{nk}$ in another way. We may write $\tilde{\varphi}_{nk} = \sum_{j=0}^n c_j^{(k)} \phi_j$ where $\mathbf{c}^{(k)} = [c_0^{(k)} \ldots c_n^{(k)}]$ is defined by $\mathbf{c}^{(k)}\Phi_n = \mathbf{e}_k$ where $\mathbf{e}_k = [0 \ldots 0\ 1\ 0 \ldots 0]$ is the kth unit vector and $\Phi_n = \Phi_n(\mathbf{x}_n)$ is the matrix (3.1). Thus $\mathbf{c}^{(k)}$ is the kth row in the inverse of Φ_n, so that we have

$$\begin{bmatrix} \tilde{\varphi}_{n0} \\ \vdots \\ \tilde{\varphi}_{nn} \end{bmatrix} = \begin{bmatrix} c_0^{(0)} & \cdots & c_n^{(0)} \\ \vdots & & \vdots \\ c_0^{(n)} & \cdots & c_n^{(n)} \end{bmatrix} \begin{bmatrix} \phi_0 \\ \vdots \\ \phi_n \end{bmatrix} = \Phi_n^{-1} \begin{bmatrix} \phi_0 \\ \vdots \\ \phi_n \end{bmatrix}. \tag{8.1}$$

It is clear that if the x_k are the zeros of the regular quasi-orthogonal function $Q_{n+1}(z, \tau_{n+1})$, then the φ_{nk} are orthogonal and $\langle \varphi_{ni}, \varphi_{nj} \rangle = \delta_{ij}\varphi_{ni}(x_i)$. In that case we have of course $L_{nk}(z) = \varphi_{nk}(z)/\varphi_{nk}(x_k)$.

Similarly, given $\mathbf{y}_n = \{y_k\}_{k=0}^{n-1}$, one can construct a biorthogonal basis for $\psi_{nk}(z) = l_n(z, y_k)$, on condition that the matrix $\Psi_n = \Psi_n(\mathbf{y}_n)$ of (7.1) is invertible. Indeed, if $\mathbf{e}_k$ is the kth unit vector, then if Ψ_n is invertible, there is exactly one solution $\mathbf{d}^{(k)} = [d_{n+1}^{(k)} \dots d_{2n}^{(k)}]$ to the equation $\mathbf{d}^{(k)}\Psi_n = \mathbf{e}_k$, for each $k = 0, \dots, n-1$. The function $\tilde{\psi}_{nk}(z) = \sum_{j=n+1}^{2n} d_j^{(k)} \psi_{nj}(z)$ is obviously in $\mathcal{K}_n$ and we have $\tilde{\psi}_{nk}(y_j) = \delta_{kj}$, $k, j = 0, \dots, n-1$. Thus the $\{\tilde{\psi}_{nk}\}_{k=0}^{n-1}$ form the FLIF of $\mathcal{K}_n$ for the interpolation points $\mathbf{y}_n = \{y_k\}_{k=0}^{n-1}$. We can write for any function $f \in \mathcal{K}_n$ that

$$f(z) = \sum_{k=0}^{n-1} f(y_k)\tilde{\psi}_{nk}(z).$$

The $\tilde{\psi}_{nk}$ are orthonormal with respect to the discrete inner product

$$\langle f, g \rangle_{\mathbf{y}_n} = \sum_{k=0}^{n-1} f(y_k)g(y_k)$$

and $\tilde{\psi}_{nk}$ is the solution to the problem

$$\min\{\|f\|_{\mathbf{y}_n}^2 : f(y_k) = 1; f \in \mathcal{K}_n\}.$$

Moreover, they form a biorthogonal basis for the ψ_{nk} in $\mathcal{K}_n$ because by the reproducing property of ψ_{nk} we have $\left\langle \tilde{\psi}_{nk}, \psi_{nj} \right\rangle = \delta_{kj}$, $k, j = 0, \dots, n-1$. The relation between the bases $\{\phi_k\}_{k=n+1}^{2n}$ and $\{\tilde{\psi}_{nk}\}_{k=0}^{n-1}$ is given by

$$\begin{bmatrix} \tilde{\psi}_{n0} \\ \vdots \\ \tilde{\psi}_{n,n-1} \end{bmatrix} = \begin{bmatrix} d_{n+1}^{(0)} & \cdots & d_{2n}^{(0)} \\ \vdots & & \vdots \\ d_{n+1}^{(n-1)} & \cdots & d_{2n}^{(n-1)} \end{bmatrix} \begin{bmatrix} \phi_{n+1} \\ \vdots \\ \phi_{2n} \end{bmatrix} = \Psi_n^{-1} \begin{bmatrix} \phi_{n+1} \\ \vdots \\ \phi_{2n} \end{bmatrix} \tag{8.2}$$

with $\Psi_n = \Psi_n(\mathbf{y}_n)$.

9 Decomposition and reconstruction

To do the wavelet analysis of a function we should be able to decompose a function $f_{2n} \in \mathcal{L}_{2n}$ into a sum $f_{2n} = f_n + g_n$ where $f_n \in \mathcal{L}_n$ and $g_n \in \mathcal{K}_n = \mathcal{L}_{2n} \ominus \mathcal{L}_n$. This is a matter of a change of basis. Assume that for each $n = 2^s$, we select a number of distinct points $\mathbf{x}_n = \{x_{nk}\}_{k=0}^{n}$ on $\mathbb{R} \setminus \mathfrak{P}_n$ and a number of points $\mathbf{y}_n = \{y_{nk}\}_{k=0}^{n-1}$ distinct and on $\mathbb{R} \setminus \mathfrak{P}_{2n}$ such that $\Psi_n(\mathbf{y}_n)$

of (7.1) is invertible. We write in short hand $\mathbf{\Phi}_n = \mathbf{\Phi}_n(\mathbf{x}_n)$ and $\Psi_n = \Psi_n(\mathbf{y}_n)$. In that case $\varphi_{nk}(z) = k_n(z, x_{nk})$ forms a basis for $\mathcal{L}_n$ and $\psi_{nk}(z) = k_{2n}(z, y_{nk}) - k_n(z, y_{nk})$ is a basis for $\mathcal{K}_n$ when $k_n(z,w)$ is the reproducing kernel for $\mathcal{L}_n$. Let the coefficients with respect to the appropriate bases be defined by

$$f_{2n} = \sum_{k=0}^{2n} p_{2n,k}\varphi_{2n,k}; \quad f_n = \sum_{k=0}^{n} p_{nk}\varphi_{n,k}; \quad g_n = \sum_{k=0}^{n-1} q_{nk}\psi_{n,k}.$$

Then setting $\mathbf{p}_{2n} = [p_{2n,0} \dots p_{2n,2n}]$, $\mathbf{p}_n = [p_{n0} \dots p_{n,n}]$, and $\mathbf{q}_n = [q_{n0} \dots q_{n,n-1}]$, we get

$$f_{2n} = \mathbf{p}_{2n}\mathbf{\Phi}_{2n}^H \begin{bmatrix} \phi_0 \\ \vdots \\ \phi_{2n} \end{bmatrix}, \quad f_n = \mathbf{p}_n \mathbf{\Phi}_n^H \begin{bmatrix} \phi_0 \\ \vdots \\ \phi_n \end{bmatrix}, \quad g_n = \mathbf{q}_n \Psi_n^H \begin{bmatrix} \phi_{n+1} \\ \vdots \\ \phi_{2n} \end{bmatrix}.$$

Equating coefficients of the corresponding basis functions in $f_{2n} = f_n + g_n$ leads to

$$\mathbf{p}_{2n}\mathbf{\Phi}_{2n}^H = [\mathbf{p}_n \quad \mathbf{q}_n] \begin{bmatrix} \mathbf{\Phi}_n^H & 0 \\ 0 & \Psi_n^H \end{bmatrix}.$$

This relation allows us to compute $\mathbf{p}_n$ and $\mathbf{q}_n$ from $\mathbf{p}_{2n}$ and conversely to reconstruct $\mathbf{p}_{2n}$ from $\mathbf{p}_n$ and $\mathbf{q}_n$.

For example, using the relation $p_{nr} = \langle f_{2n}, \tilde{\varphi}_{nr} \rangle$ and the biorthogonality relations, we get

$$p_{nr} = \sum_{k=0}^{2n} p_{2n,k}\tilde{\varphi}_{nr}(x_{2n,k}), \quad r = 0, \dots, n.$$

Similarly, it is seen from $q_{nr} = \left\langle f_{2n}, \tilde{\psi}_{nr} \right\rangle$ that

$$q_{nr} = \sum_{k=0}^{2n} p_{2n,k}\tilde{\psi}_{nr}(x_{2n,k}), \quad r = 0, \dots, n-1.$$

In the special case where the $\varphi_{n,k}$ are orthogonal, i.e., when the $x_{n,k}$ are the zeros $\xi_{n,k}$ of $Q_{n+1}(z, \tau_{n+1})$, then $\tilde{\varphi}_{n,r} = \varphi_{n,r}/\varphi_{n,r}(\xi_{n,r})$ we get for the first of the analysis formulas

$$p_{nr} = \frac{1}{\varphi_{n,r}(\xi_{n,r})} \sum_{k=0}^{2n} p_{2n,k}\varphi_{nr}(x_{2n,k}), \quad r = 0, \dots, n.$$

In general, the ψ_{nk} are not orthogonal, but if we choose $\{y_{nr}\}_{r=0}^{n-1}$ to be the last n zeros $\{\xi_{2n,n+1+r}\}_{r=0}^{n-1}$, then by definition $\tilde{\psi}_{nr}(y_{ns}) = \delta_{rs}$, $r, s = 0, \dots, n-1$. Thus, the second analysis formula reduces to

$$q_{nr} = \sum_{k=0}^{n} p_{2n,k}\tilde{\psi}_{nr}(\xi_{2n,k}) + p_{2n,n+1+r}, \quad r = 0, \dots, n-1.$$

For the reconstruction formula, we have by $p_{2n,r} = \langle f_n + g_n, \tilde{\varphi}_{2n,r} \rangle$,

$$p_{2n,r} = \sum_{k=0}^{n} p_{nk} \left\langle \varphi_{nk}, \tilde{\varphi}_{2n,r} \right\rangle + \sum_{l=0}^{n-1} q_{nl} \left\langle \psi_{nl}, \tilde{\varphi}_{2n,r} \right\rangle, \quad r = 0, \dots, 2n.$$

When the $\varphi_{2n,r}$ are orthogonal, i.e., when the $x_{2n,k}$ are the zeros $\xi_{2n,k}$ of $Q_{2n+1}(z, \tau_{2n+1})$, then $\tilde{\varphi}_{2n,r} = \varphi_{2n,r}/\varphi_{2n,r}(\xi_{2n,r})$, and the reconstruction formula becomes as in [14],

$$p_{2n,r} = \frac{1}{\varphi_{2n,r}(\xi_{2n,r})} \left(\sum_{k=0}^{n} p_{nk}\varphi_{nk}(\xi_{2n,r}) + \sum_{l=0}^{n-1} q_{nl}\psi_{nl}(\xi_{2n,r}) \right), \quad r = 0, \ldots, 2n.$$

10 Dilation and translation

The nested spaces $\{\mathcal{V}_s\}_{s=-1}^{\infty}$ are a special case of a second generation multiresolution analysis [24], but they can be interpreted in a way which is much closer to the classical definition of a multiresolution analysis. This is what we shall do here.

Before we start checking the MRA properties, we should first adapt the notion of a shift which is essential in the definition of an MRA.

10.1 Generalized shift

Recall that several of the properties in definition of classical MRA refer to shifted functions. In our case, this shift has to be given a more general meaning. It was also introduced in the polynomial case [14], and we follow here the same lines.

To introduce the idea, it is interesting to consider some classical cases. Suppose $F(\theta)$ is a 2π-periodic function. In classical MRA for 2π periodic functions, the orthogonal basis is the Fourier basis $\phi_k(t) = e^{ikt}$, a shift $F(\theta - \tau)$ has the effect that the kth Fourier coefficient of F is multiplied with $\overline{\phi_k(\tau)/\phi_k(0)} = e^{-ik\tau}$. Similarly, for the continuous Fourier transform, the basis functions are $\phi_\omega(t) = e^{i\omega t}$ and a shift $f(t - \tau)$ has the effect that the Fourier transform $f^\wedge(\omega)$ is multiplied by $\overline{\phi_\omega(\tau)/\phi_\omega(0)} = e^{-i\omega\tau}$.

The generalization we need is to consider a shift operator which is defined as above but now with respect to our orthonormal basis ϕ_k which in general is not the basis of classical Fourier analysis. So we consider the Fourier transform with respect to the basis ϕ_k namely $\mathcal{F}(f) = \{f_k^\wedge\}_{k=0}^{\infty}$ with $f_k^\wedge = \langle f, \phi_k \rangle$. A shift operator S_τ will be defined as the operator whose effect is that

$$(S_\tau f)_k^\wedge = f_k^\wedge \cdot \overline{\phi_k(\tau)}/\overline{\phi_k(\tau_0)},$$

where τ_0 was zero in the above examples. In fact the choice of a specific $\tau_0 \in \mathbb{R}$ is not really crucial and if another point is more appropriate, one can use it. This τ_0 is just a reference point with respect to which the shift is taken.

If we consider $\mathcal{F}(f)$ for $f \in \mathcal{L}_n$ as a finite dimensional vector of dimension $n+1$, then we can describe the shift operator restricted to $\mathcal{L}_n$ as a multiplication with a finite dimensional diagonal matrix. So we define the shift restricted to $\mathcal{L}_n$ as $S_{n,\tau}$, then for $f \in \mathcal{L}_n$

$$g = S_{n,\tau} f \quad \Leftrightarrow \quad \mathcal{F}(g) = \mathcal{F}(f) D_{n,\tau}$$

where $D_{n,\tau}$ is a diagonal matrix defined in terms of the orthogonal basis functions $\{\phi_k\}_{k=0}^{n}$ of $\mathcal{L}_n$.

$$D_{n,\tau} = \operatorname{diag}\left(\frac{\phi_0(\tau)}{\phi_0(\tau_0)}, \ldots, \frac{\phi_n(\tau)}{\phi_n(\tau_0)} \right)^H$$

Thus if $\mathbf{a}_n = [a_0, \ldots, a_n]$ and $f = \mathbf{a}_n[\phi_0, \ldots, \phi_n]^T \in \mathcal{L}_n$, then $S_{n,\tau} f = \mathbf{b}_n[\phi_0, \ldots, \phi_n]^T$ with $\mathbf{b}_n = [b_0, \ldots, b_n] = \mathbf{a}_n D_{n,\tau}$. Note that this corresponds to the classical shift if the ϕ_k are the classical Fourier basis functions of complex exponentials and $\tau_0 = 0$.

Now we turn to the MRA properties.

10.2 Nesting property

This one is trivial to verify. Defining $\mathcal{V}_{-1} = \mathcal{L}_0$, it is immediately seen that

$$\mathcal{V}_{-1} \subset \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_s \subset \mathcal{V}_{s+1} \subset \cdots.$$

10.3 The completeness condition

If $\mathcal{M}$ is a subspace of the Hilbert space $\mathcal{H}$, then for the $\{\mathcal{V}_k\}$ to form a MRA of $\mathcal{M}$, one should have that

$$\operatorname{clos}_{\mathcal{H}} \left(\bigcup_{s=-1}^{\infty} \mathcal{V}_s \right) = \mathcal{M}.$$

In our case, this means that we should verify in which spaces the system $\{B_k\}_{k=0}^{\infty}$ is complete.

Let $\mathcal{P}$ be the space obtained as the closure of $\operatorname{span}\{1, x, x^2, \ldots\}$ in $L_2(\hat{\mathbb{R}}, \mu)$. The following theorem is then known [4].

Theorem 10.1 *If* $\mathrm{d}\mu$ *is a finite measure on* $\hat{\mathbb{R}}$*, such that*

$$\int |B_n(x)|^2 \mathrm{d}\mu(x) < \infty, \quad B_n(x) = \prod_{k=1}^{n} \frac{x}{1 - \alpha_k x},$$

then $\mathcal{L}_n$ *as a subspace of* $L(\hat{\mathbb{R}}, \mathrm{d}\mu)$ *is in* $\mathcal{P}$ *and hence* $\mathcal{L}_\infty = \cup_{k=0}^{\infty} \mathcal{L}_k$ *is dense in* $\mathcal{P}$.

Thus in our case, we have a multiresolution of the space $\mathcal{P}$.

10.4 Scaling property

The scaling property in a classical MRA says that if one doubles the frequency then one moves from $\mathcal{V}_s$ to $\mathcal{V}_{s+1}$. Interpreting frequency again in a generalized sense as being the Fourier coefficients with respect to the basis ϕ_k, then this can be reformulated as follows. If $\mathcal{F}$ denotes the Fourier transform

$$\mathcal{F}(f) = \{f_k^\wedge\}_{k=0}^{\infty}, \quad f_k^\wedge = \langle f, \phi_k \rangle,$$

then $f \in \mathcal{V}_s \Leftrightarrow \operatorname{supp} \mathcal{F}(f) = \{0, 1, \ldots, n = 2^s\}$. Thus moving from $\mathcal{V}_s$ to $\mathcal{V}_{s+1}$ is equivalent to doubling the support of this Fourier transform

10.5 Shift invariance

The shift invariance, in the classical MRA definition says that a shifted version of a function remains at the same resolution scale, i.e., stays in the same space $\mathcal{V}_s$: $f \in \mathcal{V}_s \Rightarrow S_\tau f \in \mathcal{V}_s$ for all shifts τ. With our definition of general shift, this is obviously true because (recall $n = 2^s$)

$$f = \sum_{k=0}^{n} f_k^\wedge \phi_k \in \mathcal{V}_s \quad \Rightarrow \quad S_\tau f = \sum_{k=0}^{n} \left(\frac{\overline{\phi_k(\tau)}}{\overline{\phi_k(\tau_0)}} f_k^\wedge \right) \phi_k \in \mathcal{V}_s.$$

10.6 Riesz basis

The final condition for a stationary multiresolution is that there should exist a Riesz basis for $\mathcal{V}_0$ which consists of translates of one scaling function. By the scaling property, one can then generate bases at all resolution levels. Here we shall have one scaling function per resolution level which will generate all the basis functions for that resolution level. Indeed, using again the general shift operator defined as above, then it is clear that if $\varphi_{nk}(z) = k_n(z, x_{nk})$, $k = 0, \ldots, n = 2^s$ is a basis of reproducing kernel scaling functions (orthogonal or not) for $\mathcal{V}_s$, then we can write them as $\varphi_{nk}(z) = S_{x_{nk}}\varphi_n(z)$ with

$$\varphi_n(z) = \sum_{k=0}^{n} \overline{\phi_k(\tau_0)}\phi_k(z) = k_n(z, \tau_0).$$

Thus all the scaling basis functions φ_{nk} are generalized translates of a unique (for $\mathcal{V}_s$) scaling function φ_n.

Similarly $\psi_{nk} = S_{y_{nk}}\psi_n$ where

$$\psi_n(z) = \sum_{k=n+1}^{2n} \overline{\phi_k(\tau_0)}\phi_k(z) = l_n(z, \tau_0).$$

The bases used also have a Riesz property, i.e. there are constants $0 < A \leq B < \infty$ such that

$$A\|\mathbf{p}_n\| \leq \|f\| \leq B\|\mathbf{p}_n\|$$

where the norms are 2-norms and $f \in \mathcal{V}_s$ is given by $f = \mathbf{p}_n[\varphi_{n0}, \ldots, \varphi_{nn}]^T$. It follows from (3.2) and Parseval's equality that

$$\|f\|^2 = \|\mathbf{p}_n \Phi_n^H [\phi_0, \ldots, \phi_n]^T\|^2 = \|\mathbf{p}_n \Phi_n^H\|^2$$

and this implies that

$$\frac{1}{\|\Phi_n^{-1}\|}\|\mathbf{p}_n\| \leq \|f\| \leq \|\Phi_n\|\|\mathbf{p}_n\|.$$

Similarly, it holds that

$$\frac{1}{\|\Psi_n^{-1}\|}\|\mathbf{q}_n\| \leq \|f\| \leq \|\Psi_n\|\|\mathbf{q}_n\|$$

when $f \in \mathcal{W}_s$ with $f = \mathbf{q}_n[\psi_{n0}, \ldots, \psi_{n,n-1}]^T$.

10.7 Consequences of the generalized shift

This generalized shift operator implies that the functions in the ORK basis look like shifts of each other. The same holds for the functions in the WRK basis. This can be verified in Figure 1 where we have plotted $k_{32}(x, w)$ and $l_{32}(x, w)$ for $w = 0$ and $w = 1$.

We give yet another example.

Example 10.1 Consider the Chebyshev polynomials of the second kind. In the recurrence relation (5.1), the α_k's are all equal to 0, and the recurrence coefficients are given by

$$A_n = 2, \quad B_n = 0, \quad C_n = -1, \quad n = 1, 2, \ldots$$

The orthogonality measure is

$$d\mu(x) = \frac{\sqrt{1-x^2}}{2\pi} dx.$$

Keeping the same recurrence, but setting $\alpha_1 = 0.3$ we get orthogonal rational functions with respect to a weight function that is plotted at the right in row A of Figure 3. The dashed lines represent the Chebyshev weight, which is plotted for reference. The kernels $k_{32}(x, w)$ and $l_{32}(x, w)$ are plotted for $w = 0$, 0.5 and 0.9 respectively in row A, B, and C.

11 Symmetry

Sometimes symmetry (or antisymmetry) of the wavelet functions can be an issue in certain applications. It is obvious that if the problem is not symmetric, then the wavelets and scaling functions will not be symmetric. However, if the measure and the poles are chosen in a symmetric way, then the kernel functions that we used will be symmetric as well.

Assume that the measure is symmetric with respect to $x = 0$ and let the poles defining $\mathcal{L}_n$ be chosen symmetric with respect to $x = 0$, thus they appear in pairs $(1/\alpha_k, -1/\alpha_k)$.

Theorem 11.1 *Under the above conditions about symmetry of the measure and of the poles, the kernels $k_{2^s}(z, 0)$ and therefore also the kernels $l_{2^s}(z, 0) = k_{2^{s+1}}(z, 0) - k_{2^s}(z, 0)$ are symmetric w.r.t. $x = 0$.*

Proof. Suppose $n = 2^s$. We first observe that $k_n(z, w)$ is given by $\sum_{k=0}^n f_k(z)\overline{f_k(w)}$ where $\{f_k\}_{k=0}^n$ is any basis of orthonormal functions such that $\mathcal{L}_n = \text{span}\{f_k : k = 0, \ldots, n\}$. Now consider the measure $d\mu_n(x) = d\mu(x)/|\pi_n(x)|^2$ where $\pi_n(x) = \prod_{k=1}^n (1 - \alpha_k x)$. Define the polynomials p_k, $k = 0, 1, \ldots, n$ by orthogonalizing the functions $\{1, z, \ldots, z^n\}$ with respect to μ_n. Because μ_n is real on $\mathbb{R}$ and symmetric with respect to 0, the p_{2m} are even while the p_{2m+1} are odd functions. In particular $p_k(0) = 0$ for odd k. Hence $\tilde{k}_n(x, 0) = \sum_{k=0}^n p_k(x)p_k(0) = \sum_{k \text{ even}} p_k(x)p_k(0)$ is an even function. Furthermore, because of the symmetry of the α_k, it holds that $\pi_n(x)$ is an even function if n is even. The theorem now follows by observing that the kernel $\tilde{k}_n(z, 0)$ can be transformed into a reproducing kernel for $\mathcal{L}_n$ with respect to μ by setting $k_n(z, 0) = \tilde{k}_n(t, 0)/[\pi_n(t)]^2$. Therefore $k_n(x, 0)$ is even and hence also $l_n(x, 0)$ as a difference of even functions. □

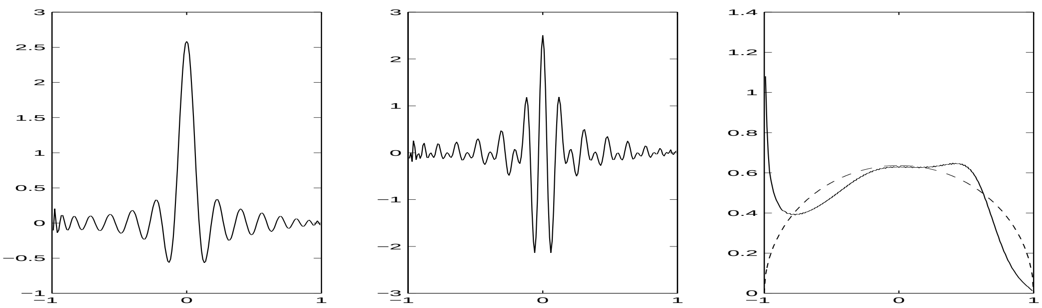
12 Computation and more experiments

It is difficult to write a general and efficient code to generate the coefficients $\{A_n\}, \{B_n\}$ and $\{C_n\}$ given the measure and the points $\{\alpha_n\}$. For example, these coefficients can be expressed as inner products, which would require the computation of integrals which may be time consuming if one wants to evaluate these accurately. There is the Nevanlinna-Pick algorithm, which is an interpolation type of algorithm to generate the coefficients from the Stieltjes transform of the given measure. This algorithm however leads to extra complications if some of the α_k do coincide. If none of them does, the algorithm is relatively simple. If all of them coincide, then a modified algorithm can be used which needs some generalized moments for the given measure.

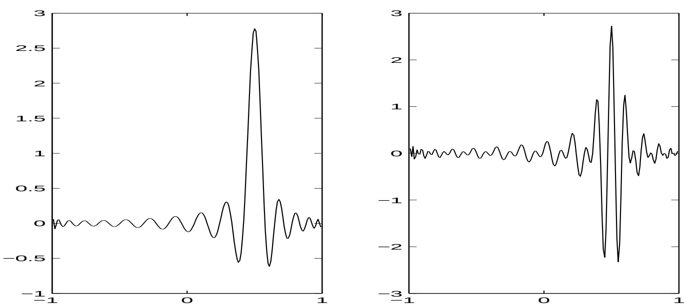
We have chosen for the opposite direction to do some experiments. Given the coefficients A_1, $\{B_n\}$ and $\{C_n\}$ and the points $\{\alpha_n\}$, it is relatively simple to generate the ORF basis

FIG. 3: Illustration of generalized shift. The weight function times the kernels $k_{32}(x,w)$ and $l_{32}(x,w)$ are plotted for $w = 0, 0.5$ and 0.9. There is only one pole at $1/\alpha_1$ with $\alpha_1 = 0.3$, all other poles are at ∞. The recurrence used is the recurrence for the Chebyshev polynomials of the second kind. The weight function is plotted on the right. For reference, the Chebyshev weight of the second kind is plotted in dashed lines.

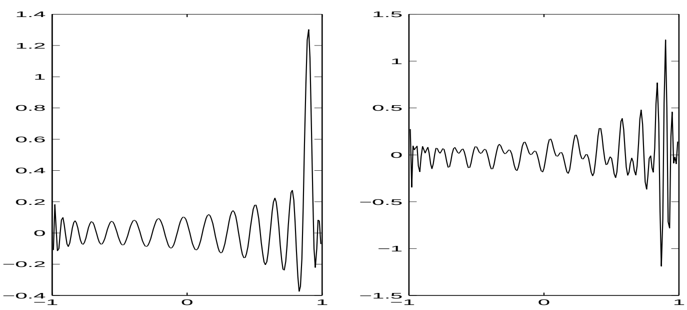
A: $k_{32}(x,0)$ and $l_{32}(x,0)$ and weight function



B: $k_{32}(x,0.5)$ and $l_{32}(x,0.5)$



B: $k_{32}(x,0.9)$ and $l_{32}(x,0.9)$



and with these, the kernels and hence also the scaling functions and the wavelets can be generated. This is what we did to see how the classical polynomial situations were influenced when some of the poles are shifted from ∞ to some finite value(s) on $\mathbb{R}$.

What is lacking here is a general theory about the construction of the weight function from the recurrence relation. Our experiments lead to certain conjectures about convergence of reproducing kernels, but a detailed proof or investigation of these is really beyond the scope of this paper which is to generalize the polynomial wavelets to the rational case. Further research will be reported in separate papers.

With respect to the problem of constructing the weight function from the orthogonal functions or reproducing kernels, there are two theorems that give a partial result. Application of these theorems in situations that are not covered by them, worked fine for the examples we include here.

First there is the case of a weight with a compact support. Here we can refer to a theorem of Máté-Nevai-Totik. for convenience of the reader. [18, Theorem 5].

Theorem 12.1 *Let all $\alpha_k = 0$, so that we are in the polynomial case. Let ϕ_k be the orthonormal polynomials with respect to a measure $\mathrm{d}\mu$ supported on $[-1,1]$. Define $k_n(x) = \sum_{k=0}^{n} |\phi_k(x)|^2$. Then if*

$$\int_{-1}^{1} \frac{\log \mu'(x)}{\sqrt{1-x^2}} \mathrm{d}x > -\infty,$$

it holds that

$$\lim_{n\to\infty} \frac{n}{k_n(x)} = \pi\mu'(x)\sqrt{1-x^2}$$

for almost every $x \in [-1,1]$.

Of course this involves the polynomial case, but we include it for the convenience of the reader. We do not know of a proof which generalizes this to the rational case where the measure has a compact support, but we conjecture that this theorem is also true for the rational case, at least when the poles are not in the support of the measure. We shall postpone a more detailed investigation and a possible proof of this conjecture to a separate paper. Anyway, our experiments gave satisfactory results for the polynomial case, and relying on the conjecture that the theorem also holds in the rational case, this is how we computed and plotted the weight function for the rational variations of the Chebyshev cases. See Figures 3.

For measures whose support is not compact, such as the Hermite or the Laguerre polynomials, the previous approach does not work, and as far as we know, the construction of the weight function is still an open problem. However, we have the following result which does hold for the rational case. Defining the functions $K_n(x) = k_n(x,\mathbf{i})/\sqrt{k_n(\mathbf{i},\mathbf{i})}$ with $k_n(z,w)$ the reproducing kernels for $\mathcal{L}_n$ and $\mathbf{i}$ the imaginary unit one can prove the following theorem where λ represents the measure defined by $\mathrm{d}\lambda(x) = [\pi(1+x^2)]^{-1}\mathrm{d}x$. The proof can again be found in [4, Theorem 11.10.15].

Theorem 12.2 *With $K_n(x) = k_n(x,\mathbf{i})/\sqrt{k_n(\mathbf{i},\mathbf{i})}$ we have*

$$\langle f,g\rangle = \int_{\hat{\mathbb{R}}} f(x)\overline{g(x)}\mathrm{d}\mu(x) = \int_{\hat{\mathbb{R}}} f(x)\overline{g(x)}\mathrm{d}\mu_n(t), \quad \forall f,g \in \mathcal{L}_n$$

where

$$\mathrm{d}\mu_n(x) = \frac{\mathrm{d}\lambda(x)}{|K_n(x)|^2} = \frac{\mathrm{d}x}{\pi|K_n(x)|^2(1+x^2)}.$$

Moreover, if all the generalized moments $m_n = \int x^n/\pi_n(x)d\mu(x)$ with $\pi_n(x) = \prod_{k=1}^{n}(1-\alpha_k x)$ are finite for all n and if $\log\mu'$ is integrable with respect to $\mathrm{d}\lambda$, i.e., $\log\mu' \in L_1(\lambda)$, then $1/K_n(z)$ will converge to the outer spectral factor $\sigma(z)$ of the measure μ uniformly in compact subsets of the upper half plane [4, Cor. 11.11.3]. The outer spectral factor is defined as

$$\sigma(z) = c\exp\left\{-\frac{\mathrm{i}}{2}\int\frac{1+xz}{x-z}\log\mu'(x)\mathrm{d}\lambda(x)\right\}, \quad c\in\mathbb{T}.$$

It is an outer function in H_2 of the upper half plane. Therefore it has a non-tangential limit to the real line such that $|\sigma(x)|^2 = \mu'(x)$ a.e. for $x\in\mathbb{R}$. Thus if the measure is absolutely continuous and given by $\mathrm{d}\mu(x) = w(x)\mathrm{d}x$, and if we denote the boundary function of $K_n(z)$, as $K_n(x)$, $x\in\mathbb{R}$, then

$$\frac{1}{\pi(1+x^2)}\frac{1}{|K_n(x)|^2}$$

will give a good idea of what $w(x)$ looks like if n is large enough. Experimental results confirm this. However, this assumes that $\log w \in L_1(\lambda)$. Although in the case of Hermite polynomials where $\log w(x) = -x^2$, we have $\int\log w(x)\mathrm{d}\lambda(x) = -\infty$, we still get good convergence as described above. However there are many important examples where the measure is only supported on a half line or on a finite interval (and is assumed to be zero everywhere else on the real line). In that case, $\log w$ will certainly not be in $L_1(\lambda)$ and it turns out that in those examples, the simple convergence does not hold in the support of the measure. But if one then replaces $K_n(x)$ by the $(C,1)$ Cesàro sums

$$S_n(x) = \frac{1}{n+1}\sum_{j=0}^{n}K_j(x)$$

then we do observe convergence and we get a reasonable picture of the weight function $w(x)$ by plotting

$$\frac{1}{\pi(1+x^2)}\frac{1}{|S_n(x)|^2}$$

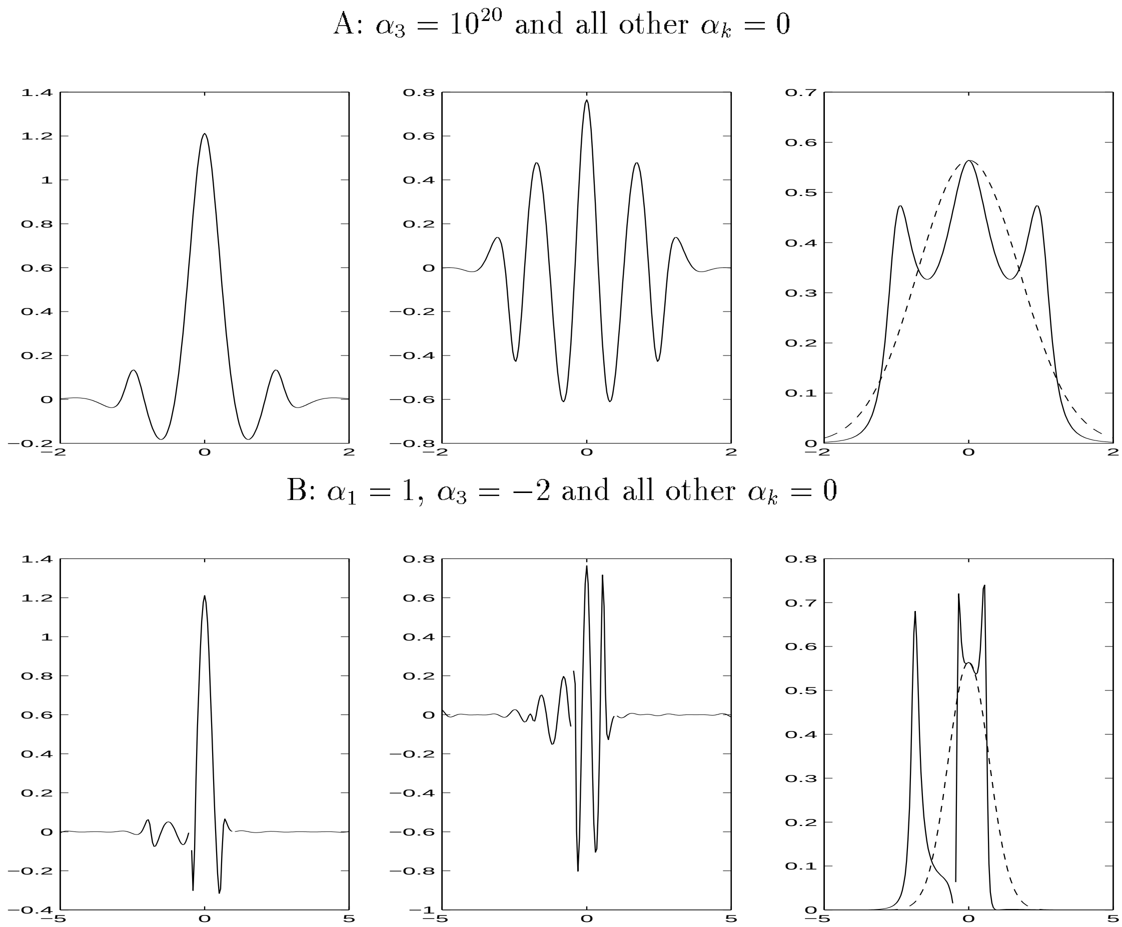
for n sufficiently large. These conclusions are drawn from many experiments but were not proved so far. A precise investigation of the conditions under which this kind of convergence holds and a proof of these results are also postponed to a separate paper. Our conjecture is that for a continuous weight function w that is positive almost everywhere on $\mathbb{R}$, the Cesàro means as described above will converge uniformly in compact subsets of $\mathbb{R}$.

We also note that the two conjectured theorems are mutually exclusive. We seem to need the positivity of the weight, or at least the existence of σ in the last one because, it was clear from the experiments that for Chebyshev-like examples where the support of the measure is an interval, the convergence of the Cesàro means was not uniform.

We used the Cesàro means of the $K_n(x)$ to compute the weight in the following examples.

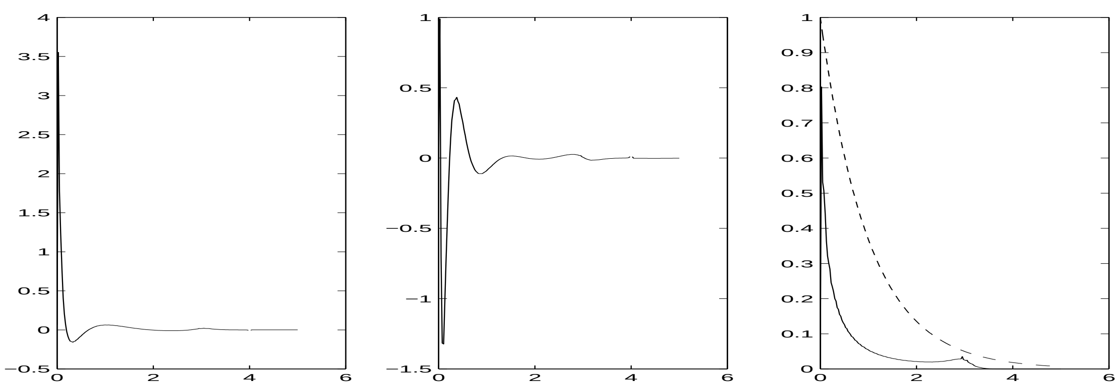
Example 12.1 The latter strategy is used for example to see what the influence is when poles are introduced. It is illustrated in Figure 4. We took the recurrence relation for the Hermite polynomials and introduced poles. In figure A, the only pole was $1/\alpha_3 = 10^{-20}$ and in figure B we chose two poles $1/\alpha_1 = 1$ and $1/\alpha_3 = -0.5$. The corresponding weight is plotted together with the scaling functions $k_{32}(x,0)$ and $l_{32}(x,0)$ both multiplied by the weight function. For comparison, the original Hermite weight is plotted in dashed lines.

FIG. 4: The functions $k_{32}(z,0)$ and $l_{32}(z,0)$ multiplied by the weight function and the weight function itself. They are generated by the recurrence relation of the Hermite polynomials, but now with $\alpha_3 = 10^{20}$ and all other $\alpha_k = 0$ (figure A) or $\alpha_1 = 1$, $\alpha_3 = -2$ and all other $\alpha_k = 0$ (figure B). The dashed line shows the Hermite weight function.



Example 12.2 As a final example, we take the recurrence relation for the Laguerre polynomials and introduced poles by setting $\alpha_k = -2$ for $k = 1, 2, 3, 4, 5$ and we chose $\alpha_k = 0.25$ for $k = 7, 8$. The result is plotted in Figure 5.

FIG. 5: Weight function times $k_{32}(z, 0)$ (left) and $l_{32}(z, 0)$ (right) where the recursion is for the Laguerre polynomials and $\alpha_k = -2$ for $k = 1, 2, 3, 4, 5$ and $\alpha_k = 0.25$ for $k = 7, 8$. The weight function is plotted on the right. The Laguerre weight function is plotted in dashed lines for comparison.



Appendix A: Proofs of some auxiliary results

In this appendix we prove some of the results that were used in the text. First we prove

Lemma A.1 *The value of A_1 in the recurrence relation for the orthogonal rational functions, as described in Theorem 5.1 is related to the orthogonality measure by a generalized standard deviation, i.e.,*

$$A_1 = \frac{\pm 1}{s}, \quad \textit{with} \quad s^2 = m_2^2 - m_1^2, \quad m_2^2 = \int_{\hat{\mathbb{R}}} |\zeta_1(x)|^2 \mathrm{d}\mu(x), \quad m_1 = \int_{\hat{\mathbb{R}}} \zeta_1(x) \mathrm{d}\mu(x).$$

Proof. First assume that $B_1 = 0$, then $\langle \phi_0, \phi_1 \rangle = 0$ implies that the generalized mean $m_1 = \int \zeta_1(x)\mathrm{d}\mu(x) = 0$. The normality condition $\|\phi_1\|^2 = 1$ then gives $A_1^2 m_2^2 = 1$.

If $B_1 \neq 0$, then the condition $\langle \phi_0, \phi_1 \rangle = 0$ gives $A_1 = -B_1/m_1$, while $\|\phi_1\|^2 = 1$ leads to

$$B_1^2(m_2^2 - m_1^2) = m_1^2.$$

In both cases, this gives the desired formulas. □

Recall that in the polynomial case where all $\alpha_k = 0$, then $\zeta_1(x) = x$ and we then get the usual definition of standard deviation.

Thus, if $\mathrm{d}\mu(x) = w(x)\mathrm{d}x$, in some sense we can say that a larger $|A_1|$ will correspond to a smaller s which means that the graph of $w(x)$ will be wider, a small $|A_1|$ will correspond to a more peaked $w(x)$.

Lemma A.2 *Consider the orthogonal rational function on the real line* $\phi_{n+1} = p_{n+1}/\pi_{n+1}$ *with* $\pi_{n+1}(z) = \prod_{k=1}^{n+1}(1-\alpha_k z)$. *Then* ϕ_{n+1} *will be singular, i.e.* $p_{n+1}(1/\alpha_n) = 0$ *iff* $A_{n+1} = 0$.

Proof. First assume $A_{n+1} = 0$. Define $f_k(z) = \phi_k(z)/\zeta_k(z)$. Then it follows from the Christoffel-Darboux relation that

$$\frac{zw}{(z-w)}[f_{n+1}(w)f_n(z) - f_{n+1}(z)f_n(w)] = A_{n+1}k_n(z,w) = 0.$$

Therefore $f_{n+1}(w)/f_n(w) = f_{n+1}(z)/f_n(z)$ for all z and w, so that $f_{n+1}(w)/f_n(w)$ is a constant. Thus

$$\tau(w) = \frac{f_{n+1}(w)}{f_n(w)} = \frac{\phi_{n+1}(w)/\zeta_{n+1}(w)}{\phi_n(w)/\zeta_n(w)} = \frac{(1-\alpha_{n+1}w)\phi_{n+1}(w)}{(1-\alpha_n w)\phi_n(w)} = \frac{p_{n+1}(w)}{(1-\alpha_n w)p_n(w)}$$

is a constant. Taking $w = 1/\alpha_n$, it follows $p_{n+1}(1/\alpha_n) = 0$. Thus $A_{n+1} = 0$ implies that ϕ_{n+1} is singular.

Conversely, assume ϕ_{n+1} is singular. It follows from the reproducing property $\langle \phi_n(z), k_n(z,w)\rangle = \phi_n(w)$ evaluated for $w = 1/\alpha_n$ and substituting the Christoffel-Darboux relation for $k_n(z,w)$ that after some calculations we get

$$-\left\langle \phi_n, \phi_{n+1}\frac{1-\alpha_{n+1}z}{1-\alpha_n z}\right\rangle p_n(1/\alpha_n)(\alpha_n - \alpha_n) + \langle \phi_n, \phi_n\rangle p_{n+1}(1/\alpha_n)\alpha_n = A_{n+1}p_n(1/\alpha_n).$$

This implies $A_{n+1} = 0$ because $p_n(1/\alpha_n) \neq 0$ and $p_n n + 1(1/\alpha_n) = 0$. □

Lemma A.3 *Let* ϕ_k *be the orthogonal rational functions on the real line. Define* $f_n(z) = \phi_n(z)/\zeta_n(z)$ *and* $\tau = f_{n+1}/f_n$. *If* $A_{n+1} \neq 0$ *then the derivative* τ' *does not change sign on the real line.*

Proof. First note that $f_n(x)$ is real for $x \in \mathbb{R} \setminus \mathfrak{P}_{n+1}$, where $\mathfrak{P}_{n+1} = \{1/\alpha_1, \ldots, 1/\alpha_{n+1}\}$. So when in the Christoffel-Darboux formula we take $w \in \mathbb{R} \setminus \mathfrak{P}_{n+1}$ we get

$$\frac{zw}{A_{n+1}(z-w)}[f_{n+1}(w)f_n(z) - f_{n+1}(z)f_n(w)] = k_n(z,w).$$

Note that $A_{n+1} \neq 0$ means that ϕ_{n+1} is regular. Now let $z \to w$, then

$$\frac{w^2}{A_{n+1}}[f_{n+1}(w)f_n'(w) - f_{n+1}'(w)f_n(w)] = k_n(w,w) > 0.$$

On the other hand

$$\tau' = \left(\frac{f_{n+1}}{f_n}\right)' = \frac{f_n f_{n+1}' - f_n' f_{n+1}}{f_n^2}.$$

Thus it follows that depending on the sign of A_{n+1}, $\tau'(w)$ is positive or negative for all real values of w such that $f_n(w) \neq 0$. □

Lemma A.4 *Let ϕ_k be the orthogonal functions on the real line. Define the function h_k by $h_k(z) = (1 - \alpha_k z)\phi_k(z)$. If ϕ_{n+1} is regular, then the functions h_{n+1} and h_n can have no common zeros.*

Proof. Like in the proof of the previous lemma, we get from the confluent form of the Christoffel-Darboux formula that

$$\frac{1}{A_{n+1}}[h'_{n+1}(w)h_n(w) - h'_n(w)h_{n+1}(w)] = k_n(w,w) > 0.$$

Therefore a common zero of h_n and h_{n+1} is impossible. □

Lemma A.5 *Let $\phi_n = p_n/\pi_n$ with $\pi_n(z) = \prod_{k=1}^n (1 - \alpha_k z)$ be the orthogonal rational functions on the real line. Define*

$$\tau(w) = \frac{\phi_{n+1}(w)/\zeta_{n+1}(w)}{\phi_n(w)/\zeta_n(w)} = \frac{(1 - \alpha_{n+1}w)\phi_{n+1}(w)}{(1 - \alpha_n w)\phi_n(w)} = \frac{p_{n+1}(w)}{(1 - \alpha_n w)p_n(w)}.$$

Then if ϕ_{n+1} is a regular, $\tau(w)$ will run $n+1$ times through all values of $\mathbb{R}$ as w runs over all values in $\mathbb{R}$.
Hence, for a given value $\tilde{\tau} \in \mathbb{R}$, there are $n+1$ values of $w_k \in \mathbb{R}$, $k = 0, \ldots, n$ such that $\tilde{\tau} = \tau(w_k)$.

Proof. Because the numerator and the denominator of τ will have no zeros in common by the previous lemma, and because $p_n(z)$ has at least n simple zeros in $\mathbb{R}$, and because $\tau'(w) \neq 0$ wherever the derivative exists, the conclusion of the lemma follows. □

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