Extremal polynomials on the unit circle with preassigned zeros and Two-point Partial Padé approximation

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Abstract

It is well known that for a general class of measures the sequence of paraorthogonal polynomials $\{B_n\}_{n\in\mathbb{N}}$ satisfies

$$\lim_{n\to\infty} |B_n(z)|^{\frac{1}{n}} = \max(|z|, 1),$$

uniformly on compact subsets of $\hat{\mathbb{C}}\backslash\mathbb{T}$. This is an essential property to obtain the exact rate of convergence for two-point Padé approximants to the Herglotz-Riesz transform of the measure when these paraorthogonal polynomials are used as denominators. In this paper we obtain a similar result when we appropriately preselect some of the zeros of these paraorthogonal polynomials. As an application, we obtain the corresponding exact rate of convergence of the approximants to the Herglotz-Riesz transform with rational perturbation when these polynomials are used as denominators.

Keywords: potential theory, extremal polynomials, Padé approximation, Herglotz-Riesz transform, quasi-paraorthogonal, unit circle

1. Introduction

The authors in [1] explained the role of paraorthogonal polynomials when these are used as denominators of two-point partial Padé approximation (2PPA) to Herglotz-Riesz transforms. If one zero of the paraorthogonal denominator is fixed, exact rates of convergence in the whole domain of analyticity were obtained in [1, Theorem 12]. When more zeros in the denominator are preselected, then these become quasi-paraorthogonal polynomials, giving 2PPA with reduced interpolation order. The objective of this paper is to obtain a similar result if we choose the preselected zeros of the denominator in an appropriate way.

Given a finite positive Borel measure μ on the unit circle $\mathbb{T}$, a paraorthogonal polynomial B_n of degree n is orthogonal to $\text{span}\{z, z^2, \ldots, z^{n-1}\}$ in L^2_μ. An essential property to obtain

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the announced results for 2PPA, is that a monic paraorthogonal polynomial B_n of degree n can be characterized by its minimal-norm property. This means that B_n is the unique monic polynomial of degree n with $B_n(0) \in \mathbb{T}$ preassigned if only if

$$\|B_n\| \leq \|z^n + zq(z) + B_n(0)\|, \quad \forall q \in \mathbb{P}_{n-2}, \tag{1}$$

where $\mathbb{P}_{n-2}$ denotes the set of polynomials of degree $n-2$ at most and $\|\cdot\|$ is the L^2_μ norm. Polynomials satisfying (1) will be called extremal.

It is also known that for a very general class of measures, (the so called regular measures, a definition will be given below), the sequence $\{B_n\}_{n\in\mathbb{N}}$ of extremal polynomials satisfies the following nth root asymptotic behavior:

$$\lim_{n\to\infty} |B_n(z)|^{\frac{1}{n}} = \max(|z|, 1), \tag{2}$$

uniformly on compact subsets of $\mathbb{C}\backslash\mathbb{T}$. Moreover, all the zeros of B_n are simple and are on $\mathbb{T}$ and the sequence of normalized counting measures of the zeros of B_n with support on $\mathbb{T}$ converges in weak-star sense as $n \to \infty$ to the equilibrium measure on $\mathbb{T}$ for the logarithmic potential. In our case, where $\mathrm{supp}(\mu) = \mathbb{T}$, the equilibrium measure is the normalized Lebesgue measure on $\mathbb{T}$ so that the zeros of B_n will be asymptotically uniformly distributed on $\mathbb{T}$. We shall call a sequence $\{P_n\}_{n\in\mathbb{N}}$ of monic polynomials with $\deg P_n = n$ asymptotically extremal if these polynomials satisfy (2) and all the limit points of the set of their zeros belong to $\mathbb{T}$.

The good news is that all these properties are preserved if we restrict the freedom of B_n by suitably fixing k_n of its zeros in advance. Thus, instead of B_n we consider polynomials of the form $\tilde{B}_n = b_n L_n$ where L_n is a monic polynomial of degree $k_n \leq n$ which is preselected, and b_n is the unique monic polynomial of degree $n - k_n$, with $b_n(0) \in \mathbb{T}$ preassigned, satisfying the following minimality condition, analogous to (1),

$$\|b_n L_n\| \leq \|\big(z^{n-k_n} + zq(z) + b_n(0)\big)L_n(z)\|, \quad \forall q \in \mathbb{P}_{n-k_n-2}. \tag{3}$$

Note that this means that b_n is a monic paraorthogonal polynomial of degree $n - k_n$ with respect to the measure $|L_n|^2 d\mu$ with $b_n(0) \in \mathbb{T}$ preassigned and therefore all its zeros are simple and are on $\mathbb{T}$. Thus b_n is an extremal polynomial with respect to the modified measure $|L_n|^2 d\mu$. The polynomials $\tilde{B}_n = b_n L_n$ will be called extremal polynomials with preassigned zeros (namely the zeros of L_n). In particular we will prove that the sequence $\{\tilde{B}_n\}_{n\in\mathbb{N}}$ of extremal polynomials with preassigned zeros will have the same nth root asymptotic behavior (2) as the sequence of extremal polynomials $\{B_n\}_{n\in\mathbb{N}}$ defined by (1) if the sequence $\{L_n\}_{n\in\mathbb{N}}$ is an asymptotically extremal sequence on $\mathbb{T}$.

To sum up we claim that if the zeros of the preassigned L_n are asymptotically uniformly distributed on $\mathbb{T}$, then the zeros of b_n determined by the paraorthogonality conditions (3) will complement them as zeros on $\mathbb{T}$ so that the sequence $\{\tilde{B}_n\}_{n\in\mathbb{N}}$ of extremal polynomials with preassigned zeros will be asymptotically extremal too and hence have the same asymptotic behavior. Our strategy to prove this statement is very close to the methods used in [2]. The results are the analogues for the unit circle of the ones that were discussed in [3] for the positive real line.

In the rest of the paper we shall use the following notation. $\mathbb{C}$ is the complex plane and $\hat{\mathbb{C}}$ the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. $\mathbb{T}, \mathbb{D}$, and $\mathbb{E}$ are the boundary, interior, and exterior of the closed unit disk, respectively. For a general set $S \subset \mathbb{C}$ we denote by $\mathring{S}$, ∂S and $\hat{S}$ its interior, its boundary and its closure, respectively. Let μ be a finite positive Borel measure on $\mathbb{T}$ we denote by $\langle \cdot, \cdot \rangle$ the inner product and by $\| \cdot \|$ the norm for $L^2_\mu(\mathbb{T})$. The supremum norm for the bounded function f over the set S is $\|f\|_S = \sup_{z \in S} |f(z)|$. The set $\mathbb{P}_n$ contains the polynomials of degree n at most, $\mathbb{P}'_n = \mathbb{P}_n \backslash \mathbb{P}_{n-1}$ are the polynomials of (strict) degree n and $\mathbb{P}^1_n \subset \mathbb{P}'_n$ is the subset of its monic polynomials of degree n. For any polynomial P, we denote the set of its zeros by $\mathcal{Z}(P)$ where we assume that zeros are repeated according to their multiplicity, and its zero counting measure is defined as $\sigma_P = \sum_{z \in \mathcal{Z}(P)} \delta_z$, where δ_z is the weight of mass 1 concentrated in z, counting multiplicity. For a polynomial $P \in \mathbb{P}'_n$ we define the reversed polynomial as $P^*(z) = z^n \overline{P(1/\overline{z})}$ and it is said to be invariant if $P = \tau P^*$ with $\tau \in \mathbb{T}$. A quasi-paraorthogonal polynomial of degree n and order $2\ell + 1$ is an invariant polynomial of degree n satisfying the orthogonality conditions $\langle P_n, z^k \rangle = 0$, $k = 2\ell + 1, \ldots, n - (2\ell + 1)$. The classical paraorthogonal polynomials correspond to quasi-paraorthogonals with $\ell = 0$. Note that a monic (quasi-)paraorthogonal polynomial B_n satisfies $B_n = B_n(0) B^*_n$ with $B_n(0) \in \mathbb{T}$. If a sequence of measures $\{\sigma_n\}_{n \in \mathbb{N}}$ converges in the weak star topology to a measure σ, as $n \to \infty$, this will be denoted as $\sigma_n \xrightarrow{*} \sigma$. For a definition of weak convergence see e.g. [4, page 9]. The notation $R^m_{n,p}$ with $0 < p < 2n + 1$ will be reserved to denote a two-point partial Padé approximant of degree n with m free poles (i.e., $n - m$ poles are preassigned) where p is the order of contact at the origin while at ∞, the order is $n + m + 1 - p$. Note that $R^m_{n,p}$ has $n + m + 1$ free parameters.

2. Extremal polynomials with preassigned zeros

We start by borrowing some concepts from potential theory that can be found in the vast literature about the subject, for example [5, 4, 6, 7] and many other references to which we refer for more details.

Among all the positive unit Borel measures supported on a compact set $S \subset \mathbb{C}$ with positive capacity there is a unique measure ν_S that minimizes the energy for the logarithmic potential. That is called the equilibrium measure of S. It is supported on the outer boundary $\partial_\infty S$, that is, the boundary of the unbounded component $D_\infty(S)$ of $\hat{\mathbb{C}} \backslash S$ up to a set of capacity zero. The Green function $g_S(z; \infty)$ for $D_\infty(S)$ with pole at ∞ is a positive and harmonic function in $D_\infty(S) \backslash \{\infty\}$ that converges to zero on the boundary of this set except for a set of capacity zero.

In our case where $S = \mathbb{T}$, the equilibrium measure is the normalized Lebesgue measure on $\mathbb{T}$, which we denote as λ. Its logarithmic potential is

$$U_\lambda(z) = \begin{cases} -\log|z|, & z \in \mathbb{E}, \\ 0, & z \in \mathbb{C} \backslash \mathbb{E}. \end{cases}$$

The capacity of $\mathbb{T}$ is $\mathrm{cap}(\mathbb{T}) = 1$ while the Green function is $g_{\mathbb{T}}(z; \infty) = \log|z|$, $z \in \mathbb{E}$.

An important property of the Green function is to control the growth of a polynomial in the exterior of the compact set S as illustrated by the following theorem.

Theorem 2.1. *(***Bernstein-Walsh inequality** *[6, Lemma 3.7]) Let $S \subset \mathbb{C}$ be a compact set of positive capacity. Then, for any monic polynomial P of degree n, we have*

$$|P(z)| \leq \|P\|_S \, e^{n g_S(z,\infty)}, \quad z \in D_\infty(S),$$

as well as,

$$\operatorname{cap}(S) \leq \|P\|_S^{\frac{1}{n}}.$$

Applied to our situation where $S = \mathbb{T}$, this implies that for a sequence $\{P_n\}_{n\in\mathbb{N}}$, $P_n \in \mathbb{P}_n^1$ for $n \in \mathbb{N}$,

$$1 \leq \liminf_{n\to\infty} \|P_n\|_{\mathbb{T}}^{\frac{1}{n}} \tag{4}$$

and

$$\limsup_{n\to\infty} \frac{|P_n(z)|^{\frac{1}{n}}}{\|P_n\|_{\mathbb{T}}^{\frac{1}{n}}} \leq \max(|z|, 1), \quad z \in \mathbb{C}.$$

The latter inequality holds for $z \in \mathbb{E}$ by the previous theorem and it is obvious in $\mathbb{D}$. Note that $\lim_{n\to\infty} \|P_n\|_{\mathbb{T}} = 1$, which as the following result shows, is equivalent with $\lim_{n\to\infty} |P_n(z)|^{\frac{1}{n}} = |z|$ uniformly on compact subsets of $\mathbb{E}$ if we assume that the zeros of $\{P_n\}_{n\in\mathbb{N}}$ have no limit point in $D_\infty(\mathbb{T})$. See Theorem 4 and Theorem 2, in chapter VII of [8]. This proves the following.

Theorem 2.2. *Consider the sequence $\{P_n\}_{n\in\mathbb{N}}$ with $P_n \in \mathbb{P}_n^1$ such that the sequence $\{\mathcal{Z}(P_n)\}_{n\in\mathbb{N}}$ has no limit point in $D_\infty(\mathbb{T})$. Then the following statements are equivalent:*

1. $\lim_{n\to\infty} |P_n(z)|^{\frac{1}{n}} = |z|$, *uniformly on compact subsets of* $\mathbb{E}$.
2. $\lim_{n\to\infty} \|P_n\|_{\mathbb{T}}^{\frac{1}{n}} = 1$.

Another important result for our objective is the characterization of the equilibrium measure in terms of nth root asymptotic behavior of a sequence of monic polynomials given in Theorem 2.1 of [9]. We rephrase it for our situation.

Theorem 2.3. *[9, Theorem 2.1] Let $\{P_n\}_{n\in\mathbb{N}}$ be a sequence of monic polynomials with $\deg(P_n) = n$ and let $\sigma_n = \frac{1}{n}\sigma_{P_n}$ be the normalized zero counting measure of P_n. If the following two conditions are satisfied:*

(a) $\limsup_{n\to\infty} \|P_n\|_{\mathbb{T}}^{\frac{1}{n}} \leq 1$.
(b) $\lim_{n\to\infty} \sigma_n(A) = 0$ *for all closed sets* $A \subset \mathbb{D}$,

then $\sigma_n \xrightarrow{*} \lambda$ as $n \to \infty$.

Conversely, suppose that the sequence of zero sets $\{\mathcal{Z}(P_n)\}_{n\in\mathbb{N}}$ is uniformly bounded and that $\sigma_n \xrightarrow{*} \lambda$ as $n \to \infty$, then the conditions (a) and (b) hold.

Note that by (4), condition (a) in the last Theorem is equivalent to

$$\lim_{n\to\infty} \|P_n\|_{\mathbb{T}}^{\frac{1}{n}} = 1.$$

Now we are ready to have a look at the asymptotic distribution of the zeros of the monic polynomials of the form $\tilde{B}_n = b_n L_n$ of minimal L^2_μ-norm that were considered in Section 1. This requires the following two definitions.

Definition 2.4. Asymptotically extremal polynomials. *A sequence of polynomials $\{P_n\}_{n\in\mathbb{N}}$ with $\{\mathcal{Z}(P_n)\}_{n\in\mathbb{N}}$ uniformly bounded, is said to be asymptotically extremal with respect to $\mathbb{T}$ if and only if it satisfies*

$$\lim_{n\to\infty} |P_n(z)|^{1/n} = \max(|z|, 1) \quad \textit{uniformly on compact subsets of } \mathbb{C}\backslash\mathbb{T}.$$

The polynomials in the sequence are said to be asymptotically extremal polynomials.

Note that a consequence of uniform convergence on compact subsets of $\mathbb{C}\backslash\mathbb{T}$ to that function, all the accumulation points of the zeros of extremal polynomials are in $\mathbb{T}$.

Definition 2.5. Uniform asymptotic zero distribution. *Let $\{P_n\}_{n\in\mathbb{N}}$ be a sequence of polynomials with $P_n \in \mathbb{P}'_n$ for all $n \in \mathbb{N}$. Then we say that the polynomial sequence has a uniform asymptotic zero distribution if the sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ of normalized counting measures $\sigma_n = \frac{1}{n}\sum_{z\in\mathcal{Z}(P_n)} \delta_z$ converges weakly to the equilibrium measure λ of $\mathbb{T}$.*

If $\{\mathcal{Z}(P_n)\}_{n\in\mathbb{N}}$ has no limit point in $\mathbb{C}\backslash\mathbb{T}$, then it is not difficult to verify by Theorem 2.3 and Theorem 2.2 that both concepts of the previous definitions are equivalent. Indeed, all sequences of asymptotically extremal polynomials have uniform asymptotic zero distribution via Theorem 2.3 and Theorem 2.2. Conversely, if a polynomial sequence $\{P_n\}_{n\in\mathbb{N}}$ has uniform asymptotic zero distribution, consider $g_z(t) = \log\frac{1}{|t-z|} \in \mathcal{C}(\mathbb{T})$, $z \in \mathbb{D}$, then by weak star convergence,

$$\lim_{n\to\infty} \sigma_n(g_z) = \lambda(g_z) = 0, \quad z \in \mathbb{D},$$

and this convergence is uniformly on compact subsets of $\mathbb{D}$ because the accumulation points of $\{\mathcal{Z}(P_n)\}_{n\in\mathbb{N}}$ are in $\mathbb{T}$. Then,

$$\lim_{n\to\infty} \exp(\sigma_n(g_z)) = \exp(\lambda(g_z)) = \exp(0), \quad z \in \mathbb{D},$$

or equivalently

$$\lim_{n\to\infty} |P_n(z)|^{\frac{1}{n}} = 1, \quad \text{uniformly on compact subsets of } \mathbb{D}.$$

The behavior in $\mathbb{E}$ follows from the converse of Theorem 2.3, which says in particular that its item (a) holds. Together with (4) this gives $\lim_{n\to\infty} \|P_n\|_{\mathbb{T}}^{1/n} = 1$, and so Theorem 2.2 gives the desired behaviour in $\mathbb{E}$.

The following result will allow us to properly complement the preassigned zeros of paraorthogonal polynomials.

Theorem 2.6. *[2, Lemma 1] Let μ be a finite positive Borel measure on $\mathbb{T}$ and let $\{L_n\}_{n\in\mathbb{N}}$ with $\deg(L_n) = k_n \leq n$ be a sequence with uniform asymptotic zero distribution. Then there exists a sequence $\{U_n\}_{n\in\mathbb{N}}$ of monic polynomials with $U_n \in \mathbb{P}^1_{n-k_n}$ and all its zeros in a fixed annulus E containing $\mathbb{T}$ such that the sequence $\{U_nL_n\}_{n\in\mathbb{N}}$ has uniform asymptotic zero distribution.*

As pointed out in the introduction, we consider polynomials of the form $\tilde{B}_n = b_nL_n$ where $\deg(L_n) = k_n \leq n$, and b_n a polynomial of degree $n-k_n$, with $b_n(0)$ preassigned. Let $\{L_n\}_{n\in\mathbb{N}}$ be an asymptotically extremal sequence of polynomials for $\mathbb{T}$ and let the sequence $\{\tilde{B}_n\}_{n\in\mathbb{N}}$ satisfy the following extremality condition

$$\|\tilde{B}_n\| \leq \left\|\left(z^{n-k_n} + zq(z) + b_n(0)\right)L_n(z)\right\|, \quad \forall q \in \mathbb{P}_{n-k_n-2}. \tag{5}$$

We wonder if the sequence $\{\tilde{B}_n\}_{n\in\mathbb{N}}$ is asymptotically extremal for $\mathbb{T}$ with preassigned zeros. We will give a positive answer.

Lemma 2.7. *Let μ be a finite positive Borel measure with support $\mathbb{T}$ and $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathbb{T}$. Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of monic asymptotically extremal polynomials on $\mathbb{T}$ of degree $k_n \leq n$, then there exists a sequence $\{M_n\}_{n\in\mathbb{N}}$ of monic polynomials of degree $m_n = n - k_n$ such that $M_n(0) = \tau_n$ and*

$$\limsup_{n\to\infty} \|M_nL_n\|_{\mathbb{T}}^{\frac{1}{n}} = 1.$$

Proof. By Theorem 2.6 there exists a sequence $\{U_n\}_{n\in\mathbb{N}}$ of monic polynomials of degree $m_n = n - k_n$ such that the zeros of $\{U_n\}_{n\in\mathbb{N}}$ are in the annulus E and the sequence of zeros of $\{U_nL_n\}_{n\in\mathbb{N}}$ has uniform asymptotic zero distribution . Now, since the zeros of $\{U_nL_n\}_{n\in\mathbb{N}}$ are uniformly bounded, the sequence $\{U_nL_n\}_{n\in\mathbb{N}}$ satisfies by Theorem 2.3,

$$\lim_{n\to\infty} \|U_nL_n\|_{\mathbb{T}}^{\frac{1}{n}} = 1.$$

Consequently

$$\limsup_{n\to\infty} |U_n(0)L_n(0)|^{\frac{1}{n}} \leq 1.$$

Since $\lim_{n\to\infty} |L_n(0)|^{1/n} = 1$ it follows that $\limsup_{n\to\infty} |U_n(0)|^{1/n} \leq 1$.

Next, define $M_n = U_n + \tau_n - U_n(0)$, so that $M_n(0) = \tau_n$. Given that

$$\lim_{n\to\infty} \|U_nL_n\|_{\mathbb{T}}^{\frac{1}{n}} \leq 1, \quad \lim_{n\to\infty} \|L_n\|_{\mathbb{T}}^{\frac{1}{n}} \leq 1,$$

and

$$\limsup_{n\to\infty} |\tau_n - U_n(0)|^{\frac{1}{n}} \leq \limsup_{n\to\infty} (1 + |U_n(0)|)^{\frac{1}{n}} \leq 1,$$

it follows that

$$\limsup_{n\to\infty} \|M_nL_n\|_{\mathbb{T}}^{\frac{1}{n}} \leq 1,$$

by Theorem 2.1 we conclude the proof. □

Before we arrive at the main result of this section, we link the polynomials that minimize the supremum norm to the ones that minimize the L^2_μ-norm. There exists a unique $T_n \in \mathbb{P}^1_n$ such that (see [10]),

$$t_n(S) = \min_{P_n \in \mathbb{P}^1_n} \|P_n\|_S = \|T_n\|_S.$$

In this context T_n is sometimes called a (monic) Chebyshev polynomial. The *Chebyshev constant* for S is defined by $\mathrm{cheb}(S) = \lim_{n\to\infty} t_n(S)^{\frac{1}{n}}$. In our case where $S = \mathbb{T}$, we have the following inequalities. Let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence of monic orthogonal polynomials with respect to μ and $\mathrm{supp}(\mu) = \mathbb{T}$, as $t_n(\mathbb{T}) = \|T_n\|_{\mathbb{T}} \leq \|p_n\|_{\mathbb{T}}$, then

$$\mathrm{cheb}(\mathbb{T}) \leq \liminf_{n\to\infty} \|p_n\|_{\mathbb{T}}^{\frac{1}{n}}, \tag{6}$$

and

$$\limsup_{n\to\infty} \|p_n\|^{\frac{1}{n}} \leq \lim_{n\to\infty} \|T_n\|_{\mathbb{T}}^{\frac{1}{n}} = \mathrm{cheb}(\mathbb{T}).$$

Note that $\mathrm{cheb}(\mathbb{T}) = \mathrm{cap}(\mathbb{T}) = 1$.

Let us introduce the following class of measures that will be our main interest.

Definition 2.8. Regular measure. *Let μ be finite positive Borel measure with support $\mathbb{T}$. Then μ is said to be regular if and only if*

$$\lim_{n\to\infty} \|p_n\|^{\frac{1}{n}} = 1,$$

where $\{p_n\}_{n\in\mathbb{N}}$ is the sequence of monic orthogonal polynomials with respect to μ.

The following theorem compares the asymptotic behavior of the norms $\|P_n\|_{\mathbb{T}}$ and $\|P_n\|$.

Corollary 2.9. *Let μ be regular measure and $\{P_n\}_{n\in\mathbb{N}}$ a polynomial sequence with $P_n \in \mathbb{P}'_n$, for all $n \in \mathbb{N}$, then*

$$\lim_{n\to\infty} \frac{\|P_n\|^{\frac{1}{n}}}{\|P_n\|_{\mathbb{T}}^{\frac{1}{n}}} = 1.$$

Proof. On the one hand, from Theorem 3.2.3 of [5],

$$\limsup_{n\to\infty} \left(\frac{\|P_n\|_{\mathbb{T}}}{\|P_n\|}\right)^{\frac{1}{n}} \leq 1$$

and on the other, $\|P_n\| \leq \mu(\mathbb{T})\, \|P_n\|_{\mathbb{T}}$ so that

$$\liminf_{n\to\infty} \left(\frac{\|P_n\|_{\mathbb{T}}}{\|P_n\|}\right)^{\frac{1}{n}} \geq \lim_{n\to\infty} \Big(\mu(\mathbb{T})\Big)^{-\frac{1}{n}} = 1,$$

which completes the proof. □

We are in a position to give the main result of this paper, that we shall apply to the convergence of two-point Padé approximation in the next sections, but it is of independent interest.

Theorem 2.10. *Let μ be a regular measure on $\mathbb{T}$. Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of asymptotically extremal polynomials with respect to $\mathbb{T}$, with L_n of degree $k_n \leq n$. Let $\tilde{B}_n = b_n L_n$ where b_n is defined by (5). Then, $\{\tilde{B}_n\}_{n\in\mathbb{N}}$ is an extremal sequence with preassigned zeros.*

Proof. First of all, recall that the sequence $\{\mathcal{Z}(b_n L_n)\}_{n\in\mathbb{N}}$ has no limit point outside of $\mathbb{T}$ because $\{L_n\}_{n\in\mathbb{N}}$ is asymptotically extremal for $\mathbb{T}$ and $\mathcal{Z}(b_n) \subset \mathbb{T}$ because b_n is paraorthogonal with respect to $|L_n|^2 d\mu$ for $n \in \mathbb{N}$. It only remains to show that σ_n, the normalized zero counting measure for $b_n L_n$ satisfies $\sigma_n \xrightarrow{*} \lambda$. For $n \in \mathbb{N}$ let M_n with $M_n(0) = b_n(0)$ be the monic polynomial as guaranteed to exist by Lemma 2.7. We have the following sequence of inequalities

$$\begin{aligned} 1 &\leq \liminf_{n\to\infty} \|b_n L_n\|_{\mathbb{T}}^{\frac{1}{n}} \leq \limsup_{n\to\infty} \|b_n L_n\|_{\mathbb{T}}^{\frac{1}{n}} \leq \limsup_{n\to\infty} \|b_n L_n\|^{\frac{1}{n}} \\ &\leq \limsup_{n\to\infty} \|M_n L_n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|M_n L_n\|_{\mathbb{T}}^{\frac{1}{n}} = 1. \end{aligned}$$

where we use (4) of Theorem 2.1 for the first inequality and that μ is regular for the third one. Next the extremal condition (5) for $b_n L_n$ implies the fourth inequality for a monic polynomial of degree $n - k_n$ with $M_n(0) = b_n(0)$. The rest of the chain follows by the regularity of the measure, in particular by Corollary 2.9, which says that

$$1 = \lim_{n\to\infty} \frac{\|M_n L_n\|^{1/n}}{\|M_n L_n\|_{\mathbb{T}}^{1/n}}$$

hence $\lim_{n\to\infty} \|M_n L_n\|_{\mathbb{T}}^{1/n} = \lim_{n\to\infty} \|M_n L_n\|^{1/n}$ on condition that one of these limits exist and this is guaranteed by Lemma 2.7 which says that $\limsup_{n\to\infty} \|M_n L_n\|_{\mathbb{T}}^{1/n} = 1$.

Since the first and the last terms in this chain of inequalities are the same, we must have equality everywhere.

So,

$$\lim_{n\to\infty} \|b_n L_n\|_{\mathbb{T}}^{\frac{1}{n}} = 1.$$

This shows that condition (a) of Theorem 2.3 is satisfied. Condition (b) holds if the sequence of normalized zero counting measures $\{\sigma_n\}_{n\in\mathbb{N}}$ of $\{b_n L_n\}_{n\in\mathbb{N}}$ verifies $\lim_{n\to\infty} \sigma_n(A) = 0$ for all closed sets $A \subset \mathbb{D}$, which is immediate by the behavior of $\{\mathcal{Z}(b_n L_n)\}_{n\in\mathbb{N}}$. □

Remark 2.11. *A strategy to construct extremal polynomials with prefixed zeros is the following. We consider a finite positive Borel measure ν on $\mathbb{T}$ and we denote by $\{\rho_n\}_{n\in\mathbb{N}}$ the sequence of orthogonal polynomials associated with ν, the corresponding Schur parameters will be denoted by $\{\delta_k\}_{k\in\mathbb{N}}$ where $\delta_0 = 1$ and $\delta_k = \rho_k(0) \in \mathbb{D}$, i.e., $\rho_k(z) = z\rho_{k-1}(z) + \delta_k \rho_{k-1}^*(z)$ for $k \geq 1$. If*

$$\limsup_{n\to\infty} |\rho_n(0)|^{\frac{1}{n}} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |\rho_k(0)| = 0,$$

then $\{\rho_n\}_{n\in\mathbb{N}}$ is asymptotically extremal, see [11]. Next, we obtain the paraorthogonal polynomials of degree $n-k_n$ associated to the modified measure $|\rho_{k_n}(e^{i\theta})|^2 d\mu(\theta)$, (see (5)).

Another alternative is to choose our preassigned zeros as the zeros of paraorthogonal polynomials for a regular measure ν, since they are extremal polynomials. If $L_n = B^{\nu}_{k_n}$ is such a paraorthogonal polynomial associated with ν of degree k_n, then we build the paraorthogonal polynomial of degree $n-k_n$ associated with $|B^{\nu}_{k_n}(e^{i\theta})|^2 d\mu(\theta)$. In particular, if ν and μ are both equal to the Lebesgue measure, then we can choose $B^{\nu}_{k_n}(z) = L_n(z) = z^{k_n} - 1$. The measure $|e^{ik_n\theta} - 1|^2 d\theta = 2(1-\cos(k_n\theta))d\theta$ is analyzed in [12].

3. Two-point Partial Padé approximation to a Herglotz-Riesz function

We consider Two-point Partial Padé Approximants (2PPA). More precisely Two-point Padé approximants to Herglotz-Riesz transforms with part of its poles prefixed. The Herglotz-Riesz transform is defined as

$$\Omega_\mu(z) = \int_{\mathbb{T}} \frac{t+z}{t-z} d\mu(t),$$

where μ is a finite positive Borel measure on $\mathbb{T}$. It is well known that Ω_μ is analytic in $\hat{\mathbb{C}}\backslash\mathbb{T}$. In particular, the following expansions hold

$$\Omega_\mu(z) = \mu_0 + 2\sum_{j\geq 1} \mu_j z^j, \qquad z\in\mathbb{D}, \tag{7}$$

$$\Omega_\mu(z) = -\mu_0 - 2\sum_{j\geq 1} \mu_{-j} z^{-j}, \quad z\in\mathbb{E}. \tag{8}$$

where $\mu_j = \int_{\mathbb{T}} t^{-j} d\mu(t)$, $j\in\mathbb{Z}$, are the moments of μ. Note that $\overline{\mu_j} = \mu_{-j}$.

Concretely, let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of monic polynomials of degree k_n with $0 \leq k_n \leq n$ and $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathbb{T}$. Let $p_n \in \mathbb{N}$ be such that $0 < p_n < 2n$. Then there exist two polynomials $b_n \in \mathbb{P}_{n-k_n}$ with $b_n(0) = \tau_n$ and $A_n \in \mathbb{P}_{n+k_n}$, such that

$$b_n(z)L_n(z)L_n^*(z)\Omega_\mu(z) - A_n(z) = \mathcal{O}(z^{p_n}), \qquad z\to 0, \tag{9}$$

$$b_n(z)L_n(z)L_n^*(z)\Omega_\mu(z) - A_n(z) = \mathcal{O}(z^{-(n-k_n-p_n)}), \qquad z\to\infty. \tag{10}$$

Any pair $(b_nL_nL_n^*, A_n)$ satisfying the above interpolation conditions defines a unique 2PPA $R^{n-k_n-1}_{n+k_n,p_n} = \frac{A_n}{b_nL_nL_n^*}$. It is a 2PPA of degree $n+k_n$ with $n-k_n-1$ free poles and its order of contact is p_n at the origin and $2n-p_n$ at infinity.

As demonstrated in [1], the denominator of a 2PPA for the Herglotz-Riesz transform Ω_μ satisfies certain orthogonality conditions. In particular,

Theorem 3.1. *Let $R^{n-k_n-1}_{n+k_n,n} = \frac{A_n}{b_nL_nL_n^*}$ be a 2PPA to the Herglotz-Riesz transform Ω_μ, then b_n is invariant and*

$$\langle b_nL_nL_n^*, z^k\rangle = 0, \quad k = k_n+1,\ldots,n-1, \tag{11}$$

or equivalently

$$\langle b_n|L_n|^2, z^k\rangle = 0, \quad k = 1,\ldots,n-k_n-1.$$

Thus b_n is a paraorthogonal polynomial of degree $n-k_n$ with respect to $|L_n|^2 d\mu$ and therefore all its zeros are simple and lie on $\mathbb{T}$. It is the unique extremal solution to the problem (5).

The authors in [1] consider the particular case $k_n = 0$ with one fixed pole $z_0 \in \mathbb{T}$ by choosing $b_n(0) \in \mathbb{T}$ such that $b_n(z_0) = 0$. The following theorem was obtained.

Theorem 3.2. *Let μ be a regular measure on $\mathbb{T}$ and $\{R_{n,n}^{n-1}\}_{n\geq 1}$ the sequence of 2PPA for Ω_μ with one prefixed pole $z_0 \in \mathbb{T}$. Then the sequence converges uniformly to Ω_μ in compact subsets of the exterior of $\mathbb{T}$. Moreover, an exact geometric rate of convergence is achieved*

$$\lim_{n\to\infty} |\Omega_\mu(z) - R_{n,n}^{n-1}(z)|^{\frac{1}{n}} = \min\left(|z|, \frac{1}{|z|}\right), \quad \forall z \notin \mathbb{T}.$$

We are now in a position to give the main result of this section.

Theorem 3.3. *Let μ be a regular measure on $\mathbb{T}$ and $\{R_{n+k_n,n}^{n-k_n-1}\}_{n\geq 1}$ the sequence of 2PPA for Ω_μ with fixed poles at zeros of $L_n L_n^*$, where $\{L_n\}_{n\in\mathbb{N}}$ is an asymptotically extremal sequence of polynomials. Then, the sequence $\{R_{n+k_n,n}^{n-k_n-1}\}_{n\geq 1}$ converges uniformly to Ω_μ in compact subsets of the exterior of $\mathbb{T}$. Moreover, an exact geometric rate of convergence is achieved*

$$\lim_{n\to\infty} \left|\Omega_\mu(z) - R_{n+k_n,n}^{n-k_n-1}(z)\right|^{\frac{1}{n}} = \min\left(|z|, \frac{1}{|z|}\right), \quad \forall z \notin \mathbb{T}.$$

Proof. To simplify the notation in this proof we denote $R_{n+k_n,n}^{n-k_n-1}$ by $R_n = \frac{A_n}{b_n L_n L_n^*} = \frac{A_n}{B_n}$.
By Theorem 8 in [1]

$$\Omega_\mu(z) - R_n(z) = \frac{2z^n}{W(z)b_n(z)L_n(z)L_n^*(z)} \int_{\mathbb{T}} \frac{W(t)b_n(t)L_n(t)L_n^*(t)}{(t-z)t^{n-1}} d\mu(t), \tag{12}$$

which holds for all $W \in \mathbb{P}_{n-k_n}$. Hence, taking $W = b_n^*$ and setting $G_n = b_n L_n$ and $d\mu_n = \frac{|G_n|^2}{\|G_n\|^2} d\mu$

$$\frac{G_n(z)G_n^*(z)}{\|G_n\|^2 z^n} \left(\Omega_\mu(z) - R_n(z)\right) = \int_{\mathbb{T}} \frac{2t}{t-z} d\mu_n(t).$$

Next, use $\frac{2t}{t-z} = 1 + \frac{t+z}{t-z}$ and define $K_n(z) = \frac{z^n \|G_n\|^2}{G_n(z)G_n^*(z)}$, then

$$\frac{\Omega_\mu(z) - R_n(z)}{K_n(z)} = \int_{\mathbb{T}} \frac{t+z}{t-z} d\mu_n(t) + 1. \tag{13}$$

Therefore, taking the real part of the right-hand side in (13) and defining the largest distance from z to $\mathbb{T}$: $d(z) = \max_{t\in\mathbb{T}} |z-t| = 1 + |z|$, then we obtain the following inequality for all $z \notin \mathbb{T}$

$$\left|\frac{\Omega_\mu(z) - R_n(z)}{K_n(z)}\right| \geq \left|\mathrm{Re}\left(\int_{\mathbb{T}} \frac{t+z}{t-z} d\mu_n(t)\right) + 1\right| = \left|1 - |z|^2\right| \int_{\mathbb{T}} \frac{d\mu_n(t)}{|t-z|^2} + 1 \geq \frac{|1-|z|^2|}{d(z)^2} + 1.$$

On the other hand, (13) gives an upper bound for $z \notin \mathbb{T}$ so that

$$1 + \left| \frac{1-|z|^2}{d(z)^2} \right|^{\frac{1}{n}} \leq \left| \frac{\Omega_\mu(z) - R_n(z)}{K_n(z)} \right|^{\frac{1}{n}} \leq \left| 1 + \max_{t\in\mathbb{T}} \left| \frac{t+z}{t-z} \right| \right|^{\frac{1}{n}}.$$

By applying the limit

$$\lim_{n\to\infty} \left| \frac{\Omega_\mu(z) - R_n(z)}{K_n(z)} \right|^{\frac{1}{n}} = 1, \quad z \notin \mathbb{T}. \tag{14}$$

As a consequence of $\{G_n\}_{n\in\mathbb{N}}$ being asymptotically extremal and the regularity of μ

$$\lim_{n\to\infty} |K_n(z)|^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{|z|^n \|G_n\|^2}{|G_n(z)|^2} \right)^{\frac{1}{n}} = \min\left(|z|, \frac{1}{|z|} \right), \quad z \notin \mathbb{T}. \tag{15}$$

The combination of (15) and (14) concludes the proof. □

Remark 3.4. *Note that if L_n is a paraorthogonal polynomial with respect to an auxiliary regular measure, then all the poles of $R_{n+k_n,n}^{n-k_n-1}$ are on $\mathbb{T}$.*

Remark 3.5. *It is well known that rational approximants to a Carathéodory function can be obtained as convergents of a positive Perron-Carathéodory continued fraction (PC-fraction). The even and odd approximants are alternatingly partial Padé approximants at the origin and at infinity (the denominators are ρ_n^* and ρ_n respectively). The proper 2-point partial Padé approximation, approximating simultaneously at the origin and infinity have a paraorthogonal polynomial as denominator. They are called modified approximants, but they do not appear as convergents of a continued fraction. See [13], [1] for more details. Extremal polynomials with preassigned zeros are quasi-paraorthogonal polynomials, introduced in [14], and a fortiori, also approximants associated with such denominators are not simply linked with continued fractions.*

4. Two-point Partial Padé approximation to a Herglotz-Riesz function with rational perturbation

We analyse the convergence of Two-point Partial Padé to functions of the type

$$f(z) = \Omega_\mu(z) + R(z), \tag{16}$$

where $R(z) = \frac{P(z)}{Q(z)}$, such that $R(z) = -\overline{R}(\frac{1}{z})$, $\forall z \in \mathbb{C}\backslash\mathbb{T}$, with Q being an invariant monic polynomial of even degree $d = 2\ell$, and such that $Q(z) \neq 0$, $\forall\, z \in \mathbb{T}$ and $R(\infty) \in \mathbb{R}$. We prove that if the prescribed poles are chosen conveniently, as in the previous section, then sequences of 2PPA for f converge geometrically to f on compact subsets of $\mathbb{C}\backslash(\mathbb{T}\cup\mathcal{Z}(Q))$. We obtain the same rate of convergence as for $R \equiv 0$.

Concretely, for all $n \in \mathbb{N}$, let $L_n \in \mathbb{P}^1_{k_n}$ with $0 \leq k_n \leq n$ and let integers p_n with $0 < p_n < 2n$ be given and let $b_n \in \mathbb{P}^1_{n-k_n}$ with prescribed value $b_n(0) \in \mathbb{T}$ and $A_n \in \mathbb{P}_{n+k_n}$ satisfy

$$\begin{aligned} b_n(z)L_n(z)L_n^*(z)f(z) - A_n(z) &= \mathcal{O}(z^{p_n}), \qquad z \to 0, \\ b_n(z)L_n(z)L_n^*(z)f(z) - A_n(z) &= \mathcal{O}(z^{-(n-k_n-p_n)}), \qquad z \to \infty. \end{aligned}$$

We multiply both relations by Q and if we define $D_n = QA_n - b_n L_n L_n^* P \in \mathbb{P}_{n+k_n+d}$, then

$$b_n(z)L_n(z)L_n^*(z)Q(z)\Omega_\mu(z) - D_n(z) = \mathcal{O}(z^{p_n}), \qquad z \to 0, \tag{17}$$
$$b_n(z)L_n(z)L_n^*(z)Q(z)\Omega_\mu(z) - D_n(z) = \mathcal{O}(z^{-(n-k_n-d-p_n)}), \qquad z \to \infty. \tag{18}$$

So, any pair $(b_n L_n L_n^* Q, QA_n - b_n L_n L_n^* P)$ defines a unique rational function $R_{n+k_n+d,p_n}^{n-k_n-1} = \frac{QA_n - b_n L_n L_n^* P}{b_n L_n L_n^* Q}$. It is a 2PPA to Ω_μ of degree $n+k_n+d$ with $n-k_n-1$ free poles and with $2k_n+d$ fixed poles at the zeros of $L_n L_n^* Q$, with $b_n(0) \in \mathbb{T}$ fixed and order of contact is p_n at the origin and $2n-p_n$ at infinity.

Now, from [1, Theorem 7], we have

Theorem 4.1. *Let $R_{n+k_n,n}^{n-k_n-1} = \frac{A_n}{b_n L_n L_n^*}$ be a 2PPA to f as given in (16), then b_n is invariant and*

$$\langle b_n Q |L_n|^2, z^k \rangle = 0, \quad k = d+1, \ldots, n-k_n-1. \tag{19}$$

Since Q is monic, invariant and has no zeros on $\mathbb{T}$, it follows that $Q(z) = \gamma_Q \hat{Q}(z)\hat{Q}^*(z) \in \mathbb{P}_d$ with $\hat{Q} \in \mathbb{P}_\ell$, having only zeros in $\mathbb{D}$, $\gamma_Q = 1/\overline{\hat{Q}(0)}$ and $d = 2\ell$. Then, $b_n Q$ is a quasi-paraorthogonal polynomial of degree $n-k_n+d$ and order $2\ell+1$ with respect to the measure $\tilde{\mu}$ with $d\tilde{\mu} = |L_n|^2 d\mu$ and $b_n Q$ has at least $n-k_n-d$ zeros of odd multiplicity on $\mathbb{T}$ ([14, Theorem 3.2]) and since Q has no zeros on $\mathbb{T}$, these must be zeros of b_n. If $b_n = \hat{b}_n O_n$ where $\hat{b}_n$ monic of degree $n-k_n-d$ whose zeros are simple and coincide with the zeros of b_n that have odd multiplicity on $\mathbb{T}$ guaranteed by the orthogonality conditions, then O_n has degree d. Since O_n is invariant, $O_n = \gamma_n \hat{O}_n \hat{O}_n^*$, where $\hat{O}_n$ of degree ℓ, has only zeros in $\hat{\mathbb{D}}$ and $\gamma_n = 1/\overline{\hat{O}_n(0)} \in \mathbb{E}$ because $O_n(0) \in \mathbb{T}$. It's easy to see that $\hat{b}_n$ is paraorthogonal with respect to $d\hat{\mu} = |\hat{O}_n \hat{Q} L_n|^2 d\mu$. Therefore we have:

Corollary 4.2. *Let $R_{n+k_n,n}^{n-k_n-1} = \frac{A_n}{b_n L_n L_n^*}$ be a 2PPA to f, such that $Q = \gamma_Q \hat{Q}\hat{Q}^*$ with $\mathcal{Z}(\hat{Q}) \subset \mathbb{D}$ and $b_n = \hat{b}_n O_n$, where $O_n = \gamma_n \hat{O}_n \hat{O}_n^*$, with $\mathcal{Z}(\hat{O}_n) \subset \hat{\mathbb{D}}$ and $b_n(0) \in \mathbb{T}$ preassigned. Then $\hat{b}_n$ is a paraorthogonal polynomial of degree $n-k_n-d$ with respect to $|\hat{O}_n \hat{Q} L_n|^2 d\mu$ and $\hat{b}_n(0) \in \mathbb{T}$ preassigned.*

Theorem 4.3. *Let μ be a regular measure on $\mathbb{T}$ and f given as in (16). Let $\{R_n\}_{n\geq 1}$ with $R_n = R_{n+k_n,n}^{n-k_n-1} = \frac{A_n}{b_n L_n L_n^*}$ be the sequence of 2PPA for f with fixed poles at zeros of $L_n L_n^*$, $n \in \mathbb{N}$, where $\{L_n\}_{n\in\mathbb{N}}$ is an asymptotically extremal sequence where $\deg(L_n) = k_n$. Then the sequence $\{R_n\}_{n\geq 1}$ converges uniformly to f in compact subsets of the exterior of $\mathbb{T} \cup \mathcal{Z}(Q)$. Moreover, an exact geometric rate of convergence is achieved*

$$\lim_{n\to\infty} |f(z) - R_n(z)|^{\frac{1}{n}} = \min\left(|z|, \frac{1}{|z|}\right) \quad \forall z \notin \mathbb{T} \cup \mathcal{Z}(Q),$$

and each pole of f attracts as many poles of R_n as its order of multiplicity.

Proof. We reproduce the ideas of the paper [15], translating the one-point variant to our two-point version. This involves the following steps. First we obtain uniform convergence

to zero on compact subsets of the exterior of $\mathbb{T}$ for the sequence $\{\gamma_n^{-1}O_nQ(f-R_n)\}_{n\in\mathbb{N}}$, as $n\to\infty$. Secondly, this convergence enables us to obtain convergence in capacity of R_n to f on compact subsets of $\mathbb{C}\backslash\mathbb{T}$. Finally, a result of Gonchar (see [15, Lemma 1]) on convergence in capacity for rational functions allows us to settle the proof.

If $R_n=\frac{A_n}{b_nL_nL_n^*}$ is a 2PPA of type $R^{n-k_n-1}_{n+k_n,n}$ for f, then $\frac{QA_n-b_nL_nL_n^*P}{Qb_nL_nL_n^*}$ is a 2PPA to Ω_μ of type $R^{n-k_n-1}_{n+d+k_n,n}$ to which Theorem 8 in [1] applies, and gives an expression for the interpolation error:

$$f(z)-R_n(z)=\frac{2z^n}{W(z)Q(z)b_n(z)L_n(z)L_n^*(z)}\int_{\mathbb{T}}\frac{b_n(t)Q(t)L_n(t)L_n^*(t)W(t)}{t-z}\frac{d\mu(t)}{t^{n-1}},$$

which holds for all $W\in\mathbb{P}_{n-k_n-d}$. Hence taking $W=\hat{b}_n^*\in\mathbb{P}_{n-k_n-d}$ and setting $G_n=\hat{b}_nL_n$ ($\deg(G_n)=n-d$), we get for all $z\notin\mathbb{T}\cup\mathcal{Z}(Q)\cup\mathcal{Z}(O_n)\cup\mathcal{Z}(L_n)\cup\mathcal{Z}(L_n^*)$

$$\begin{aligned}f(z)-R_n(z) &= \frac{2z^n}{Q(z)O_n(z)G_n(z)G_n^*(z)}\int_{\mathbb{T}}\frac{Q(t)O_n(t)G_n(t)G_n^*(t)}{t-z}\frac{d\mu(t)}{t^{n-1}}\\ &= \frac{2z^n}{Q(z)O_n(z)G_n(z)G_n^*(z)}\int_{\mathbb{T}}\frac{Q(t)O_n(t)|G_n(t)|^2t^{n-d}}{t-z}\frac{d\mu(t)}{t^{n-1}}\\ &= \frac{2z^n}{Q(z)O_n(z)G_n(z)G_n^*(z)}\int_{\mathbb{T}}\frac{Q(t)O_n(t)|G_n(t)|^2}{t-z}\frac{d\mu(t)}{t^{d-1}}.\end{aligned}$$

Recall that $Q=\gamma_Q\hat{Q}\hat{Q}^*$ and $O_n=\gamma_n\hat{O}_n\hat{O}_n^*$ with $\deg(\hat{Q}\hat{O}_n)=2\ell=d$, then

$$f(z)-R_n(z)=\frac{2z^n\gamma_Q\gamma_n}{Q(z)O_n(z)G_n(z)G_n^*(z)}\int_{\mathbb{T}}\frac{t}{t-z}|\hat{O}_n(t)\hat{Q}(t)G_n(t)|^2d\mu(t).$$

Equivalently, for $z\notin(\mathbb{T}\cup\mathcal{Z}(L_n)\cup\mathcal{Z}(L_n^*))$

$$\frac{1}{\gamma_n}O_n(z)Q(z)\left(f(z)-R_n(z)\right)=\frac{2z^n\gamma_Q}{G_n(z)G_n^*(z)}\int_{\mathbb{T}}\frac{t}{t-z}|\hat{O}_n(t)\hat{Q}(t)G_n(t)|^2d\mu(t). \tag{20}$$

In order to show that $\{\gamma_n^{-1}O_n(z)Q(z)(f(z)-R_n(z))\}_{n\in\mathbb{N}}$ tends to zero as n grows to infinity, we define $H_n(z)=\frac{z^{n-d}\|\hat{O}_n\hat{Q}G_n\|^2}{G_n(z)G_n^*(z)}$ and $d\nu_n(t)=\frac{|\hat{O}_n(t)\hat{Q}(t)G_n(t)|^2}{\|\hat{O}_n\hat{Q}G_n\|^2}d\mu(t)$, then

$$\gamma_n^{-1}O_n(z)Q(z)\left(f(z)-R_n(z)\right)=H_n(z)F_n(z), \tag{21}$$

where

$$F_n(z)=2z^d\gamma_Q\int_{\mathbb{T}}\frac{t}{t-z}d\nu_n(t). \tag{22}$$

It is easy to see that the conclusions of Theorem 2.6 are true if we replace for $n\in\mathbb{N}$ the polynomials L_n of degree k_n by polynomials $\hat{O}_n\hat{Q}L_n$ of degree k_n+d.

Indeed, since $\{L_n\}_{n\in\mathbb{N}}$ is an asymptotically extremal sequence of polynomials, they have a uniform asymptotic zero distribution. Then $\{\hat{O}_n\hat{Q}L_n\}_{n\in\mathbb{N}}$ also has a uniformly zero distribution since the number of zeros of $\hat{O}_n\hat{Q}$ is d and $\lim_{n\to\infty}\frac{d}{n}=0$. Then we apply

Theorem 2.6 and there exists a polynomial U_n with zeros in an annulus containing $\mathbb{T}$ such that $\{U_n\hat{O}_n\hat{Q}L_n\}_{n\in\mathbb{N}}$ has uniform asymptotic zero distribution. Lemma 2.7 it true for $\{U_n\hat{O}_n\hat{Q}L_n\}_{n\in\mathbb{N}}$ rather $\{L_n\}_{n\in\mathbb{N}}$ because the number of zeros of $\{\hat{O}_n\hat{Q}\}_{n\in\mathbb{N}}$ is $d = o(n)$, the proof essentially the same since both $\hat{O}_n(0)$ and $\hat{Q}(0)$ are in $\mathbb{D}$. Therefore also Lemma 2.7 is still valid for $\{M_n\hat{O}_n\hat{Q}\}_{n\in\mathbb{N}}$ where M_n is a polynomial of degree $n - k_n - d$ such that $M_n(0) = \hat{b}_n(0)$, then

$$\lim_{n\to\infty} \|M_n\hat{O}_n\hat{Q}L_n\|_{\mathbb{T}}^{\frac{1}{n}} = 1.$$

Now, proceeding as in the proof of Theorem 2.10, we deduce that,

$$\lim_{n\to\infty} \|\hat{b}_n\hat{O}_n\hat{Q}L_n\|_{\mathbb{T}}^{\frac{1}{n}} = 1.$$

Consequently, as $\deg(\hat{Q}\hat{O}_n) = d$ and $\mathcal{Z}(\hat{Q}\hat{O}_n) \subset \hat{\mathbb{D}}$,

$$\lim_{n\to\infty} \|\hat{b}_nL_n\|_{\mathbb{T}}^{\frac{1}{n}} = 1,$$

since the zeros of $\{\hat{b}_nL_n\}_{n\in\mathbb{N}}$ have no limit point outside of $\mathbb{T}$. Therefore, with $G_n = \hat{b}_nL_n$ for all $n \in \mathbb{N}$, the sequence $\{G_n\}_{n\in\mathbb{N}}$ is asymptotically extremal, and we thus have

$$\lim_{n\to\infty} |G_n(z)|^{\frac{1}{n}} = \max(|z|, 1),$$

uniformly on compact subset of $\mathbb{C}\backslash\mathbb{T}$. Since the sequence $\{F_n\}_{n\in\mathbb{N}}$ is uniformly bounded on compact subsets of $\mathbb{C}\backslash\mathbb{T}$ both above and below, we deduce

$$\lim_{n\to\infty} \left|\gamma_n^{-1}Q(z)O_n(z)(f(z) - R_n(z))\right|^{\frac{1}{n}} = \min\left(|z|, \frac{1}{|z|}\right),$$

uniformly on compact subset of $\mathbb{C}\backslash\mathbb{T}$. Then,

$$\lim_{n\to\infty} \gamma_n^{-1}Q(z)O_n(z)(f(z) - R_n(z)) = 0,$$

uniformly on compact subsets of $\mathbb{C}\backslash\mathbb{T}$. Let E be an arbitrary compact subset of $\mathbb{C}\backslash\mathbb{T}$, then for all $\epsilon > 0$ we have that

$$\{z \in \mathbb{C} : |f(z) - R_n(z)| \geq \epsilon\} \subset \left\{z \in \mathbb{C} : \frac{|(\gamma_n^{-1}O_nQ(f - R_n))(z)|}{\epsilon} \geq \left|(\gamma_n^{-1}O_nQ)(z)\right|\right\}$$

and hence,

$$E \cap \{z \in \mathbb{C} : |f(z) - R_n(z)| \geq \epsilon\} \subset \left\{z \in E : \frac{|(\gamma_n^{-1}O_nQ(f - R_n))(z)|}{\epsilon} \geq |(\gamma_n^{-1}O_nQ)(z)|\right\}.$$

Since the logarithmic capacity of the compact set $\{z \in \mathbb{C} : |P_n(z)| \leq \gamma\}$, with $P_n \in \mathbb{P}_n^1$, is given by $\gamma^{\frac{1}{n}}$ (see [10]), we have

$$\begin{aligned}
&\operatorname{cap}\left(\{z \in E : |f(z) - R_n(z)| \geq \epsilon\}\right) \leq \\
&\operatorname{cap}\left(\left\{z \in E : \frac{\|\gamma_n^{-1}O_nQ\,(f - R_n)\,\|_E}{\epsilon} \geq \left|(\gamma_n^{-1}O_nQ)(z)\right|\right\}\right) = \left(\frac{\|\gamma_n^{-1}O_nQ\,(f - R_n)\,\|_E}{\epsilon}\right)^{\frac{1}{2d}},
\end{aligned}$$

where $d = \deg(\gamma_n^{-1} O_n) = \deg(Q)$. Therefore,

$$\limsup_{n\to\infty} \operatorname{cap}\left(\{z \in E : |f(z) - R_n(z)| \geq \epsilon\}\right) \leq \limsup_{n\to\infty} \left(\frac{\|\gamma_n^{-1} O_n Q\,(f - R_n)\,\|_E}{\epsilon} \right)^{\frac{1}{2d}} = 0,$$

since $\lim_{n\to\infty} \|\gamma_n^{-1} O_n Q\,(f - R_n)\,\|_E = 0$. Hence, the sequence $\{R_n\}_{n\in\mathbb{N}}$ converges in capacity to f on compact subsets of $\mathbb{C}\backslash\mathbb{T}$.

Note that R_n has at most d poles in $\mathbb{C}\backslash\mathbb{T}$ and f (counting multiplicity) has d poles on $\mathbb{C}\backslash\mathbb{T}$, given by $\mathcal{Z}(Q)$, it follows that the convergence in capacity of the rational functions R_n to the meromorphic function f in $\mathbb{C}\backslash\mathbb{T}$ implies convergence uniformly on compact subsets of $\mathbb{C}\backslash(\mathbb{T} \cup \mathcal{Z}(Q))$ and each pole of f attracts as many zeros of b_n as its multiplicity (see [15, Lemma 1]). Therefore

$$\lim_{n\to\infty} O_n(z) = Q(z) \tag{23}$$

and one concludes that d zeros of $\{b_n\}_{n\in\mathbb{N}}$ approach the zeros of Q and the others are on $\mathbb{T}$ and $\lim_{n\to\infty} \gamma_n = \gamma_Q$. Now, from (21)

$$\begin{aligned} \frac{1}{H_n(z) z^d \gamma_Q \gamma_n} O_n(z) Q_n(z)\,(f(z) - R_n(z)) &= \int_{\mathbb{T}} \frac{2t}{t - z} d\nu_n(t) \\ &= \int_{\mathbb{T}} \frac{t + z}{t - z} d\nu_n(t) + \int_{\mathbb{T}} d\nu_n(t). \end{aligned}$$

Proceeding as in the proof of Theorem 3.3 of the previous section, to find that

$$\lim_{n\to\infty} \left| \frac{1}{H_n(z) z^d \gamma_Q \gamma_n} O_n(z) Q(z)\,(f(z) - R_n(z)) \right|^{\frac{1}{n}} = 1,$$

uniformly on compact subsets of $\mathbb{C}\backslash(\mathbb{T} \cup \mathcal{Z}(Q))$. Equivalently,

$$\lim_{n\to\infty} \left| \frac{f(z) - R_n(z)}{H_n(z)} \right|^{\frac{1}{n}} = 1,$$

uniformly on compact subsets of $\mathbb{C}\backslash(\mathbb{T} \cup \mathcal{Z}(Q))$. As $\{G_n\}_{n\in\mathbb{N}}$ is asymptotically extremal where G_n has degree $n - d$, and the measure μ is regular,

$$\lim_{n\to\infty} |f(z) - R_n(z)|^{\frac{1}{n}} = \min\left(|z|, \frac{1}{|z|}\right),$$

uniformly on compact sets of $\mathbb{C}\backslash(\mathbb{T} \cup \mathcal{Z}(Q))$. □

Theorem 4.4. *Under the same assumptions as in the previous Theorem 4.3, we have*

$$\limsup_{n\to\infty} \|O_n - Q\|_d^{\frac{1}{n}} = \max\{|q| : q \in \mathcal{Z}(Q) \cap \mathbb{D}\},$$

for any norm $\|\cdot\|_d$ *in* $\mathbb{P}_d$.

Proof. Since $Q(z) = \prod_{j=1}^{s}(z-q_j)^{m_j}$, we consider the Hermite interpolation polynomial at the nodes q_j taking their multiplicities m_j into account.

$$O_n(z) - Q(z) = \sum_{j=1}^{s}\sum_{k=0}^{m_j-1} O_n^{(k)}(q_j)A_{j,k}(z),$$

where $A_{j,k} \in \mathbb{P}_{d-1}$ satisfies $A_{j,l}^{(k)}(q_i) = \delta_{j,i}\delta_{k,l}$ ($\delta_{j,k}$ denotes the Kronecker delta). Since all norms in a finite dimensional space $\mathbb{P}_d$ are equivalent, it suffices to show the theorem for a particular norm. In this case, since $\{A_{j,l}\}$ is a basis, we can take for $P = \sum_{j,k} a_{j,k}A_{j,k} \in \mathbb{P}_d$ the norm $\|P\|_\infty = \max_{j,k}(|a_{j,k}|)$. Thus it holds

$$\|O_n - Q\|_\infty = \max\{|O_n^{(k)}(q_j)| : 0 \le k \le m_j - 1; j = 1, \dots, s\}.$$

We know

$$\lim_{n\to\infty} |O_n(z)Q(z)(f(z) - R_n(z))|^{\frac{1}{n}} = \min\left(|z|, \frac{1}{|z|}\right)$$

uniformly on compact subsets of $\mathbb{C}\backslash\mathbb{T}$. In particular, at $z = q_i$, since

$$O_n(z)Q(z)(f(z) - R_n(z)) = Q_n(z)Q(z)\left(\Omega_\mu(z) - \frac{A_n(z)}{b_n(z)L_n(z)L_n^*(z)}\right) + O_n(z)P(z), \quad (24)$$

we obtain

$$\lim_{n\to\infty} |O_n(q_i)P(q_i)|^{\frac{1}{n}} = \min\left(|q_i|, \frac{1}{|q_i|}\right) \quad i = 1, \dots, s.$$

Therefore, since $\lim_{n\to\infty} |P(q_i)|^{\frac{1}{n}} = 1$,

$$\lim_{n\to\infty} |O_n(q_i)|^{\frac{1}{n}} = \min\left(|q_i|, \frac{1}{|q_i|}\right), \quad i = 1, 2, \dots, s.$$

Recall that $O_n = \gamma_n \hat{O}_n \hat{O}_n^*$, so that

$$\max_{i=1,\dots,s} \min\left(|q_i|, \frac{1}{|q_i|}\right) = \max_{q\in\mathcal{Z}(Q)\cap\mathbb{D}} |q|.$$

Now suppose that Q has only simple zeros, then $s = d$ and $m_j = 1$ for $j = 1, \dots, s$ and thus

$$\lim_{n\to\infty} \|O_n - Q\|_\infty^{\frac{1}{n}} = \max_{q\in\mathcal{Z}(Q)\cap\mathbb{D}} |q|$$

proves the theorem for simple zeros.

If Q has coinciding zeros we need more preparation. Suppose first that q_i is a zero of Q with multiplicity $m_i = 2$. Let $G_n = \hat{b}_n L_n$ be given by $G_n(z) = \prod_{i=1}^{\kappa_n}(z - z_{i,n})^{\nu_i}$ with $\sum_{i=1}^{\kappa_n} \nu_i = n - d$, then

$$\frac{G_n'(z)}{G_n(z)} = \sum_{i=1}^{\kappa_n} \frac{\nu_i}{z - z_{i,n}}.$$

Since $\{L_n\}_{n\in\mathbb{N}}$ is an extremal sequence, there exists an integer n_0 large enough such that for all $n \geq n_0$: $\mathcal{Z}(L_n) \subset A_r = \{z : r \leq |z| \leq 1/r\}$ and $\mathcal{Z}(Q) \cap A_r = \varnothing$. Then

$$\frac{n-d}{D(q_i, A_r)} \leq \left|\frac{G_n'(q_i)}{G_n(q_i)}\right| \leq \frac{n-d}{d(q_i, A_r)}, \quad i = 1, \ldots, \kappa_n, \quad \forall\, n \geq n_0, \tag{25}$$

where $D(q_i, A_r)$ is the diameter of the set $\{q_i\} \cup A_r$ and $d(q_i, A_r)$ is the distance from q_i to A_r. From (24) and (21)

$$\begin{aligned} O_n(z)Q(z)\left(\Omega_\mu(z) - \frac{A_n(z)}{\hat{b}_n(z)L_n(z)L_n^*(z)}\right) + O_n(z)P(z) &= \gamma_n \frac{z^{n-d}\|\hat{O}_n\hat{Q}G_n\|^2}{G_n(z)G_n^*(z)}F_n(z) \\ &= \gamma_n H_n(z)F_n(z). \end{aligned}$$

Differentiating gives

$$\begin{aligned} &\left[O_nQ\left(\Omega_\mu - \frac{A_n}{\hat{b}_nL_nL_n^*}\right)\right]'(z) + O_n'(z)P(z) + O_n(z)P'(z) = \\ &\qquad \gamma_n H_n(z)\left(F_n'(z) + \left(\frac{n-d}{z} - \frac{G_n'(z)}{G_n(z)} - \frac{(G_n^*)'(z)}{G_n^*(z)}\right)F_n(z)\right). \end{aligned}$$

In particular, we deduce,

$$\begin{aligned} &O_n'(q_i)P(q_i) + O_n(q_i)P'(q_i) = \\ &\qquad \gamma_n H_n(q_i)\left(F_n'(q_i) + \left(\frac{n-d}{q_i} - \frac{G_n'(q_i)}{G_n(q_i)} - \frac{(G_n^*)'(q_i)}{G_n^*(q_i)}\right)F_n(q_i)\right). \end{aligned}$$

Using the previous result for simple zeros of Q. i.e.,

$$\lim_{n\to\infty} H_n(q_i) = \min\left(|q_i|, \frac{1}{|q_i|}\right), \quad i = 1, \ldots, s,$$

and

$$\lim_{n\to\infty} |O_n(q_i)|^{\frac{1}{n}} = \min\left(|q_i|, \frac{1}{|q_i|}\right), \quad i = 1, \ldots, s,$$

and by (25) both $\frac{G_n'(z)}{G_n(z)}$, $\frac{G_n^*(z)}{G_n^(z)}$ and $F_n'(z)$ are absorbed by the nth root. Therefore we obtain

$$\lim_{n\to\infty} |O_n'(q_i)|^{\frac{1}{n}} = \max\left(\lim_{n\to\infty} |O_n(q_i)|^{\frac{1}{n}}, \lim_{n\to\infty} H_n(q_i)|^{\frac{1}{n}}\right) = \min\left(|q_i|, \frac{1}{|q_i|}\right).$$

This proves the result when $q_i \in \mathcal{Z}(Q)$ has multiplicity $m_i = 2$.

If the multiplicity is greater than 2, we reproduce the same strategy to obtain

$$\limsup_{n\to\infty} |O_n^{(k)}(q_i)|^{\frac{1}{n}} = \min\left(|q_i|, \frac{1}{|q_i|}\right), \quad k = 0, 1, \ldots, \quad m_i - 1, \quad i = 1, 2, \ldots, s, \tag{26}$$

since

$$\left(\frac{G_n'(z)}{G_n(z)}\right)^{(k)} = \sum_{k=1}^{\kappa_n} \frac{(-1)^k\ k!\ \nu_i}{(z-z_{i,n})^{k+1}},$$

where the superscript $^{(k)}$ refers to the kth derivative. □

Remark 4.5. *Note that if $k_n = 0$ in the last two results, we generalize the results obtained in [1] to rational perturbation of the Herglotz-Riesz transform.*

5. Conclusion

We proved that, given a regular measure of $\mathbb{T}$, the sequence of quasi-paraorthogonal polynomials whose preassigned zeros that are asymptotically uniformly distributed on $\mathbb{T}$ will be asymptotically extremal, which means that its zeros are asymptotically uniformly distributed (Theorem 2.10). This result is used to obtain an exact geometric rate of convergence of two-point partial Padé approximants to the Herglotz-Riesz transform of the measure if these quasi-paraorthogonal polynomials are used as their denominators (Theorem 3.2). With a rational perturbation of the Herglotz-Riesz transform, it is proved that the same geometric rate of convergence holds and that the poles of the perturbation attract a number of the poles of the approximant corresponding to their multiplicity (Theorem 4.3) and with geometric rate (Theorem 4.4).

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