Szegő-type quadrature formulas on an arc of the unit circle

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`http://nalag.cs.kuleuven.be/papers/ade/CAM23`

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Szegő-type quadrature on an arc of the unit circle

- Generalize:
  Gauss quadrature on an interval of $\mathbb{R}$
  Nodes = zeros of OPRL (in support)
  $\leftrightarrow$
  Szegő quadrature on an arc of $\mathbb{T}$
  Nodes = zeros of POPUC (in support?)
- Similar generalizations for Radau and Lobattao quadrature

Para-orthogonal polynomials

Theorem

- *monic OPUC:*
  $\rho_k(z) = z\rho_{k-1}(z) + \delta_k \rho_{k-1}^*(z), \quad \delta_k \in \mathbb{D}, \quad p_k^*(z) = z^k \overline{p_k(1/\overline{z})}$
  monic POPUC:
  $P_n(z) = z\rho_{n-1}(z) + \tau_n \rho_{n-1}^*(z) = \nu_n^{-1}[\rho_n(z) + \hat{\tau}_n \rho_n^*(z)],$
  $\tau_n, \hat{\tau}_n \in \mathbb{T}$, $\nu_n = 1 + \overline{\delta_n}\hat{\tau}_n$
- $\varphi_\delta(z) = \dfrac{z+\delta}{1+\overline{\delta}z}$ *then* $\tau_n = \varphi_{\delta_n}(\hat{\tau}_n) \quad \Leftrightarrow \quad \hat{\tau}_n = \varphi_{-\delta_n}(\tau_n)$
- P_n *is* τ_n*-invariant:* $P_n(z) = \tau_n P_n^*(z)$. *(to get* $\mathcal{Z}(P_n) \subset \mathbb{T}$*)*
- *orthogonality* $P_n \perp \operatorname{span}\{z, \ldots, z^{n-1}, z^n - \hat{\tau}_n\}$
  (to get max degree exactness)
- $\mathcal{Z}(P_n)$ *are simple and at least* $n-1$ *are in* $\mathring{\Gamma}$, $\Gamma = \operatorname{supp}(\mu) =$ *arc on* $\mathbb{T}$.

Purpose of this talk

Theorem

Use $\mathcal{Z}(P_n) = \{z_{k,n}\}_{k=1}^n$ *as nodes for a Szegő quadrature formula*

$$\int_\Gamma f(z)d\mu(z) \approx \sum_{k=1}^{n} \lambda_{k,n} f(z_{k,n}), \quad \Gamma = [a,b] \subset \mathbb{T}$$

is exact in $\operatorname{span}\{z^{-(n-1)}, \ldots, z^{n-1}, z^n - \frac{\omega_n}{z^n}\}$, $\omega_n = \tau_n \hat{\tau}_n$.

- How to choose $\tau_n \leftrightarrow \hat{\tau}_n$ such that all zeros are in $\operatorname{supp}(\mu) = [a,b] \subset \mathbb{T}$? $(\Rightarrow \lambda_{k,n} > 0, \quad k = 1, \ldots, n)$
- What if we use $\tau_n \leftrightarrow \hat{\tau}_n$ to fix a or b as node? (Radau)
  [Give up orthogonality $P_n \perp z^n - \hat{\tau}_n$]
- Can we fix both a and b as nodes? (Lobatto)
  [Give up orthogonality $Q_n \perp \{z, z^{n-1}\}$ but keep $Q_n \perp z^{n-1} - \hat{\tau}_{n,1} z$]

Szegő-Radau quadrature

Here the parameters τ_n and/or $\hat{\tau}_n$ are fixed by fixing a or b when $\mathrm{supp}(\mu) = [a, b]$.

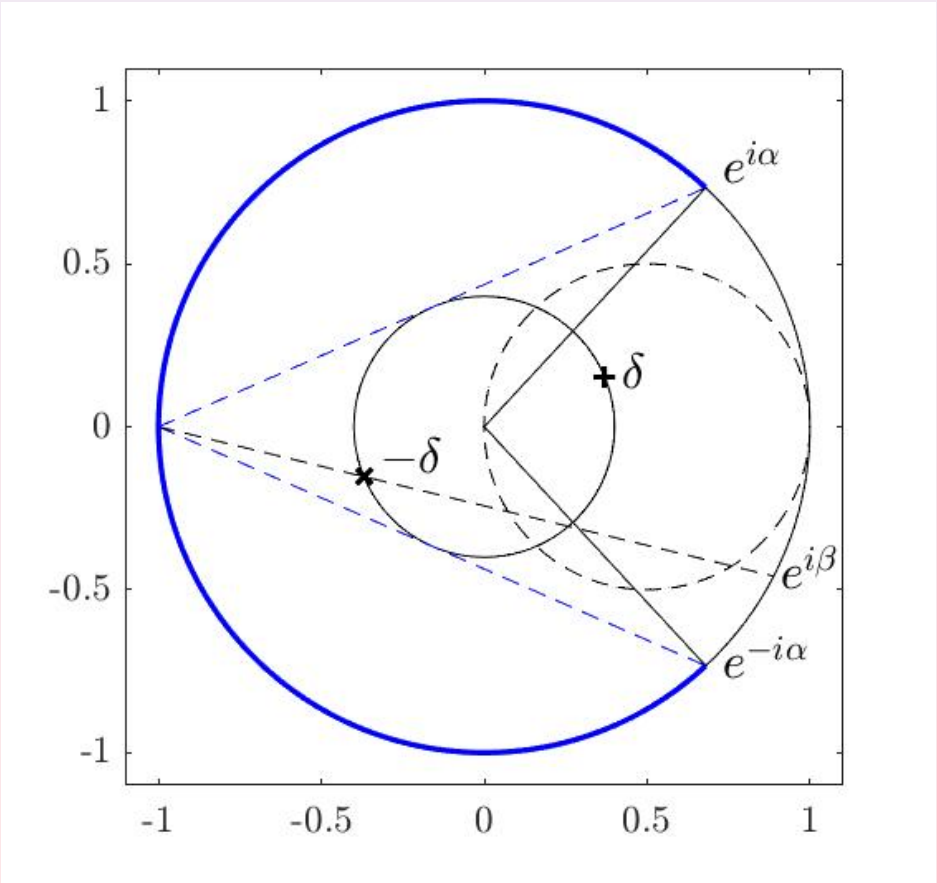
$$P_n(z) = \nu_n^{-1}[\rho_n(z) + \hat{\tau}_n \rho_n^*(z)]_{z=c} = 0 \Leftrightarrow \hat{\tau}_n = -\frac{\rho_n(c)}{\rho_n^*(c)} = -G_n(c).$$

Theorem

Let μ be a measure with $\mathrm{supp}(\mu) = [a, b] \subset \mathbb{T}$. Define the monic POPUC $P_n(z) = \nu_n^{-1}[\rho_n(z) + \hat{\tau}_n \rho_n^*(z)]$, $\hat{\tau}_n = -G_n(c)$, $c \in \{a, b\}$.
Then a Szegő-Radau quadrature exists as $I_n(f) := \sum_{n=1}^{n} \lambda_{k,n} f(z_{k,n})$ with $\lambda_{k,n} > 0$ and $z_{k,n} \in \mathcal{Z}(P_n)$, are simple and in Γ.
The quadrature is exact for all $f \in \mathrm{span}\{z^{-(n-1)}, \ldots, z^{n-1}, z^n - \frac{\hat{\tau}_n}{\overline{\tau}_n} z^{-n}\}$.
Moreover the sets $\mathcal{Z}(P_n) \setminus \{c\}$ and $\mathcal{Z}(P_{n+1}) \setminus \{c\}$ interlace (in Γ).
$\lim_{n\to\infty} I_n(f) = I(f)$ for all Riemann-Stieltjes integrable f.

Geronimus class

Standard example of measure supported on an arc:
$\rho_k(z) = z\rho_{k-1}(z) + \delta\rho^*_{k-1}(z)$, i.e., $\delta_k = \delta \in \mathbb{D}$ for all $k \in \mathbb{N}$.



$$\mu'(e^{i\theta}) = \frac{\sqrt{\sin(\frac{\theta+\alpha}{2})\sin(\frac{\theta-\alpha}{2})}}{\sin(\frac{\theta-\beta}{2})}$$

$$\sin(\tfrac{\alpha}{2}) = |\delta|, \quad \beta = 2\arg(1-\delta)$$

Mass point in $e^{i\beta} \in \Gamma^c$

if $|1-2\delta| > 1$

We assume $|1-2\delta| \leq 1$.

$$\Gamma = \mathrm{supp}(\mu) = [a, \bar{a}]$$

Szegő quadrature

Zeros and finite Blaschke products

$$P_n(z) = 0 \Leftrightarrow \rho_n(z) + \hat{\tau}_n \rho_n^*(z) = 0 \Leftrightarrow G_n(z) = -\hat{\tau}_n, \qquad G_n(z) = \frac{\rho_n(z)}{\rho_n^*(z)}$$

$$G_n(z) \quad = \quad \varphi_{\delta_n}(zG_{n-1}(z))$$

Theorem

$-\hat{\tau}_n \in G_n(\Gamma) \Rightarrow \mathcal{Z}(P_n) \subset \Gamma = \mathrm{supp}(\mu)$

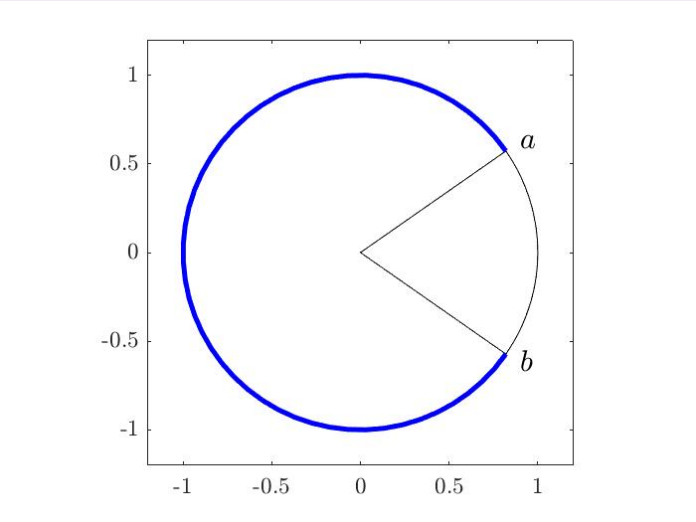
Proof. $\mathcal{Z}(P_n) = \mathcal{Z}(G_n + \hat{\tau}_n)$. Hence

$$-\hat{\tau}_n \in G_n(\Gamma) \Rightarrow \mathcal{Z}(P_n) = G_n^{-1}(-\hat{\tau}_n) \subset G_n^{-1}(G_n(\Gamma)) \subset \Gamma$$

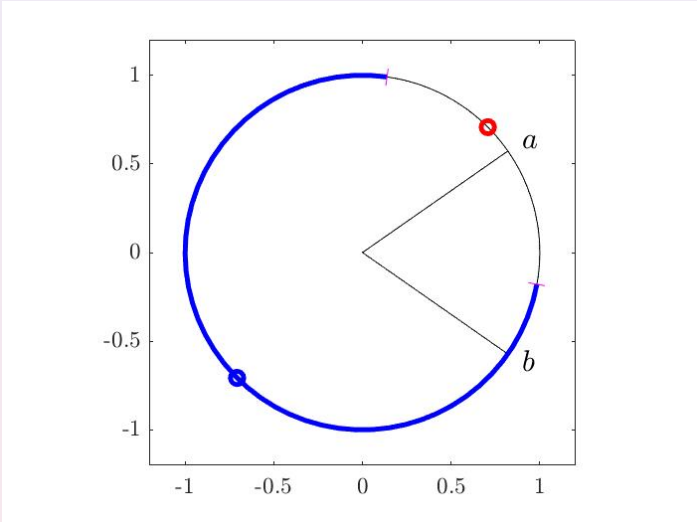
□

$-\hat{\tau}_n \in G_n(\Gamma) \Rightarrow \mathcal{Z}(P_n) \subset \Gamma = \mathrm{supp}(\mu)$

Proof: $(n = 4)$



$\xrightarrow{G_n}$



$\xrightarrow{G_n^{-1}}$

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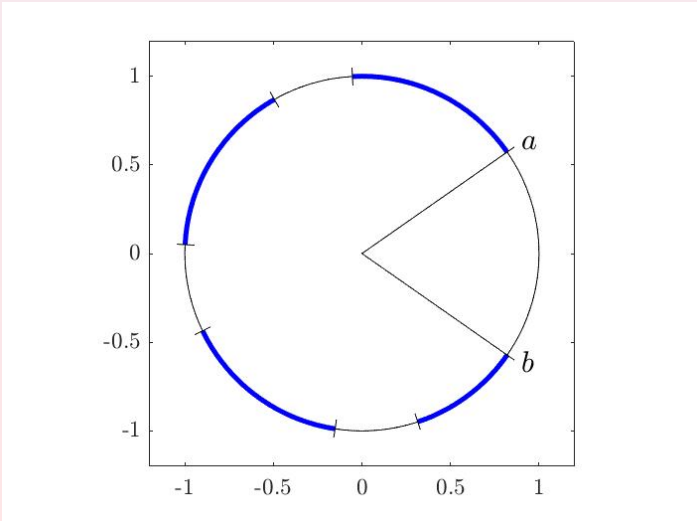
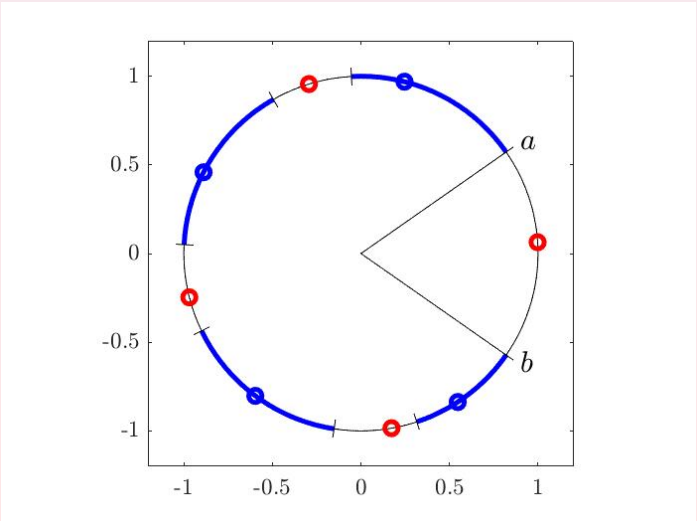


$\Rightarrow$



Szegő quadrature

Question

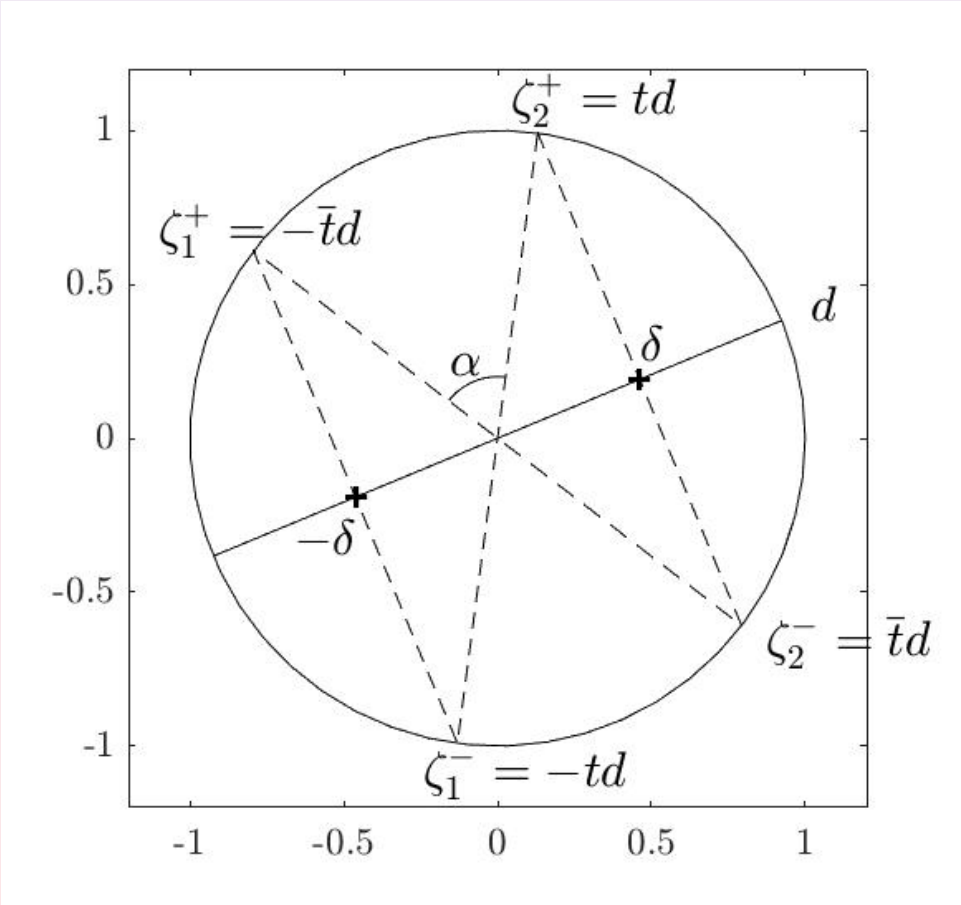
Can we find a $\hat{\tau}$ independent of n such that in Geronimus case $-\hat{\tau} \in G_n(\Gamma)$ for all $n \in \mathbb{N}$?

Therefore we need a closer look ar the Moebius transform

$$\varphi_\delta(z) = \frac{z+\delta}{1+\overline{\delta}z}$$

The Moebius transform

$d = \delta/|\delta|$; $|\delta| = \mathrm{Re}\,(t)$; $\sin(\alpha/2) = |\delta|$, $a = e^{i\alpha}$; $\varphi_\delta(z) = \frac{\delta+z}{1+\bar{\delta}z}$;



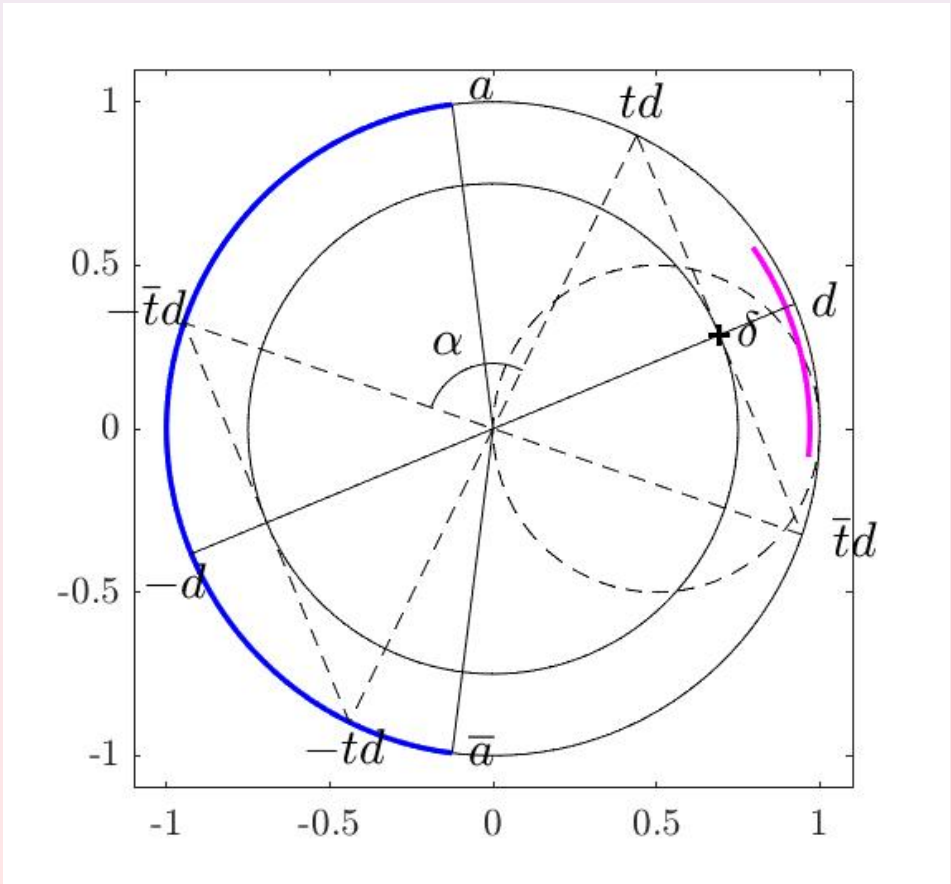
- $\varphi_\delta(\pm d) = \pm d$
  d attracts; $-d$ repels
- $\varphi_\delta(\zeta_1^\pm) = \zeta_2^\pm$
- $\varphi_\delta(\zeta_1^-, \zeta_1^+) = (\zeta_2^-, \zeta_2^+)$
- $|\varphi'_\delta(\zeta_1^\pm)| = 1$
- $|\varphi'_\delta(z)| < 1, \quad \forall z \in (\zeta_1^-, \zeta_1^+)$
- $\forall z \in (\zeta_2^-, \zeta_2^+)$:
  $\zeta_2^- \prec \varphi_\delta(\bar{a}z) \prec z \prec \varphi_\delta(az) \prec \zeta_2^+$

Szegő quadrature (Geronimus case)

Theorem (Geronimus case)

Set $d = \delta/|\delta|$, then $-d \in G_n(\Gamma)$ for all $n \in \mathbb{N}$.

Proof. $\Gamma = [a, \bar{a}]$, $\Gamma^c = (\bar{a}, a)$,



$1 \in (\bar{t}, t) \Rightarrow -d \in -(\bar{t}, t)d$

For $n = 1$: $G_1(z) = \frac{z+\delta}{1+\bar{\delta}z} = \varphi_\delta(z)$

$|1 - 2\delta| < 1 \Rightarrow a \in (t, -\bar{t})d$
$\Rightarrow (-\bar{t}, t)d \subset \Gamma \subset (t, \bar{t})d$
$\Rightarrow G_1(\Gamma^c) \subset (\bar{t}, t)d$
$\Rightarrow -d \in G_1(\Gamma)$

For $n > 1$: $G_{n+1}(z) = \varphi_\delta(zG_n(z))$
$G_n(\Gamma^c) \subset G_{n+1}(\Gamma^c) \to (t, \bar{t})d$
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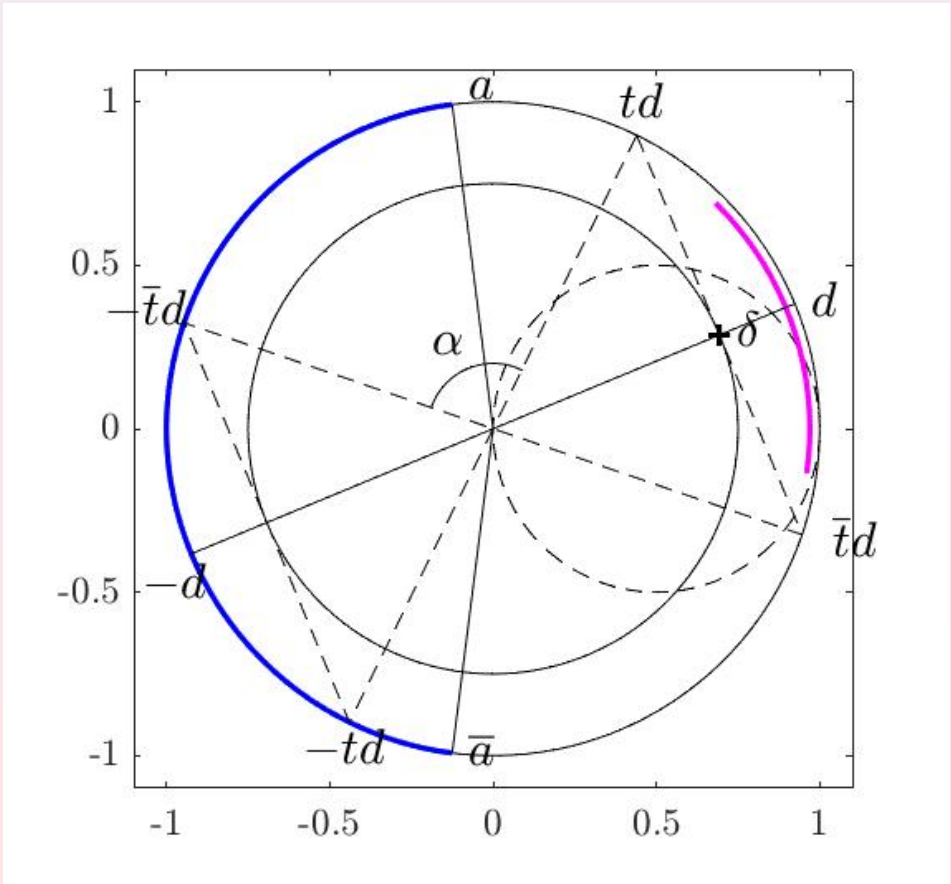

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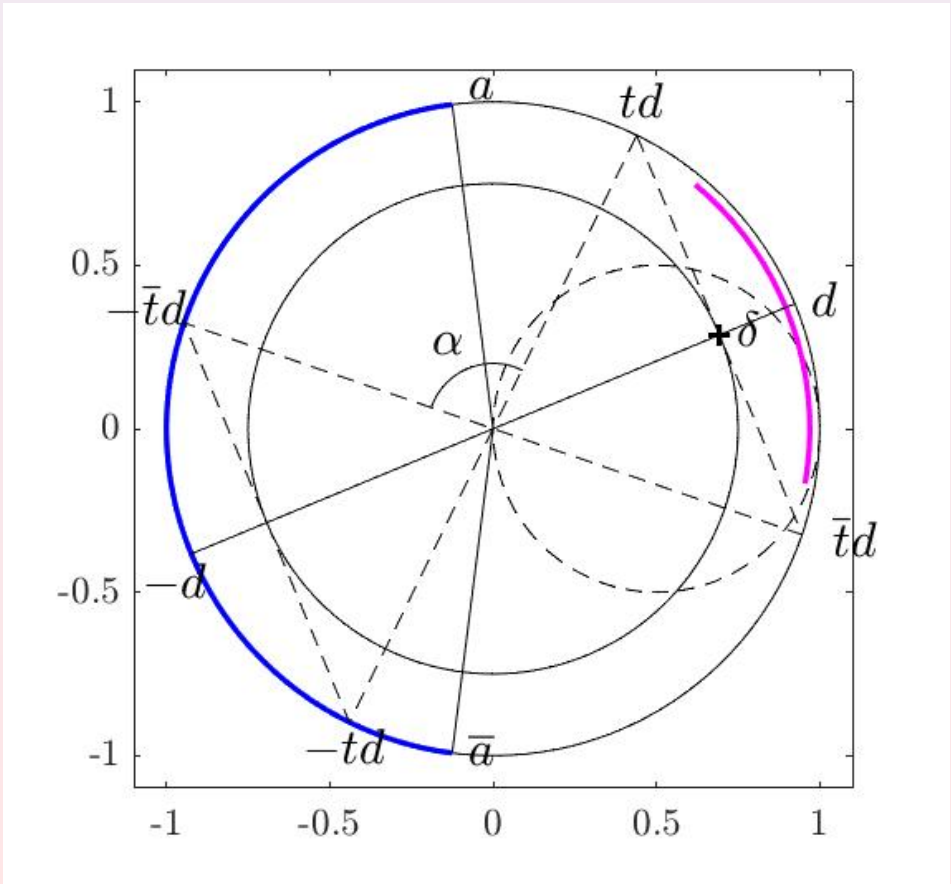

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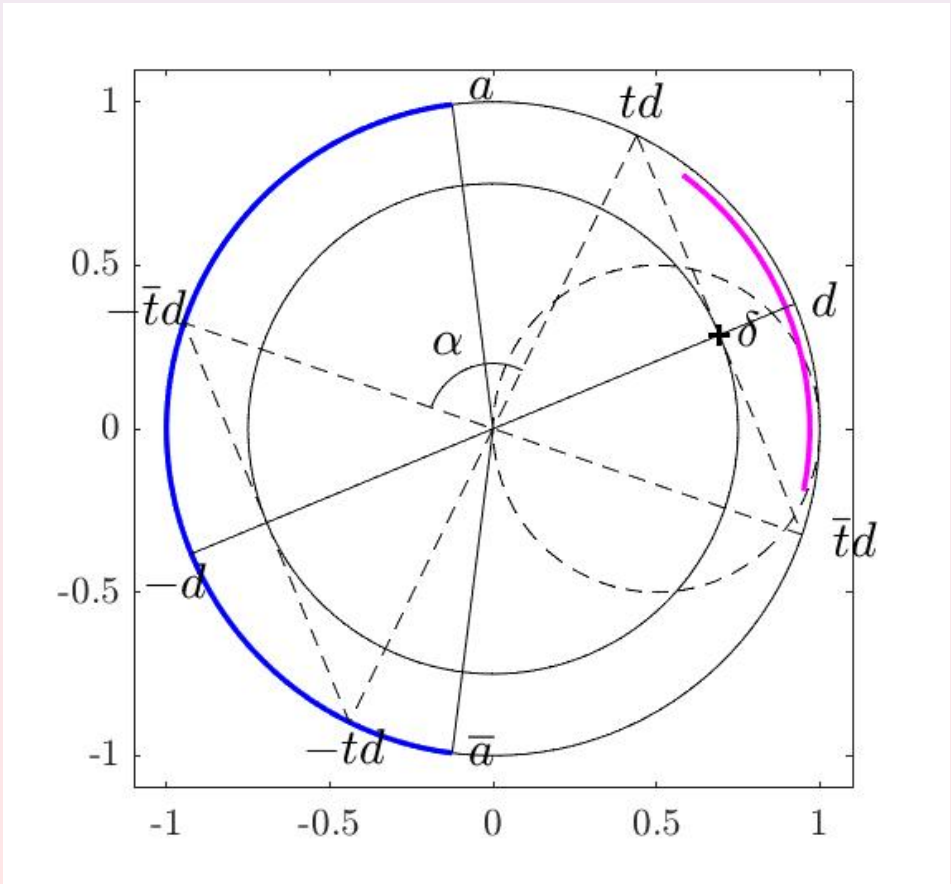

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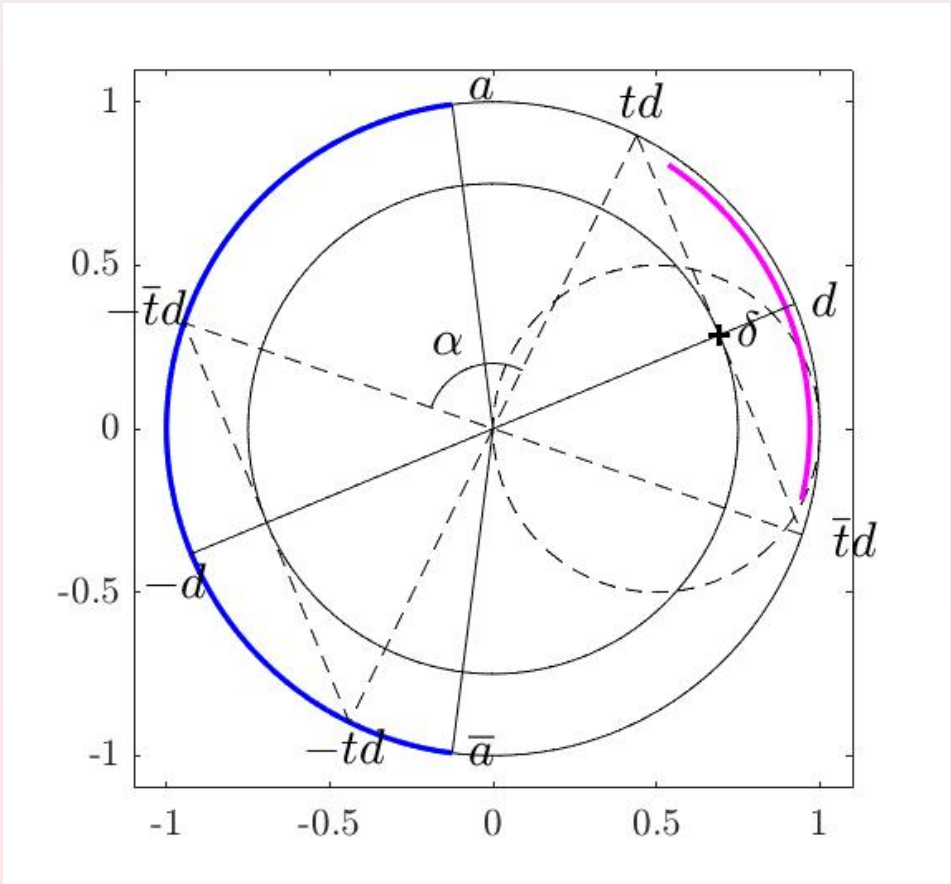

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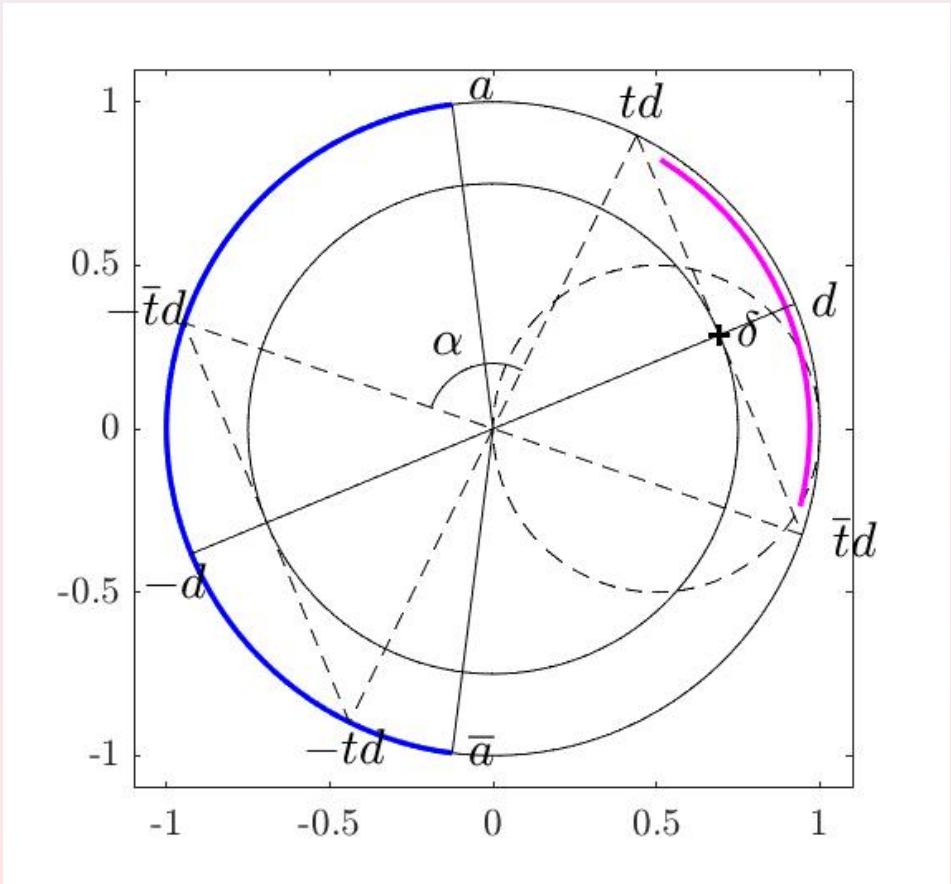

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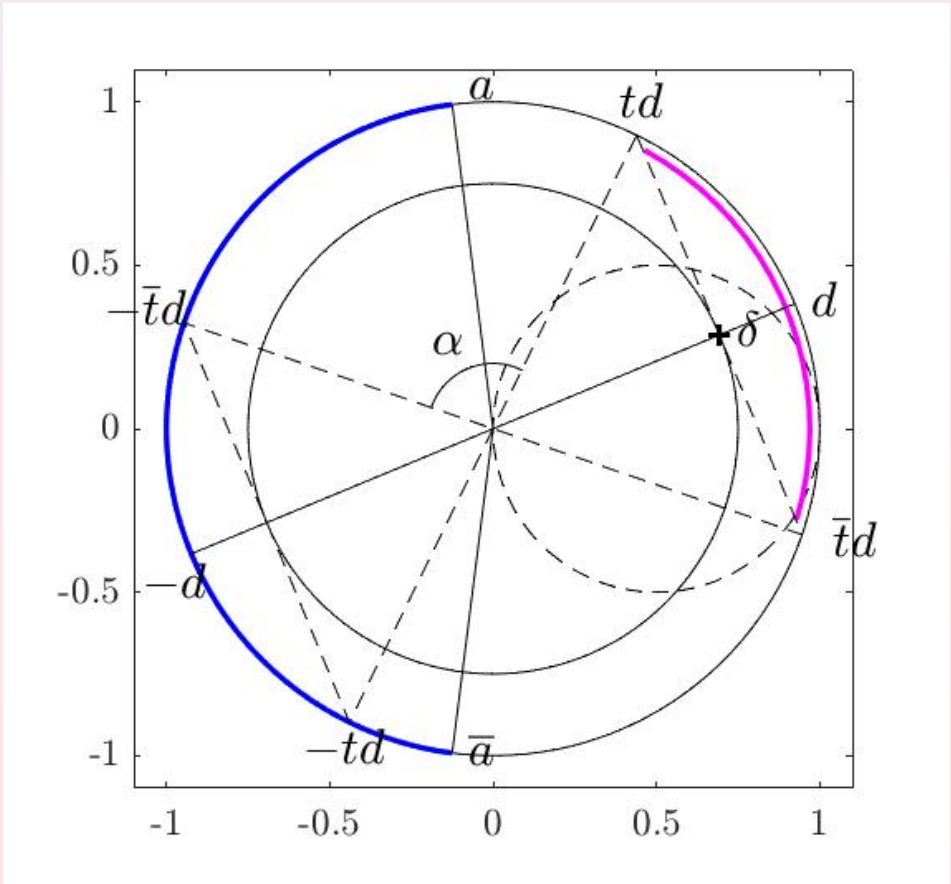

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Szegő quadrature

Theorem (Geronimus case)

Let μ be a Geronumus measure with parameter δ such that $|1-2\delta|<1$ and let $d=\delta/|\delta|$. The monic OPUC are ρ_n and define the POPUC $P_n(z)=z\rho_{n-1}(z)+d\rho_{n-1}^*(z)=\nu_n^{-1}[\rho_n(z)+d\rho_n^*(z)]$.
Then a Szegő quadrature exists as

$$I(f):=\int_\Gamma f(z)d\mu(z)\approx I_n(f):=\sum_{n=1}^{n}\lambda_{k,n}f(z_{k,n})$$

with $\lambda_{k,n}>0$ and $z_{k,n}\in\mathcal{Z}(P_n)$, are simple and in Γ.
The quadrature is exact for all $f\in\operatorname{span}\{z^{-(n-1)},\ldots,z^{n-1},z^n-d^2z^{-n}\}$.
Moreover the sets $\mathcal{Z}(P_n)$ and $\mathcal{Z}(P_{n+1})$ interlace (in Γ).
$\lim_{n\to\infty}I_n(f)=I(f)$ for all Riemann-Stieltjes integrable f.

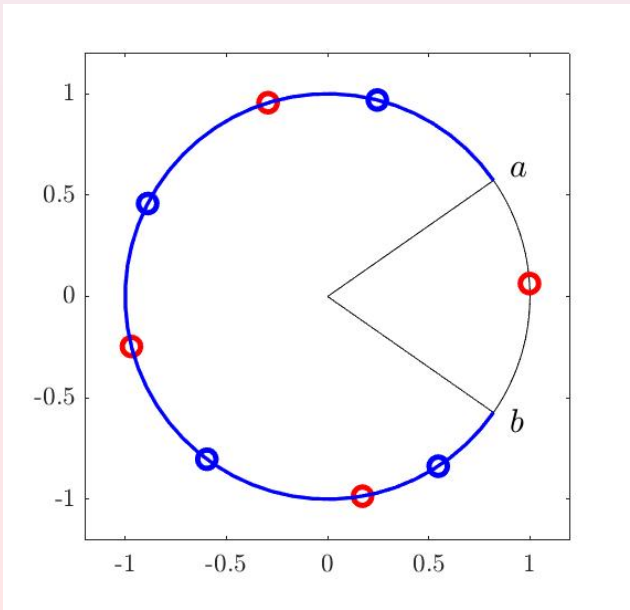
Proof. Simple and in Γ: see above;
positive and exact: see e.g. Díaz Mendoza et al. *J. Math. Anal. Appl*, 2017.

Szegő quadrature

Interlacing:

$$\frac{1+zG_n(z)\overline{d}}{(1-z)(1+G_n(z)\overline{d})} = \frac{1}{2}\left(\frac{1-G_n(z)\overline{d}}{1+G_n(z)\overline{d}} + \frac{1+z}{1-z}\right),$$

RHS = rational Carathéodory functions $\Rightarrow$
zeros of numerator and denominator in LHS interlace.
Numerator zeros are $\mathcal{Z}(G_{n+1})$, denominator zeros are $\mathcal{Z}(G_n) \cup \{1\}$
Since $1 \in G_n(\Gamma^c)$ and $\mathcal{Z}(G_{n+1}) \cup \mathcal{Z}(G_n) \subset \Gamma$, we have interlacing.



Convergence proof similar to Gauss as in Szegő's book. □

Szegő-Lobatto quadrature

Fixing 2 nodes requires a downgrade from POPUC to quasi-POPUC
$Q_n \perp \operatorname{span}\{z^2, \ldots, z^{n-2}, z^{n-1} - \hat{\tau}_n z\}$.

$$\begin{aligned} Q_n(z) &= z(z-\eta_n)\rho_{n-2}(z) + \tau_n(1-\overline{\eta}_n z)\rho^*_{n-2}(z) \\ &= \nu_n^{-1}[(z-\xi_n)\rho_{n-1}(z) + \hat{\tau}_n(1-\overline{\xi}_n z)\rho^*_{n-1}(z)] \\ \hat{\tau}_n &= -\frac{\delta_n+\tau_n\overline{\eta}_n}{\eta_n+\tau_n\overline{\delta}_n}, \quad \xi_n = \frac{\eta_n\delta_n+\tau_n}{\delta_n+\tau_n\overline{\eta}_n} \end{aligned}$$

$\eta_n \in \mathbb{D}$ is defined by 2 linear equations $Q_n(a) = 0 = Q_n(b)$.
$Q_n(z) = 0 \Leftrightarrow \varphi_{-\eta}(z) z G_{n-2}(z) = -\tau_n$, where $G_{n-2}(z) = \frac{\rho_{n-2}(z)}{\rho^*_{n-2}(z)}$

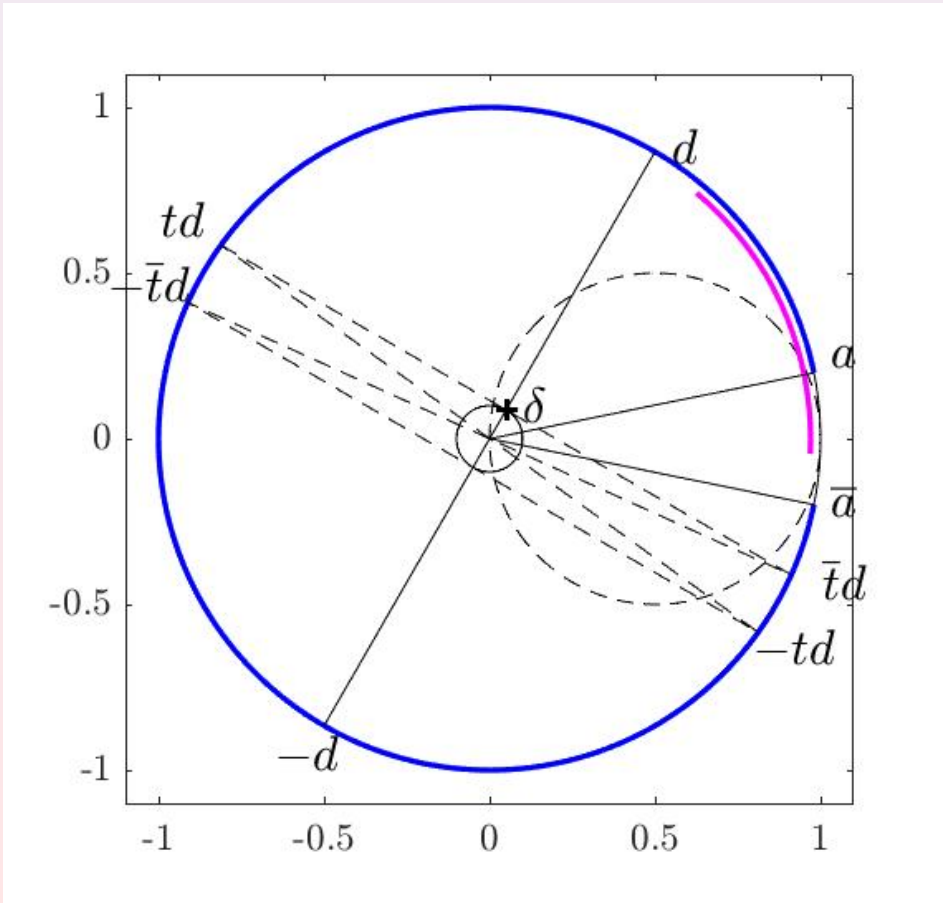
Question

Will the same choice $\tau_n = d$ guarantee
in the Geronimus case that $\mathcal{Z}(Q_n) \subset \Gamma$?

Szegő-Lobatto quadrature

$$Q_n(a) = Q_n(b) = 0 \Rightarrow \eta_n = \eta_n(\tau_n) \in \mathbb{T}_n = \{z : C_n + \tau_n R_n\},$$

$$\left.\begin{array}{l} \tau_n = G_{n-2}(b) \Rightarrow \eta_n = a \\ \tau_n = G_{n-2}(a) \Rightarrow \eta_n = b \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \mathbb{T}_n \cap \mathbb{T} = \{a, b\} \\ \eta_n \in \mathbb{D} \ \text{ iff } \ \tau_n \in G_{n-2}(\Gamma^c) \end{array}\right.$$



$-\tau_n = d \in G_n(\Gamma), \quad \forall n \in \mathbb{N}$ but
not $\tau_n = d \in G_n(\Gamma^c), \quad \forall n \geq 3$
but $\lim_{n\to\infty} G_n(\Gamma^c) = (\bar{t}, t)d$
Hence $\exists n_0 : d \in G_n(\Gamma^c), \quad \forall n \geq n_0$

But if $|\delta| > 0.5$ then $n_0 = 3$.

Szegő-Lobatto quadrature

For $\tau_n = d$:

$$\begin{aligned} Q_n(z) = 0 \quad &\Leftrightarrow \quad \varphi_{-\eta}(z) z G(z) \overline{d} + 1 = 0 \\ P_{n-1}(z) = 0 \quad &\Leftrightarrow \quad z G(z) \overline{d} + 1 = 0 \end{aligned}$$

We define the rational function

$$\frac{1 + \varphi_{-\eta}(z) \overline{d} z G(z)}{(1 - \varphi_{-\eta}(z))(1 + \overline{d} z G(z))} = \frac{1}{2} \left(\frac{1 - \overline{d} z G(z)}{1 + \overline{d} z G(z)} + \frac{1 + \varphi_{-\eta}(z)}{1 - \varphi_{-\eta}(z)} \right).$$

RHS is Carathéodory $\Rightarrow$ LHS $\mathcal{Z}$(numerator) and $\mathcal{Z}$(denominator) interlace
$1 = \varphi_{-\eta}(z) \Leftrightarrow z = \varphi_{\eta}(1) = \frac{1+\eta}{1+\overline{\eta}} = \tilde{\eta}$.

$$LHS \ \frac{Q_n(z)}{(z - \tilde{\eta}') P_{n-1}(z)} = \frac{(z-a)(z-\overline{a}) \tilde{P}_{n-2}(z)}{(z - \tilde{\eta}) P_{n-1}(z)}$$

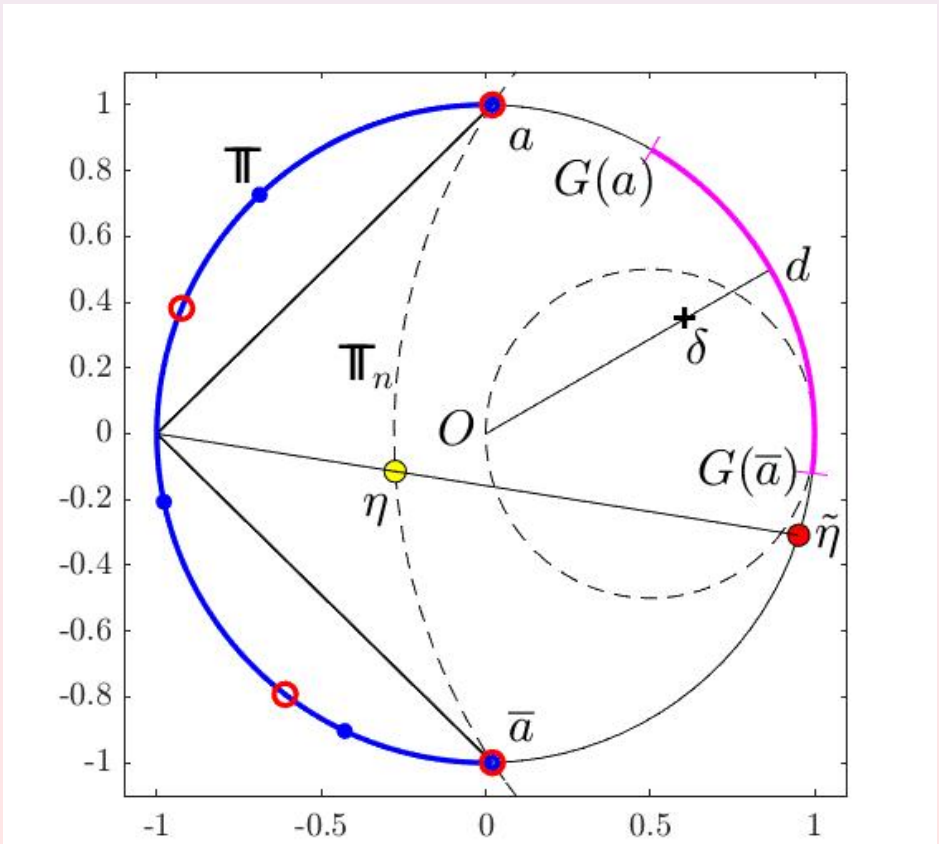
$\tilde{P}_{n-2}$ is paraorthogonals for $d\tilde{\mu}(z) = (z-a)(z-\overline{a}) \overline{z} d\mu(z)$

Szegő-Lobatto quadrature

The zeros of numerator and denominator interlace

$$\frac{(z-a)(z-\overline{a})\tilde{P}_{n-2}(z)}{(z-\tilde{\eta})P_{n-1}(z)}$$

as POPUC $\tilde{P}_{n-2}$ and P_{n-1} can have at most 1 zero in (b,a).
Hence if $\tilde{\eta} \in (b,a)$ then $\mathcal{Z}(\tilde{P}_{n-2})$ and $\mathcal{Z}(P_{n-1})$ interlace in $\mathring{\Gamma}$.



$$\tilde{\eta} = \tfrac{1+\eta}{1+\overline{\eta}} = \mathbb{T} \cap L_\eta$$

$$L_\eta = (-1)(1-p) + \eta p,$$

$$p = \tfrac{2(1+\mathrm{Re}\,(\eta))}{|1+\eta|^2} \in \mathbb{R}$$

note

$\eta \in \mathbb{D} \Rightarrow \tilde{\eta} \in (b,a)$ is automatic

Szegő-Lobatto

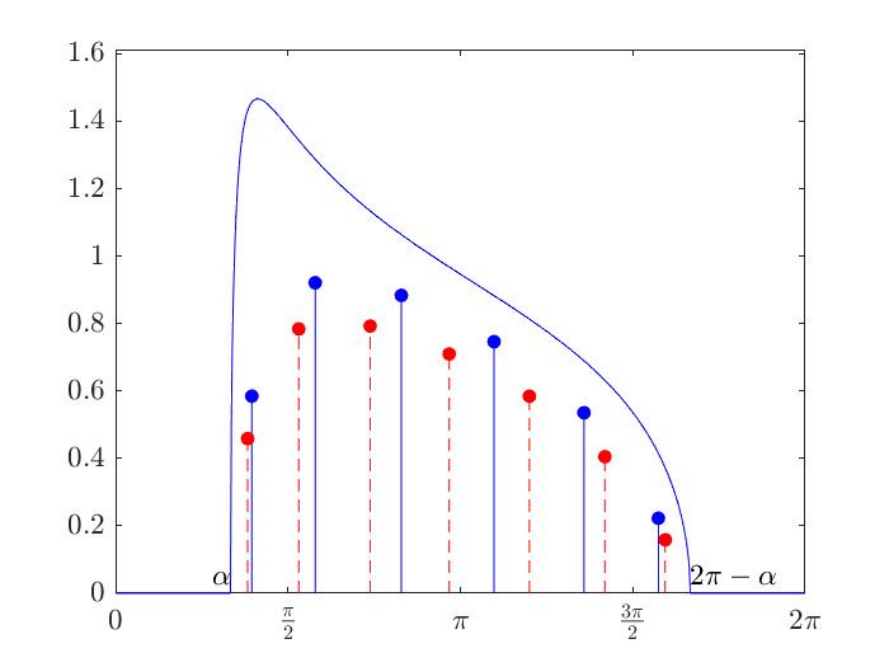
Theorem (Geronimus case)

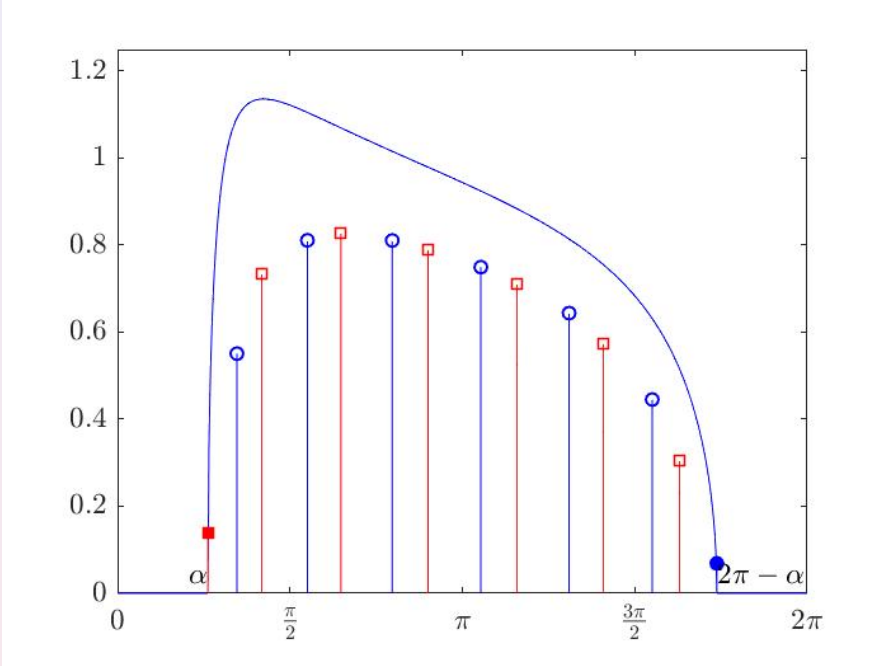
Let μ be a Geronumus measure with parameter δ such that $|1-2\delta|<1$ and let $d=\delta/|\delta|$. $Q_n(z)=z(z-\eta)\rho_{n-2}(z)+d(1-\overline{\eta}z)\rho^*_{n-2}(z)$.
Then for $n\geq n_0$, a Szegő-Lobatto quadrature exists as

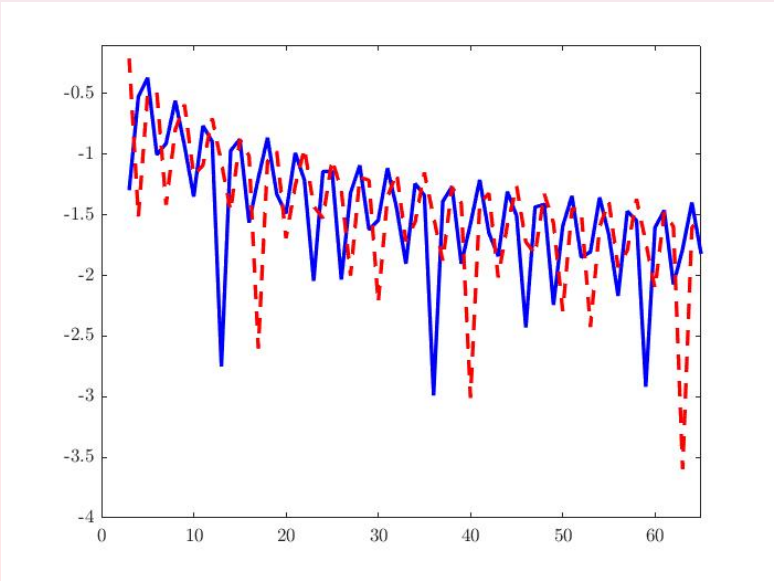
$$I(f):=\int_\Gamma f(z)d\mu(z)\approx I_n(f):=\sum_{n=1}^{n}\lambda_{k,n}f(z_{k,n})$$

with $\lambda_{k,n}>0$ and $z_{k,n}\in\mathcal{Z}(Q_n)$, are simple and in Γ.
If $|\delta|>0.5$, $n_0=3$.
There is some $\omega_n\in\mathbb{T}$ such that the quadrature is exact for all $f\in\operatorname{span}\{z^{-(n-2)},\ldots,z^{n-2},z^{n-1}-\omega_n z^{-(n-1)}\}$.
Moreover the sets $\mathcal{Z}(Q_n)\setminus\{a,\overline{a}\}$ and $\mathcal{Z}(Q_{n+1})\setminus\{a,\overline{a}\}$ interlace (in $\mathring{\Gamma}$).
$\lim_{n\to\infty}I_n(f)=I(f)$ for all Riemann-Stieltjes integrable f.

Numerical examples







$\nwarrow$: Szegő quadrature

$\delta = 0.5e^{-i\pi/6}$, $n = 6$, $n = 7$

$\nearrow$: Szegő-Radau quadrature

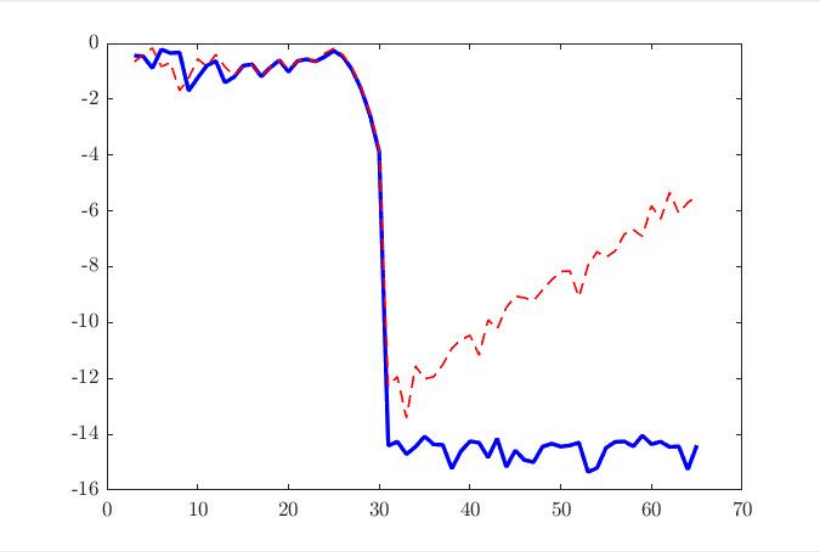
$\delta = 0.4e^{-i\pi/8}$, $c = \overline{a}$, $c = a$

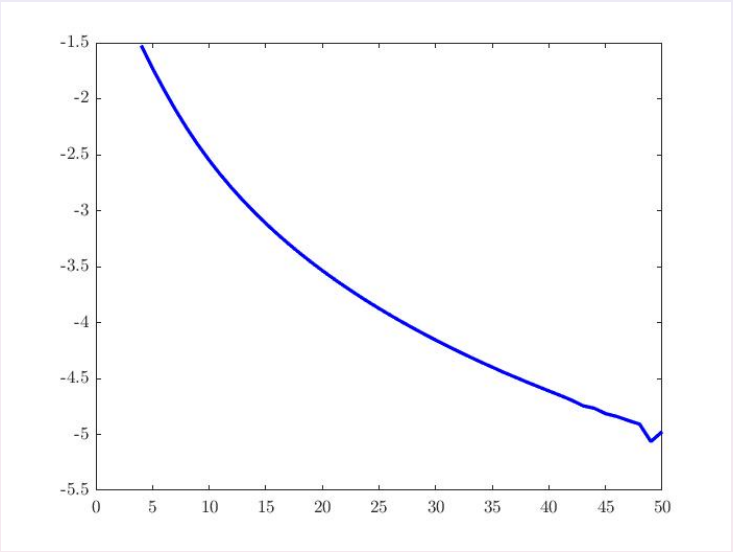
$\leftarrow$: $\int \sqrt{z-a}\, d\mu(z)$, $\log_{10} \left| \frac{I_n - I}{I} \right|$

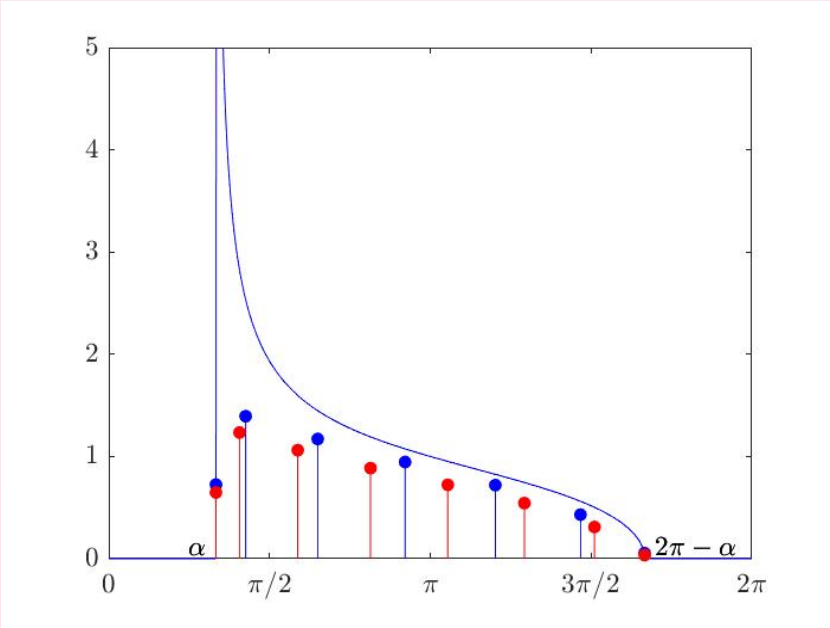
$n = 3, 4, \ldots, 65$

Szegő, Szegő-Radau

Numerical examples







↖: Szegő-Radau cvg

$f(z) = z^{30} + z^{-30}$

↗: Szegő-Lobatto cvg

$$f = \sqrt{(z-a)(z-\bar{a})}$$

$\delta = 0.6e^{i\pi/5}$, $n = 4, \ldots, 50$

←: Szegő-Lobatto, $\delta = 0.5e^{-i\pi/3}$

$n = 7$, $n = 8$ $(|1 - 2\delta| = 1)$







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