Generalizations of para-orthogonal polynomials and quadrature

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Celebrating the retirement of Bernd Kirstein and Bernd Fritzsche

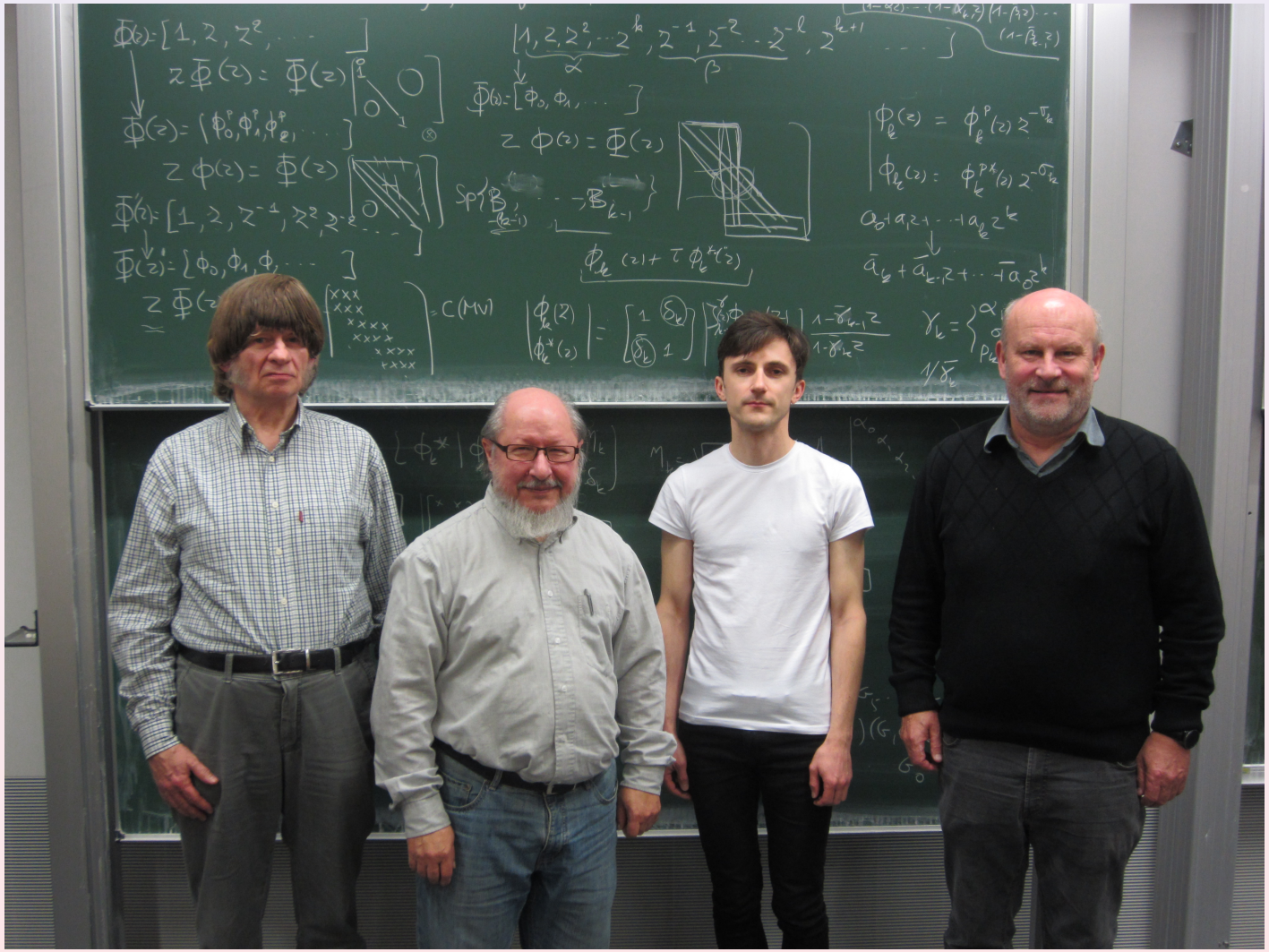
`http://nalag.cs.kuleuven.be/papers/ade/LEIPZIG20`

More than 20 years



1996

More than 20 years



2016

Motivation

Spectral Duality $\mathbb{R}$ vs $\mathbb{T}$

$\mathbb{R}$	$\mathbb{T}$
$\{P_n\}$ OPRL $P_n \perp \mathbb{P}_{n-1}$ zero on $\mathbb{R}$	$\{\rho_n\}$ OPUC $\rho_n \perp \mathbb{P}_{n-1}$ zeros in $\mathbb{D}$
$\{P_n\}$ OPRL $P_n \perp \mathbb{P}_{n-1}$ zeros on $\mathbb{R}$	$\{B_n\}$ Para-OPUC $B_n \perp \mathbb{B}_{n-1}$ zeros on $\mathbb{T}$

Motivation

Spectral Duality $\mathbb{R}$ vs $\mathbb{T}$

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$\{P_n\}$ OPRL $P_n \perp \mathbb{P}_{n-1}$ zero on $\mathbb{R}$	$\{\rho_n\}$ OPUC $\rho_n \perp \mathbb{P}_{n-1}$ zeros in $\mathbb{D}$
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Motivation (symmetry of the zeros)

$$\{P_n(z) = \textstyle\sum_{k=0}^{n} a_{n,k} z^k\} \textbf{ OPRL w.r.t. } \nu$$

- Zeros P_n on $\mathbb{R}$ or mirror pairs $(\alpha_k, \overline{\alpha_k}) \Rightarrow P_n$ is real (real coefficients)
- P_n real & $P_n \perp \mathbb{P}_{n-1}$
  $\Rightarrow$ zeros are on $\mathbb{R}$, simple, $n-1$ in open interval containing $\operatorname{supp} \nu$
- As $n \to \infty$, the zeros fill $\operatorname{supp} \nu$,
  Gauss QF $I_n(f) \overset{*}{\to} I(f) = \int f(x) \mathrm{d}\nu(x)$

These properties do not hold for OPUC. Need ParaOPUC!

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These properties do not hold for OPUC. Need ParaOPUC!

Motivation (symmetry of the zeros)

$$\{B_n(z) = \sum_{k=0}^{n} b_{n,k} z^k\} \textbf{ ParaOPUC w.r.t. } \mu$$

- Zeros B_n on $\mathbb{T}$ or mirror pairs $(\alpha_k, \frac{1}{\overline{\alpha_k}}) \Rightarrow B_n$ is (τ-)invariant, $\tau \in \mathbb{T}$

$$B_n(z) = \tau B_n^*(z) := \tau z^n \overline{B_n(\tfrac{1}{\overline{z}})} \Leftrightarrow b_{n,k} = \tau \overline{b_{n,n-k}}, \quad k = 0 : n$$

- B_n para-orthogonal to $\mathbb{B}_{n-1} \Rightarrow$
  all zeros on $\mathbb{T}$, simple, $n-1$ in open arc containing $\operatorname{supp} \mu$
- As $n \to \infty$, the zeros fill $\operatorname{supp} \mu$,
  Szegő QF $I_n(f) \overset{*}{\rightharpoonup} I(f) = \int f(x) \mathrm{d}\mu(x)$

What is Para-orthogonality and where does it come from?

P_n OPRL

$$\langle P_n, z^k \rangle = r_k, \quad k = 0:n: \left\{ \begin{array}{ll} \mathcal{H}_n P_n = R & \Rightarrow \mathcal{H}_n \overline{P_n} = \overline{R} \\ \mathcal{H}_n \text{ Hankel} & \overline{P_n} = P_n \end{array} \right\} \Rightarrow R = \overline{R}$$

$$P_n \perp \mathbb{P}_{n-1}: \mathcal{H}_n P_n = [\underbrace{0, \ldots, 0}_{z^0, \ldots, z^{n-1}}, a_n]^T, \; a_n = \langle P_n, z^n \rangle = \|P_n\|^2$$

B_n POPUC

$$\langle B_n, z^k \rangle = r_k, \quad k = 0:n: \left\{ \begin{array}{ll} \mathcal{T}_n B_n = R & \Rightarrow \mathcal{T}_n \tau B_n^* = \tau R^* \\ \mathcal{T}_n \text{ Toeplitz} & B_n = \tau B_n^* \end{array} \right\} \Rightarrow R = \tau R^*$$

$$B_n \perp \mathbb{B}_{n-1}: \mathcal{T}_n B_n \underset{n-1}{=} [\tau \overline{a_n}, \underbrace{0, \ldots, 0}_{z^1, \ldots, z^{n-1}}, a_n]^T, \; a_n = \langle B_n, z^n \rangle \neq 0$$

$$\text{and } \langle B_n, z^n - \underbrace{\tau \tfrac{\overline{a_n}}{a_n}}_{\hat{\tau}} \rangle = a_n - \overline{\tau \tfrac{\overline{a_n}}{a_n}} \tau \overline{a_n} \underset{0}{=} 0$$

$$B_n \perp \mathbb{B}_{n-1} = \text{span}\{z^1, \ldots, z^{n-1}, z^n - \hat{\tau} z^0\} \qquad \tau \underset{1}{=} \hat{\tau} \Leftrightarrow a_n \in \mathbb{R}$$

What is Para-orthogonality and where does it come from?

P_n OPRL

$$\langle P_n, z^k \rangle = r_k, \quad k = 0:n: \left\{ \begin{array}{ll} \mathcal{H}_n P_n = R & \Rightarrow \mathcal{H}_n \overline{P_n} = \overline{R} \\ \mathcal{H}_n \text{ Hankel} & \overline{P_n} = P_n \end{array} \right\} \Rightarrow R = \overline{R}$$

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B_n POPUC

$$\langle B_n, z^k \rangle = r_k, \quad k = 0:n: \left\{ \begin{array}{ll} \mathcal{T}_n B_n = R & \Rightarrow \mathcal{T}_n \tau B_n^* = \tau R^* \\ \mathcal{T}_n \text{ Toeplitz} & B_n = \tau B_n^* \end{array} \right\} \Rightarrow R = \tau R^*$$

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and $\langle B_n, z^n - \underbrace{\tau \frac{\overline{a_n}}{a_n}}_{\hat{\tau}} \rangle = a_n - \overline{\tau \frac{\overline{a_n}}{a_n}} \tau \overline{a_n} \underset{0}{=} 0$

$$B_n \perp \mathbb{B}_{n-1} = \operatorname{span}\{z^1, \ldots, z^{n-1}, z^n - \hat{\tau} z^0\} \qquad \tau \underset{1}{=} \hat{\tau} \Leftrightarrow a_n \in \mathbb{R}$$

Characterization of $B_{n,1}$

Theorem

For $n \geq 2$ and $\tau \in \mathbb{T}$ there is a unique τ-invariant, monic $B_{n,1} \in \mathbb{P}_n \setminus \mathbb{P}_{n-1}$ orthogonal to $\mathbb{B}_{n,1} = \operatorname{span}\{z, \ldots, z^{n-1}\}$. It is given by

$$B_{n,1}(z) = z\rho_{n-1}(z) + \tau\rho^*_{n-1}(z), \quad \tau = B_{n,1}(0).$$

It is also orthogonal to $z^n - \hat{\tau}$, $\hat{\tau} = \tau\frac{\overline{a_n}}{a_n}$, $a_n = \langle B_{n,1}, z^n \rangle$.
Its zeros are simple and on $\mathbb{T}$ and $n-1$ are in $\mathring{A}(\mu)$
($\mathcal{A}(\mu) = arc = \operatorname{supp}\mu$)

W.B. Jones, O. Njåstad, W.J. Thron, *Moment theory, orthogonal polynomials, quadrature and continued fractions associated with the unit circle*, Bull. Lonon Math, Soc, **21** (1989) 113-152.

Exploit $\tau \in \mathbb{T}$

Note $\tau \in \mathbb{T}$ is still free. It can be used to

- fix a zero somewhere on $\mathbb{T}$
- satisfy an extra orthogonality condition to $z^n - \tau z^0$.

Characterization of $B_{n,0}$

Theorem

For $n \geq 2$ and $\tau \in \mathbb{T}$ there is a unique τ-invariant $B_{n,0} \in \mathbb{P}_n \setminus \mathbb{P}_{n-1}$ orthogonal to $\mathbb{B}_{n,0} = \operatorname{span}\{z, \ldots, z^{n-1}, z^n - \tau\}$ with $\langle B_{n,0}, z^n \rangle = 1$.
It is given by

$$B_{n,0} = \nu_{n,0}[\rho_n(z) + \tau \rho_n^*(z)], \quad \nu_{n,0} = \frac{\det \mathcal{T}_{n-1}}{\det \mathcal{T}_n} > 0$$

Its zeros are simple and on $\mathbb{T}$, ($n-1$ are in $\mathring{A}(\mu)$)

O. Njåstad, J.C. Santos-León, *Domain of validity of Szegő quadrature formulas*, J. Comput. Appl. Math., **202**(2) (2007), 440-449.

Quasi-para-orthogonal polynomials

Looking for
τ-invariant polynomials $B_{n,d} \in \mathbb{B}_{n,d}^{\perp}$ with $\mathbb{B}_{n,d}$ of dimension $n-d$
$B_{n,d}$ QPOPUC of degree n and order $d = 0, : n-1$.

$$
\begin{aligned}
B_{n,0} &\perp \mathbb{B}_{n,0} = \operatorname{span}\{z, \ldots, z^{n-1}, z^n - \tau\} \\
B_{n,1} &\perp \mathbb{B}_{n,1} = \operatorname{span}\{z, \ldots, z^{n-1}\} \quad \cup \operatorname{span}\{z^n - \hat{\tau}\} \\
\\
B_{n,2} &\perp \mathbb{B}_{n,2} = \operatorname{span}\{z^2, \ldots, z^{n-2}, z^{n-1} - \tau z\} \\
B_{n,3} &\perp \mathbb{B}_{n,3} = \operatorname{span}\{z^2, \ldots, z^{n-2}\} \quad \cup \operatorname{span}\{z^{n-1} - \hat{\tau} z\} \\
&\vdots \\
B_{n,2\ell} &\perp \mathbb{B}_{n,2\ell} = \operatorname{span}\{z^{\ell+1}, \ldots, z^{n-\ell-1}, z^{n-\ell} - \tau z^{\ell}\} \\
B_{n,2\ell+1} &\perp \mathbb{B}_{n,2\ell+1} = \operatorname{span}\{z^{\ell+1}, \ldots, z^{n-\ell-1}\} \quad \cup \operatorname{span}\{z^{n-\ell} - \hat{\tau} z^{\ell}\} \\
&\vdots \\
B_{n,n-1}: \quad B_{2s+1,2s} &\perp \mathbb{B}_{n,n-1} = \operatorname{span}\{z^{s+1} - \tau z^s\} \\
B_{2s+2,2s+1} &\perp \mathbb{B}_{n,n-1} = \operatorname{span}\{z^{s+1}\}
\end{aligned}
$$

Quasi-para-orthogonal polynomials

Looking for given $\tau_0, \tau_1, \tau_2, \ldots, \tau_s, \quad s = \lfloor \frac{n-1}{2} \rfloor$
τ-invariant polynomials $B_{n,d} \in \mathbb{B}_{n,d}^{\perp}$ with $\mathbb{B}_{n,d}$ of dimension $n-d$
$B_{n,d}$ QPOPUC of degree n and order $d = 0 : n-1$.

$$
\begin{aligned}
B_{n,0} \quad &\perp \quad \mathbb{B}_{n,0} = \operatorname{span}\{z, \ldots, z^{n-1}, z^n - \tau_0\} \;\&\; \tau = \tau_0 \\
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\end{aligned}
$$

$$
\begin{aligned}
B_{n,2} \quad &\perp \quad \mathbb{B}_{n,2} = \operatorname{span}\{z^2, \ldots, z^{n-2}, z^{n-1} - \tau_1 z\} \;\&\; \tau = \tau_1 \\
B_{n,3} \quad &\perp \quad \mathbb{B}_{n,3} = \operatorname{span}\{z^2, \ldots, z^{n-2}\} \qquad \cup \operatorname{span}\{z^{n-1} - \hat{\tau} z\}
\end{aligned}
$$

$$\vdots$$

$$
\begin{aligned}
B_{n,2\ell} \quad &\perp \quad \mathbb{B}_{n,2\ell} = \operatorname{span}\{z^{\ell+1}, \ldots, z^{n-\ell-1}, z^{n-\ell} - \tau_\ell z^\ell\} \;\&\; \tau = \tau_\ell \\
B_{n,2\ell+1} \quad &\perp \quad \mathbb{B}_{n,2\ell+1} = \operatorname{span}\{z^{\ell+1}, \ldots, z^{n-\ell-1}\} \qquad \cup \operatorname{span}\{z^{n-\ell} - \hat{\tau} z^\ell\}
\end{aligned}
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$$\vdots$$

$B_{n,n-1}$:

$$
\begin{aligned}
B_{2s+1,2s} \quad &\perp \quad \mathbb{B}_{n,n-1} = \operatorname{span}\{z^{s+1} - \tau_s z^s\} \;\&\; \tau = \tau_s \\
B_{2s+2,2s+1} \quad &\perp \quad \mathbb{B}_{n,n-1} = \operatorname{span}\{z^{s+1}\}
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For $n > d+1$, and $d = 2\ell+1$ there is a d-parametric family of τ-invariant, monic $B_{n,d} \in \mathbb{P}_n \setminus \mathbb{P}_{n-1}$ orthogonal to $\mathbb{B}_{n,d} = \operatorname{span}\{z^{\ell+1}, \ldots, z^{n-\ell-1}\}$. It is given by

$$B_{n,d}(z) = z\pi_\ell(z)\rho_{n-\ell-1}(z) + \tau\pi_\ell^*(z)\rho_{n-\ell-1}^*(z),$$
$$\pi_\ell(z) = (z-\eta_1)\cdots(z-\eta_\ell), \quad \eta_i \in \mathbb{C}$$

It is also orthogonal to $z^{n-\ell} - \hat{\tau}z^\ell$, $\hat{\tau} = \tau\frac{\overline{a_{n-\ell}}}{a_{n-\ell}}$, $a_{n-\ell} = \langle B_{n,d}, z^{n-\ell}\rangle$.

Theorem

For $n > d+1$, $d = 2\ell$, and $\tau_\ell \in \mathbb{T}$ there is a d-parametric family of τ-invariant $B_{n,d} \in \mathbb{P}_n \setminus \mathbb{P}_{n-1}$ with $a_{n-\ell} = \langle B_{n,d}, z^{n-\ell}\rangle = 1$, orthogonal to $\mathbb{B}_{n,d} = \operatorname{span}\{z^{\ell+1}, \ldots, z^{n-\ell-1}, z^{n-\ell} - \tau_\ell z^\ell\}$, $\tau = \tau_\ell$. It is given by

$$B_{n,d}(z) = \nu_{n,d}[\sigma_\ell(z)\rho_{n-\ell}(z) + \tau_\ell\sigma_\ell^*(z)\rho_{n-\ell}^*(z)],$$
$$\sigma_\ell(z) = (1-\overline{\xi_1}z)\cdots(1-\overline{\xi_\ell}z), \quad \xi_i \in \mathbb{C}$$
$$\nu_{n,d}(z) = \det \mathcal{T}_{n-\ell-1} / \det \mathcal{T}_{n-\ell}$$

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Zeros

Theorem

For $n \geq 2$ & $d \in \{2\ell, 2\ell+1\}$, then any invariant (n,d)-quasi-para-orthogonal polynomial $B_{n,d}$ has $n-2\ell$ zeros of odd multiplicity in $\mathbb{T}$ and at least $n-2\ell-1$ of them are in $\mathring{A}(\mu)$.

Proof (idea) By symmetry of zeros of invariant polynomial + orthogonality conditions: $B_{n,d} \in \mathbb{B}_{n,d}^{\perp}$

By contradiction: If there are less than $n-2\ell-1$ of these zeros, then there exists a $Q \in \mathbb{B}_{n,d}$ such that $B_{n,d}\overline{Q}$ does not change sign on $\mathring{A}(\mu)$ so that

$$0 = \langle B_{n,d}, Q \rangle = \int B_{n,d}(z)\overline{Q(z)}\mathrm{d}\mu(z) > 0 \qquad \square$$

Corollary

If $d \in \{0,1\}$ all zeros are simple and on $\mathbb{T}$, while at least $n-1$ are in $\mathring{A}(\mu)$.

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Quadrature

$$I(f) = \int_{-\pi}^{\pi} f(e^{i\theta}) \mathrm{d}\mu(\theta) \approx I_n(f) = \sum_{k=1}^{n} \lambda_k f(z_k)$$

- $\{z_k\}_{k=1}^n$: nodes on $\mathbb{T}$ [better: in $\mathrm{supp}\,(\mu)$]
  $W_n(z) = (z - z_1) \cdots (z - z_n)$ = nodal polynomial
- $\{\lambda_k\}_{k=1}^n$: weights (preferably positive)

Szegő quadrature

$W_n(z) = B_{n,1}(z) = z\rho_{n-1}(z) + \tau\rho_{n-1}^*(z)$, all weights are positive
Exact for all
$f \in \mathcal{L}_{2(n-1)} = \mathbb{L}_{-(n-1),n-1} = \mathrm{span}\{z^k : k = -(n-1) : n-1\}$.

the free $\tau \in \mathbb{T}$ useful to either

- increase exactness to $\mathcal{L}_{2n-1} = \mathcal{L}_{2n-2} \cup \mathrm{span}\{z^n + \frac{\omega_n}{z^n}\}$, $\omega_n \in \mathbb{T}$
- fix a node at some position $\alpha \in \mathbb{T}$

Quadrature

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- fix a node at some position $\alpha \in \mathbb{T}$

Quadrature ($\mathrm{supp}\,(\mu) = \mathbb{T}$)

Define nested sequence of subspaces $\{\mathcal{L}_k\}_{k\geq 1}$, $\dim \mathcal{L}_k = k+1$

- $\mathcal{L}_{2(k-1)} = \mathbb{L}_{-(k-1),k-1}$
- $\mathcal{L}_{2k-1} = \mathcal{L}_{2k-2}(\omega_k) = \mathcal{L}_{2(k-1)} \cup \mathrm{span}\{z^k + \frac{\omega_k}{z^k}\}$, $\omega_k \in \mathbb{T}$
- $\mathcal{L}_{2n-1} \supset \mathcal{L}_{2n-2} \supset \cdots \supset \mathcal{L}_0 = \mathrm{span}\{1\}$

Quadrature ($\mathrm{supp}(\mu) = \mathbb{T}$)

Define nested sequence of subspaces $\{\mathcal{L}_k\}_{k\geq 1}$, $\dim \mathcal{L}_k = k+1$

- $\mathcal{L}_{2(k-1)} = \mathbb{L}_{-(k-1),k-1}$
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- $\mathcal{L}_{2n-1} \supset \mathcal{L}_{2n-2} \supset \cdots \supset \mathcal{L}_0 = \mathrm{span}\{1\}$

Theorem

For $n \geq 2$, $\tau \in \mathbb{T}$, W_n nodal polynomial for qf $I_n(f)$

- $W_n = B_{n,1} \Leftrightarrow I_n(f)$ *exact* $\forall f \in \mathcal{L}_{2n}$, $\quad \tau \in \mathbb{T}$ *is free*
- $W_n \sim B_{n,0} \Leftrightarrow I_n(f)$ *exact* $\forall f \in \mathcal{L}_{2n+1}(\underbrace{-\tau W_n(0)}_{\omega_n})$, $\quad \tau = \tau_0 \leftrightarrow \omega_n$
- $\{\lambda_k\} \subset \mathbb{R}$, *and all n are positive*

Quadrature ($\mathrm{supp}(\mu) = \mathbb{T}$)

Define nested sequence of subspaces $\{\mathcal{L}_k\}_{k\geq 1}$, $\dim \mathcal{L}_k = k+1$

- $\mathcal{L}_{2(k-1)} = \mathbb{L}_{-(k-1),k-1}$
- $\mathcal{L}_{2k-1} = \mathcal{L}_{2k-2}(\omega_k) = \mathcal{L}_{2(k-1)} \cup \mathrm{span}\{z^k + \frac{\omega_k}{z^k}\}$, $\omega_k \in \mathbb{T}$
- $\mathcal{L}_{2n-1} \supset \mathcal{L}_{2n-2} \supset \cdots \supset \mathcal{L}_0 = \mathrm{span}\{1\}$

Theorem (Jacobi theorem for $\mathbb{T}$)

For $n \geq 2$, $\tau \in \mathbb{T}$, W_n nodal polynomial for qf $I_n(f)$ — *2ℓ free $\mathbb{R}$ pars*

- $W_n = B_{n,2\ell+1} \Leftrightarrow I_n(f)$ *exact* $\forall f \in \mathcal{L}_{2(n-\ell)}$, — *$\tau \in \mathbb{T}$ is free*
- $W_n \sim B_{n,2\ell} \Leftrightarrow I_n(f)$ *exact* $\forall f \in \mathcal{L}_{2(n-\ell)+1}(\underbrace{-\tau W_n(0)}_{\omega_{n-\ell}})$, $\tau = \tau_\ell \leftrightarrow \omega_{n-\ell}$
- $\{\lambda_k\} \subset \mathbb{R}$, *and* $(n-\ell)$ *are positive*

R. Cruz-Barroso, C. Díaz Mendoza and F. Perdomo-Pío, *Szegő-type quadrature formulas*, J. Math. Anal. Appl. 455(1) (2017) 592–605.

Quadrature $(\mathrm{supp}\,(\mu) = \mathbb{T})$ Let's skip this

Recall even if we go for maximal degree of exactness $\varepsilon_{n,d}$, then

- $B_{n,2\ell}$ has 2ℓ free parameters ($\tau = \tau_\ell$ fixed), $\varepsilon_{n,2\ell} = 2(n-\ell)+1$
- $B_{n,2\ell+1}$ has 2ℓ free parameters plus $\tau \in \mathbb{T}$, $\varepsilon_{n,2\ell+1} = 2(n-\ell+1)$

Use this freedom to fix some nodes.

Theorem

Szegő-Radau: use free τ in $B_{n,1}$ to fix 1 node on $\mathbb{T}$ (positive qf)
Szegő-Lobatto: use 2 free parameters in $B_{n,2}$ to fix nodes $\alpha_1, \alpha_2 \in \mathbb{T}$.
Positive qf exact in $\mathcal{L}_{2\ell-2}(\omega)$ exist for some $\alpha_1, \alpha_2, \omega$.

R. Cruz-Barroso, C. Díaz Mendoza and F. Perdomo-Pío, *Szegő-type quadrature formulas*, J. Math. Anal. Appl. 455(1) (2017) 592–605.

Prefixed nodes $\ell = 1$

$B_{n,2}(z) = \nu_{n,2}[(1-\overline{\xi}z)\rho_{n-1}(z) + \tau_1(z-\xi)\rho^*_{n-1}(z)] = 0$ for $z \in \{\alpha_1, \alpha_2\} \subset \mathbb{T}$

$\therefore$ find $\xi \in \mathbb{C}$ such that $\frac{z-\xi}{1-\overline{\xi}z} = -\overline{\tau_1}\frac{\rho_{n-1}(z)}{\rho^*_{n-1}(z)}$, $z \in \{\alpha_1, \alpha_2\}$.

unimodular interpolation problem on $\mathbb{T}$: If $\exists \xi \in \mathbb{E} \Rightarrow$ positive weights.

$B_{n,3}(z) = z(z-\eta)\rho_{n-2}(z) + \tau(1-\overline{\eta}z)\rho^*_{n-2}(z)] = 0$ for $z \in \{\alpha_1, \alpha_2\} \subset \mathbb{T}$

$\therefore$ find $\eta \in \mathbb{C}$ such that $\frac{(z-\eta)}{1-\overline{\eta}z} = -\tau\frac{\rho^*_{n-2}(z)}{z\rho_{n-2}(z)}$, $z \in \{\alpha_1, \alpha_2\}$

unimodular interpolation problem on $\mathbb{T}$: $\exists\ \infty$ many $\eta \in \mathbb{D} \Rightarrow$ positive weights.

C. Jagels and L. Reichel, Szegő-Lobatto quadrature rules, *J. Comp. Appl. Math.* 200 (2007) 116-126.

use τ to fix a 3rd node on $\mathbb{T}$ (requires extra conditions)

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$B_{n,2\ell}(z) = \nu_{n,\ell}[\sigma_\ell(z)\rho_{n-\ell}(z) + \tau_\ell \sigma_\ell^*(z)\rho_{n-\ell}^*(z)] = 0$ for $z \in \{\alpha_k\}_{k=1}^{2\ell} \subset \mathbb{T}$

$\sigma_\ell(z) = (1 - \overline{\xi_1}z)\cdots(1 - \overline{\xi_\ell}z)$, set $f_\ell(z) = -\overline{\tau_\ell}\frac{\rho_{n-\ell}(z)}{\rho_{n-\ell}^*(z)} \in \mathbb{T}$ for $z \in \mathbb{T}$

$\therefore$ find $\{\xi_k\}_{k=1}^{\ell} \subset \mathbb{C}$ such that $\frac{\sigma_\ell^*(z)}{\sigma_\ell(z)} = f_\ell(z)$, $\forall z \in \{\alpha_k\}_{k=1}^{2\ell}$

unimodular interpolation problem on $\mathbb{T} \Leftrightarrow \sigma_\ell^*(z) = f_\ell(z)\sigma_\ell(z)$, $z \in \{\alpha_k\}_{k=1}^{2\ell}$

positive weights if all $\xi_k \in \mathbb{E}$

[because $\sigma_\ell \rho_{n-\ell}$ all zeros in $\mathbb{D}$ + Favard $\Rightarrow$ (positive) Szegő qf]

$B_{n,2\ell+1}(z) = z\pi_\ell(z)\rho_{n-\ell-1}(z) + \tau\pi_\ell^*(z)\rho_{n-\ell-1}^*(z) = 0$ for $z \in \{\alpha_k\}_{n=1}^{2\ell} \subset \mathbb{T}$

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- E.g. $d = 2\ell + 1$ $B_{n,2\ell+1}(z) = z\pi_\ell(z)\rho_{n-\ell-1}(z) + \tau\pi_\ell^*(z)\rho_{n-\ell-1}^*(z)$
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$$(\mathbb{C}) \quad \prod_{1\le i<j\le \ell} \frac{(\alpha_{\ell+i} - \alpha_{\ell+j})(\alpha_i - \alpha_j)}{(\overline{\alpha_{\ell+i}} - \overline{\alpha_{\ell+j}})(\alpha_i - \alpha_j)} \neq \prod_{i=1}^{\ell} \frac{f_{\ell+i}\alpha_{\ell+i}^{\ell}}{f_i \alpha_i^{\ell}}, \quad f_k = f_k(\alpha_k).$$

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Prefixed nodes $\ell > 1$

$d = 2\ell + 1$

(C) holds for $\tau \in T \subset \mathbb{T}$, i.e. a.e. on $\mathbb{T}$

- $\Rightarrow$ there is almost always a $\tau \in T$ such that $B_{n,2\ell+1}(\alpha_{2\ell+1}) = 0$ but zeros of π_ℓ for that τ may not be in $\mathbb{D}$
  $\Rightarrow$ no positive qf with $2\ell + 1$ prefixed nodes.
- if zeros π_ℓ are in $\mathbb{D}$ for $\tau \in T' \subset T \subset \mathbb{T}$
  the τ for which $B_{n,2\ell+1}(\alpha_{2\ell+1}) = 0$ may not be in T'
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$d = 2\ell$ is similar.

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Example: Rogers-Szegő weight on $\mathbb{T}$

$$\mu(\theta, q) = \mu(\theta) = \frac{1}{\sqrt{2\pi \log(1/q)}} \sum_{j=-\infty}^{\infty} \exp\left(-\frac{(\theta - 2\pi j)^2}{2\log(1/q)}\right), \quad 0 < q < 1$$

Schur parameters: $\delta_n = (-1)^n q^{\frac{n}{2}}$, $n = 1, 2, \ldots$ (we take $q = 0.5$)

Let us look at some matlab movies of the zeros of $B_{n,d}$ as we are varying $\tau \in \mathbb{T}$

$$\begin{aligned} B_{n,2\ell+1}(z) &= z\pi_\ell(z)\rho_{n-\ell-1}(z) + \tau\pi_\ell^*(z)\rho_{n-\ell-1}^*(z) \\ B_{n,2\ell}(z) &= \nu_{n,2\ell}[\sigma_\ell(z)\rho_{n-\ell}(z) + \tau\sigma_\ell^*(z)\rho_{n-\ell}^*(z)] \end{aligned}$$

with 2ℓ prefixed nodes and check if we can find

- a positive qf with an extra prefixed node for $d = 2\ell + 1$
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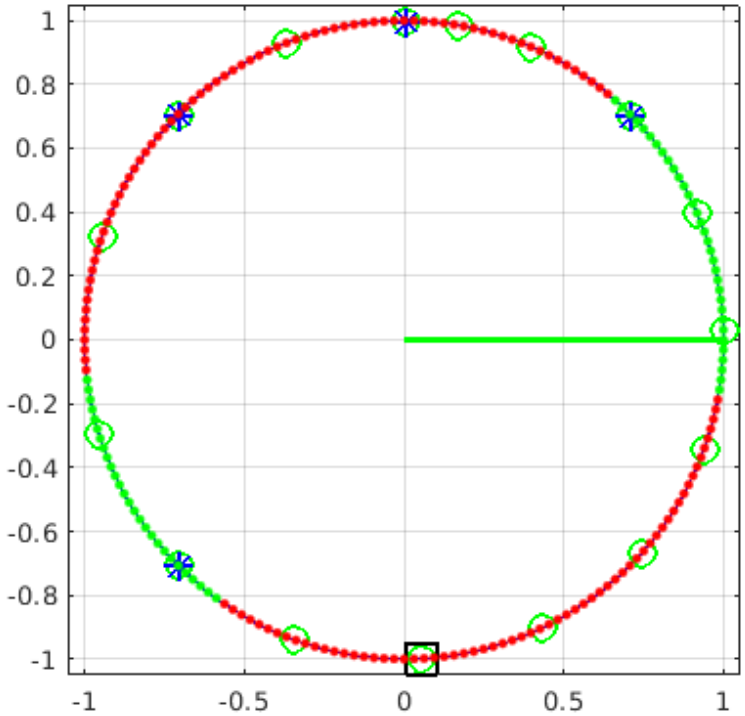
Example: Rogers-Szegő weight on $\mathbb{T}$

$$B_{16,4}(z) = \nu_{16,4}[\sigma_2(z)\rho_{14}(z) + \tau\sigma_2^*(z)\rho_{14}^*(z)]$$

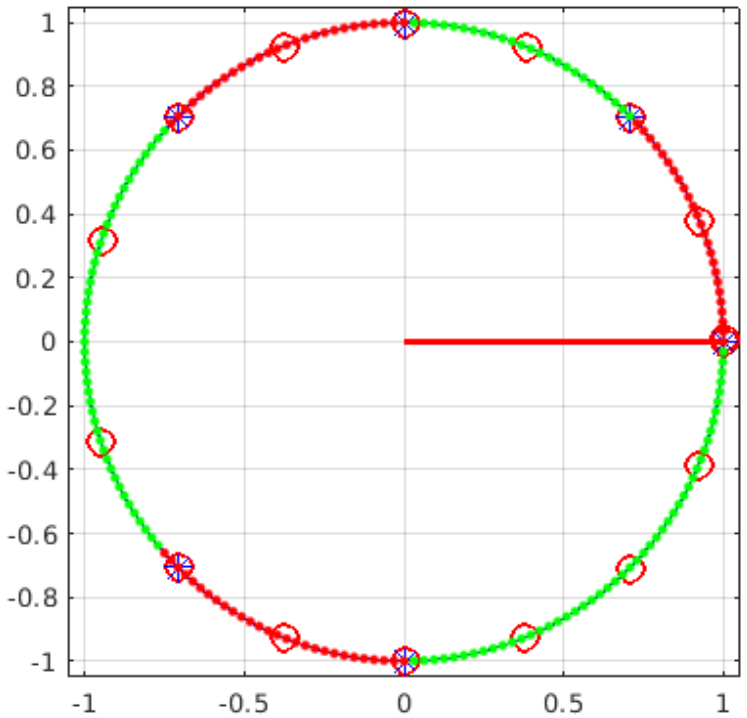


$B_{n,d}, n = 16, d = 4, \ell = 2$
circles = 16 zeros of $B_{16,4}$
stars = 4 prefixed nodes
green radius = τ (here 1)
$\tau \in$ green arcs = pos qf
$\tau \in$ red arcs = no pos qf
black square = ω

movie = everything moves as the radius (τ) rotates over 360°

Example: Rogers-Szegő weight on $\mathbb{T}$

$B_{16,7}(z) = z\pi_3(z)\rho_{12}(z) + \tau\pi_3^*(z)\rho_{12}^*(z)$



$B_{n,d}, n = 16, d = 7, \ell = 3$
circles $=$ 16 zeros of $B_{16,7}$
stars $=$ 6 prefixed nodes
red ray $= \tau$ (here 1)
$\tau \in$ green arcs $=$ pos qf
$\tau \in$ red arcs $=$ no pos qf

movie $=$ everything moves as the radius (τ) rotates over 360°
moving nodes will cover the whole circle $=$ a 7th node can be fixed.

The End

Thank you for your attention



Ad multos annos B & B