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Szegő-type quadrature formulas on an arc of the unit circle. Markov's approach

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Abstract

We recover the Szegő, Szegő-Radau and Szegő-Lobatto quadrature formulas when the measure is supported on the arc of the unit circle, making use of Hermite interpolation. We guarantee that the minimum norm paraorthogonal polynomial has its zeros in the support of a measure when it is less than or equal to the semicircle. Estimates of the error in the approximation depending on the analyticity of the integrand also be given.

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1. Introduction

When on the real line we consider the estimation of integrals of the form

$$J[f] = \int_a^b f(x)\mathrm{d}\rho(x),$$

where f is a Riemann-Stieltjes integrable function with respect to a finite positive Borel measure ρ supported on the interval $[a, b]$ of the real line, it is very common to use a quadrature formula of the form

$$J_n[f] := \sum_{k=1}^{n} \lambda_k f(x_k), \quad \{\lambda_k\}_{k=1}^n \subset \mathbb{R}, \quad \{x_k\}_{k=1}^n \subset [a, b], \quad x_k \neq x_j, \quad k \neq j,$$

whose parameters to be determined are called weights $\{\lambda_k\}_{k=1}^n$ and nodes $\{x_k\}_{k=1}^n$ of quadrature formula. It is highly recommended to have positive rules, i.e., rules with positive

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weights, mainly for convergence and computational stability reasons. The usual strategy to determine the weights and the nodes is to impose $J[p] = J_n[p]$ for all polynomials p of degree as high as possible given that the polynomials are dense in the space of continuous functions on $[a, b] \subset \mathbb{R}$ with the uniform norm. Gauss-Radau and Gauss-Lobatto formula quadrature mean that one (Radau) or both (Lobatto) endpoints of $[a, b]$ have been fixed as nodes in $J_n[f]$ such that the rest of the nodes and all the weights are to be determined by imposing that $J[p] = J_n[p]$, for all polynomials p of degree up to $2n - 1 - r$ with $r = 1, 2$, respectively. For $r = 0$ we obtain the Gauss formula quadrature which has no node fixed in advance. In its construction the quasi-orthogonal polynomials associated with ρ play an essential role.

In [3], some of the authors studied the counterpart when the measure is supported on an arc of the unit circle $\mathbb{T}$, i.e., they consider a positive finite Borel measure μ supported on $\Gamma = [a, b] = \{e^{\mathrm{i}\theta} : \theta_a \leq \theta \leq \theta_b < \theta_a + 2\pi\} \subset \mathbb{T}$. They addressed the approximation of integrals of the form

$$I[f] = \int_{\theta_a}^{\theta_b} f(e^{\mathrm{i}\theta}) \mathrm{d}\mu(\theta), \tag{1}$$

where $g(\theta) = f(e^{\mathrm{i}\theta})$ is a Riemann-Stieltjes integrable function with respect to μ. To do so, they considered a quadrature formula like

$$I_n[f] := \sum_{k=1}^{n} \lambda_k f(z_k), \qquad \{z_k\}_{k=1}^n \subset [a, b], \quad z_j \neq z_k, \;\; j \neq k, \quad \{\lambda_k\}_{k=1}^n \subset \mathbb{R}. \tag{2}$$

A similar strategy is considered, Laurent polynomials are used instead of polynomials in the case of an interval of the real line, for determining the weights $\{\lambda_k\}_{k=1}^n$ and nodes $\{z_k\}_{k=1}^n$. This makes sense since they are dense with respect to the uniform norm in the set $\mathcal{C}(\Gamma)$ of continuous functions on the arc $\Gamma \subset \mathbb{T}$ so that it allows to take full advantage of the theory of quasi-paraorthogonal polynomials, which are known to have zeros on $\mathbb{T}$ depending on the orthogonality conditions, as some of the authors proved in [1]. We will impose that $I[L] = I_n[L]$ for all Laurent polynomials L belonging to some subspace with the highest possible dimension.

To achieve our goal on the unit circle, we shall, like in [3] and also [4, 1], consider a certain nested sequence $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ of subspaces of Laurent polynomials, such that $\mathcal{L}_0 = \mathrm{span}\{1\}$, $\mathcal{L}_{n-1} \subset \mathcal{L}_n$, $\dim(\mathcal{L}_n) = n + 1$ and $\bigcup_{n\in\mathbb{N}} \mathcal{L}_n = \mathbb{L}$, the space of all Laurent polynomials. In particular, given a sequence $\{\omega_k\}_{k\geq 1} \subset \mathbb{T}$ we define for $k \geq 0$

$$\mathcal{L}_{2k} := \mathrm{span}\{z^{-k}, \ldots, z^k\}, \quad \mathcal{L}_{2k-1} = \mathcal{L}_{2k-1}(\omega_k) := \mathcal{L}_{2(k-1)} \oplus \mathrm{span}\left\{z^k + \frac{\omega_k}{z^k}\right\}.$$

The choice of these subspaces allows us to recover for an arc of the unit circle many of the classical properties known for an interval of the real line. For example, the weights are always real and at least half of them are positive (see [4, Theorem 2.3]); and a theorem analogous of the well known Jacobi theorem can be formulated (see [3, Theorem 2.6]), here the quadrature formulas are characterized in terms of orthogonality conditions. In

fact, one of the interest of this contribution is to give a constructive proof of the most representative cases, Szegő, Szegő-Radau and Szegő-Lobatto.

We say that the quadrature formula $I_n[f]$ given by (2) is exact in $\mathcal{L}_k$ iff $I_n[L] = I[L]$, $\forall L \in \mathcal{L}_k$, i.e., $\mathcal{L}_k$ is a domain of validity for $I_n[f]$. Furthermore, let $I_n[f]$ be the quadrature formula given by (2), exact in $\mathcal{L}_k$ with $n-1 \leq k \leq 2n-1$, then it has all its weights $\{\lambda_k\}_{k=1}^n$ real and at least $\lfloor \frac{k}{2} \rfloor + 1$ of them are positive, see [4], in other words, the number of positive weights is linked to the degree of accuracy.

The existence of positive rules of the form (2) whose maximal domain of validity has dimension $2n$, is proved in [3, Theorem 4.2] when μ belong to the class of Geronimus measures. Such formula is called a Szegő quadrature formula and it is the analogue to the Gaussian formula on an interval. If one of the ends point of the arc $[a, b]$ is fixed, a quadrature rule exists for every positive finite Borel measure μ supported on the arc (see [3]). This formula is called a Szegő-Radau formula and it is the analogue to Gauss-Radau formula on an interval.

The existence of a quadrature formula with two preassigned nodes at the endpoints of the arc is also proved in [3], under certain conditions, for every measure of the Geronimus class. In this case, it is called the Szegő-Lobatto quadrature formula which is the analogue to the Gauss-Lobatto formula. As we shall see, the quasi-paraorthogonal polynomials play an crucial role.

Convergence results for these quadrature formulas were obtained depending on the differentiability of the integrand in [3], see Theorem 4.5, Theorem 5.5 and Theorem 6.6, respectively of that paper.

An objective of this contribution is to recover the Szegő, Szegő-Radau and Szegő-Lobatto quadratures by considering Hermite interpolation, what Gautschi in his book (see [5, p. 165]) coined the *Markov approach* (following Markov's method used in [7]). This is not a trivial generalization, since the Hermitian orthogonality of invariant polynomials on the unit circle will fulfill the role of the real orthogonality of polynomials on the real line, and these must be identified with Laurent polynomials that have to fit the proposed nesting of Laurent polynomial subspaces, which is not as simple as the nesting of real polynomials.

Another objective is to show that the zeros of the minimum norm monic paraorthogonal polynomials have their zeros in the arc for any positive measure when the arc Γ has a length less than or equal to the semicircle that allows their zeros to be considered as nodes of Szegő quadrature formulas. As an immediate consequence, the existence of Szegő-Lobatto quadrature formulas is also guaranteed.

Before continuing, we introduce some notations. For any function f we denote the parahermitean conjugate as $f_*(z) = \overline{f(1/\overline{z})}$. Note that $f_*(t) = \overline{f(t)}$ if $t \in \mathbb{T}$. For a finite positive Borel measure μ, supported on an arc $\Gamma = [a, b] = [e^{i\theta_a}, e^{i\theta_b}] \subset \mathbb{T}$, with length $|\Gamma| = \theta_b - \theta_a$, we consider the Hilbert space $L^2_\mu(\Gamma)$ of μ-square integrable functions, equiped with the inner product $\langle f, g \rangle = \int_{\theta_a}^{\theta_b} f(e^{\mathrm{i}\theta})\overline{g(e^{\mathrm{i}\theta})}\mathrm{d}\mu(\theta)$ and the corresponding norm $\|\cdot\|$. The sequence of monic orthogonal polynomials, known as Szegő polynomials, with respect to μ is denoted as $\{\rho_n\}_{n\in\mathbb{N}}$. Their Verblunsky coefficients are $\delta_n = \rho_n(0) \in \mathbb{D}$.

By the class of Geronimus measures we refer to the set of absolutely continuous measures $\mathrm{d}\mu(\theta) = \omega(\theta)\mathrm{d}\theta$ that are characterized by $\rho_n(0) = \delta$ for all $n \in \mathbb{N}$ with $|2\delta - 1| < 1$. They are supported on an arc symmetric with respect to the real line that does not contain 1. It is uniquely defined by $|\delta|$, see [6] and also [3]. The set of Geronimus weights will be denoted by $\mathcal{G}$. For $u, v, w \in \mathbb{T}$, we denote $u \prec v \prec w$, if $u = e^{\theta_u}$, $v = e^{i\theta_v}$, $w = e^{i\theta_w}$ and $\theta_u < \theta_v < \theta_w < \theta_u + 2\pi$.

The supremum norm of a bounded function f on Γ is $\|f\|_\infty$. $\mathbb{P}_n$ is the set of polynomials of degree at most n and $\mathbb{P}_n^1$ is the set of monic polynomials of exact degree n. For any polynomial P of degree n, define $P^*(z) = z^n P_*(z)$ as the reciprocal polynomial. If it is invariant, i.e., if $P(z) = \tau P^*(z)$ with $\tau \in \mathbb{T}$, then if P is monic, $\tau = P(0)$. A quasi-paraorthogonal polynomial of degree n and order $2\ell + 1$ is an invariant monic polynomial of degree n satisfying the orthogonality conditions $\langle P_n, z^k \rangle = 0$, $k = 2\ell+1, \ldots, n-(2\ell+1)$. The classical paraorthogonal polynomials correspond to quasi-paraorthogonals with $\ell = 0$. Note that a monic (quasi-)paraorthogonal polynomial B_n satisfies $B_n = B_n(0)B_n^*$ with $B_n(0) \in \mathbb{T}$.

If S is a set, then $\hat{S}$ denotes its closure, in particular $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, while $\mathrm{cap}(S)$ is its logarithmic capacity. A level set for a function H at level r is defined as $\gamma(r) = \{z : |H(z)| = r\}$, and its interior is the corresponding sublevel set $\mathrm{Int}(\gamma(r)) = \{z : |H(z)| < r\}$.

We shall develop the Markov approach in three sections separately, one for each quadrature formula: in Section 2 we recover the Szegő quadrature and show that the minimum norm paraorthogonal polynomials have their zeros in the support of the measure Γ when the arc has a length less than or equal to the semicircle, Szegő-Radau is covered in Section 3 and finally, we study Szegő-Lobatto in Section 4. We shall also obtain convergence results for integrand analytic in a simply connected neighborhood of Γ with the possibly exception of some poles.

2. Markov's approach to Szegő quadrature formula

2.1. Existence and uniqueness

We shall recover the Szegő quadrature formulas exact in $\mathcal{L}_{2n-1}(\omega_n)$ by Hermite interpolation for any $\omega_n \in \mathbb{T}$. To do this, we must solve a specific Hermite interpolation problem. Concretely, given distinct numbers $\{z_1, z_2, \ldots, z_n\} \subset \mathbb{T}$, we are going to get the unique Laurent polynomial $H_{2n-1}^f \in \mathcal{L}_{2n-1}(\omega_n)$ such that

$$H_{2n-1}^f(z_j) = f(z_j) \quad \text{and} \quad (H_{2n-1}^f)'(z_j) = f'(z_j), \quad j = 1, \ldots, n.$$

It is advantageous to divide it into two interpolation problems. On the one hand, find the unique Laurent polynomial $H_{1,2n-1}^f \in \mathcal{L}_{2n-1}(\omega_n)$ such that

$$H_{1,2n-1}^f(z_j) = f(z_j) \quad \text{and} \quad (H_{1,2n-1}^f)'(z_j) = 0, \quad j = 1, \ldots, n.$$

On the other hand, determine the unique Laurent polynomial $H_{2,2n-1}^f \in \mathcal{L}_{2n-1}(\omega_n)$ such that

$$H_{2,2n-1}^f(z_j) = 0 \quad \text{and} \quad (H_{2,2n-1}^f)'(z_j) = f'(z_j), \quad j = 1, \ldots, n.$$

Then, we can write $H^f_{2n-1} = H^f_{1,2n-1} + H^f_{2,2n-1}$, i.e.,

$$H^f_{2n-1}(z) = \sum_{j=1}^{n} \tilde{A}_j(z) f(z_j) + \sum_{j=1}^{n} \tilde{B}_j(z) f'(z_j) \tag{3}$$

with $\tilde{A}_j,\ \tilde{B}_j \in \mathcal{L}_{2n-1}(\omega_n)$, verifying the conditions

$$\begin{cases} \tilde{A}_j(z_k) = \delta_{jk}, & \tilde{A}'_j(z_k) = 0, \\ \tilde{B}_j(z_k) = 0, & \tilde{B}'_j(z_k) = \delta_{jk}, \end{cases} \qquad k = 1, \ldots, n, \quad j = 1, \ldots, n.$$

Let us denote by $\{\ell_j\}_{j=1}^n$ the Lagrange fundamental polynomials associated with nodes $\{z_j\}_{j=1}^n$ and $W_n(z) = \prod_{k=1}^n (z - z_k)$ the nodal polynomial (note that it is invariant).

Theorem 2.1. *Let $W_n^2(0) \neq \omega_n$. Under the above interpolation conditions, we have the following expressions for $\tilde{A}_j,\ \tilde{B}_j \in \mathcal{L}_{2n-1}(\omega_n)$, $j = 1, \ldots, n$,*

1. $\tilde{B}_j(z) = \ell_j^2(z)(z - z_j) \dfrac{z - \frac{z_j \omega_n}{W_n^2(0)}}{z_j \left(1 - \frac{\omega_n}{W_n^2(0)}\right)} \left(\dfrac{z_j}{z}\right)^n.$
2. $\tilde{A}_j(z) = \ell_j^2(z) \left(\dfrac{z_j}{z}\right)^{n-1} + \dfrac{n - 1 - 2 z_j \ell'_j(z_j)}{z_j} \tilde{B}_j(z).$

Proof. For $j = 1, \ldots, n$. Let $\tilde{B}_j \in \mathcal{L}_{2n-1}(\omega_n)$ verify $\tilde{B}_j(z_k) = 0$ and $\tilde{B}'_j(z_k) = \delta_{jk}$ for $k = 1, \ldots, n$ and define $B_j \in \mathbb{P}_{2n}$ by

$$\tilde{B}_j(z) = B_j(z)/z^n, \tag{4}$$

with

$$B_j(z) = \tilde{b} z^{2n} + z q(z) + \tilde{b} \omega_n, \quad q \in \mathbb{P}_{2n-2}, \quad \tilde{b} \in \mathbb{C}. \tag{5}$$

Then

$$B_j(z_k) = 0, \quad k = 1, \ldots, n, \tag{6}$$

and by (4)

$$\tilde{B}'_j(z) = (B'_j(z) z - B_j(z) n)/z^{n+1}.$$

Since $\tilde{B}'_j(z_j) = 1$ and by (6), one has $B'_j(z_j) = z_j^n$ while $B'_j(z_k) = 0$, for all $k \neq j$. Therefore, z_j is a simple zero of B_j, while all the other nodes are double zeros. Thus, we can write

$$B_j(z) = \tilde{b} \frac{W_n^2(z)}{z - z_j} p_j(z), \quad p_j \in \mathbb{P}_1^1.$$

From (5) with $z = 0$, we obtain $\tilde{b}\omega_n = -\tilde{b}\frac{W_n^2(0)}{z_j} p_j(0)$, so that $p_j(z) = z - \frac{z_j \omega_n}{W_n^2(0)}$ and therefore, given that $W_n(z) = \ell_j(z)(z - z_j) W'_n(z_j)$, we obtain

$$B_j(z) = \tilde{b} \frac{W_n^2(z)}{z - z_j} p_j(z) = \tilde{b} \left(W'_n(z_j)\right)^2 \ell_j^2(z_j)(z - z_j) p_j(z).$$

It remains to impose $B_j'(z_j) = z_j^n$ to determine $\tilde{b}$. Therefore, given that

$$B_j'(z) = \tilde{b}\big(W_n'(z_j)\big)^2 \ell_j(z) \left[2\ell_j'(z)(z-z_j)p_j(z) + \ell_j(z)\left(p_j(z) + (z-z_j)\right)\right],$$

and since $p_j(z_j) \neq 0$, we deduce

$$z_j^n = \tilde{b}\big(W_n'(z_j)\big)^2 p_j(z_j) \Rightarrow \tilde{b}\left(W_n'(z_j)\right)^2 = \frac{z_j^n}{p_j(z_j)}.$$

Finally

$$\tilde{B}_j(z) = \ell_j^2(z)(z-z_j)\frac{p_j(z)}{p_j(z_j)}\left(\frac{z_j}{z}\right)^n, \quad p_j(z) = z - \frac{z_j\omega_n}{W_n^2(0)}, \quad j = 1, \ldots, n.$$

Similarly we can obtain expressions for $\tilde{A}_j$. Let $\tilde{A}_j \in \mathcal{L}_{2n-1}(\omega_n)$, then

$$\tilde{A}_j(z) = A_j(z)/z^n, \quad A_j \in P_{2n} \tag{7}$$

with

$$A_j(z) = \tilde{a}z^{2n} + zq(z) + \tilde{a}\omega_n, \quad q \in \mathbb{P}_{2n-2}, \quad \tilde{a} \in \mathbb{C}. \tag{8}$$

As a consequence of $\tilde{A}_j(z_k) = \delta_{jk}$, it follows that

$$A_j(z_k) = z_k^n \delta_{jk}, \quad k = 1, \ldots, n. \tag{9}$$

By (7), we have

$$\tilde{A}_j'(z) = \big(A_j'(z)z - nA_j(z)\big)/z^{n+1},$$

as $\tilde{A}_j'(z_k) = 0$, for $k = 1, \ldots, n$ and by (9), one has $A_j'(z_k) = nz_k^{n-1}\delta_{jk}$, $k = 1, \ldots, n$. Therefore, all nodes are double zeros of A_j except z_j. Thus,

$$A_j(z) = \tilde{a}\left(W_n'(z_j)\right)^2 \ell_j^2(z)p_2(z), \tag{10}$$

with $p_2 \in \mathbb{P}_2^1$. Since we want to express A_j in terms of B_j, we will use the previously defined $p_j \in \mathbb{P}_1^1$ and take the basis $\{1, z-z_j, (z-z_j)p_j(z)\}$ in $\mathbb{P}_2$. Then

$$p_2(z) = c + b(z-z_j) + (z-z_j)p_j(z), \quad c, b \in \mathbb{C}.$$

On the one hand, by (8), $A_j(0) = \tilde{a}\omega_n$. On the other hand, if we focus at $z = 0$ in (10)

$$\tilde{a}\omega_n = \tilde{a}\left(\frac{W_n(0)}{z_j}\right)^2 \left(c - bz_j + z_j p_j(0)\right),$$

it follows that

$$\frac{z_j^2\omega_n}{W_n^2(0)} = c - bz_j + \frac{z_j^2\omega_n}{W_n^2(0)},$$

and we deduce $c = bz_j$ and $c + b(z - z_j) = c - bz_j + bz = bz$. Recapitulating, we find that the polynomial A_j takes the form

$$A_j(z) = \tilde{a}\,(W_n'(z_j))^2\,\ell_j^2(z)\big(bz + (z - z_j)p_j(z)\big).$$

It remains to impose the interpolation conditions associated with z_j. On the one hand $A_j(z_j) = z_j^n$, i.e.

$$z_j^n = \tilde{a}\,(W_n'(z_j))^2\,bz_j,$$

so we infer,

$$A_j(z) = z_j^{n-1}\ell_j^2(z)\left(z + \frac{1}{b}(z - z_j)p_j(z)\right). \tag{11}$$

On the other hand, since $A_j'(z_j) = nz_j^{n-1}$, and as

$$A_j'(z) = z_j^{n-1}\ell_j(z)\left\{2\ell_j'(z)\Big(z + \tfrac{1}{b}(z - z_j)p_j(z)\Big) + \ell_j(z)\Big(1 + \tfrac{1}{b}\left[p_j(z) + (z - z_j)\right]\Big)\right\},$$

we get by evaluating at $z = z_j$,

$$nz_j^{n-1} = z_j^{n-1}\Big(2\ell_j'(z_j)z_j + 1 + \tfrac{1}{b}p_j(z_j)\Big).$$

Thus, by (11), we obtain

$$\begin{aligned} A_j(z) &= z_j^{n-1}\ell_j^2(z)\left(z + \frac{n - 1 - 2z_j\ell_j'(z_j)}{p_j(z_j)}(z - z_j)p_j(z)\right) \\ &= z_j^{n-1}\ell_j^2(z)z + \big(n - 1 - 2z_j\ell_j'(z_j)\big)\,z_j^{n-1}\ell_j^2(z)(z - z_j)\frac{p_j(z)}{p_j(z_j)} \\ &= z_j^{n-1}z\ell_j^2(z) + \frac{n - 1 - 2z_j\ell_j'(z_j)}{z_j}B_j(z). \end{aligned}$$

Finally,

$$\tilde{A}_j(z) = \ell_j^2(z)\left(\frac{z_j}{z}\right)^{n-1} + \frac{n - 1 - 2z_j\ell_j'(z_j)}{z_j}\tilde{B}_j(z), \quad j = 1, \ldots, n.$$

The proof is concluded. □

Now, if we integrate both sides of formula (3), we obtain the following

$$I[H_{2n-1}^f] = \sum_{j=1}^{n} I[\tilde{A}_j]f(z_j) + \sum_{j=1}^{n} I[\tilde{B}_j]f'(z_j).$$

Since our goal is to construct a quadrature formula only involving function values, we should impose that

$$I[\tilde{B}_j] = 0, \quad j = 1, \ldots, n. \tag{12}$$

Remark 2.2. *Clearly, $\mathcal{L}_{2k-1}(\omega_k)$ is not always a Hermite system on $\mathbb{C}\backslash\{0\}$. Since for k distinct nodes $\{z_{i,k}\}_{i=1}^{k} \subset \mathbb{C}\backslash\{0\}$ the interpolation problem in $\mathcal{L}_{2k-1}(\omega_k)$ in the sense of Hermite has a unique solution if and only if $z_{1,k}^2 \cdots z_{k,k}^2 \neq \omega_k$.*

In the following theorem, we recover the positivity of the weights $\{\lambda_j\}_{j=1}^{n}$ and a new computable expression for them.

Theorem 2.3. *Let $W_n^2(0) \neq \omega_n$ and $\lambda_j = I[\tilde{A}_j]$, $j = 1, \ldots, n$. If (12) holds, then*

$$\lambda_j = I\left[|\ell_j|^2\right] > 0, \quad j = 1, \ldots, n.$$

Proof. By Theorem 2.1, and using (12) we have

$$\lambda_j = I\left[\tilde{A}_j\right] = \int_{\theta_a}^{\theta_b} \left(\frac{z_j}{t}\right)^{n-1} \ell_j^2(t) \mathrm{d}\mu(\theta), \quad t = e^{\mathrm{i}\theta}, \quad j = 1, \ldots, n.$$

Since clearly $\overline{\ell_j(t)} = \left(\frac{z_j}{t}\right)^{n-1} \ell_j(t)$, for $t \in \Gamma$, the proof follows immediately. □

From a practical point of view, it remains to see if the conditions (12) is compatible with the fact that W_n has its distinct zeros on the arc $[a, b]$. A first step is the following theorem.

Theorem 2.4. *Let $W_n^2(0) \neq \omega_n$. If (12) holds, then the nodal polynomial W_n satisfies the following orthogonality conditions*

$$\langle W_n, z^k \rangle = 0, \quad k = 1, \ldots, n-1, \quad \text{and} \quad \left\langle W_n, z^n + \tfrac{\omega_n}{W_n(0)} \right\rangle = 0. \tag{13}$$

Proof. For $j = 1, \ldots, n$,

$$\begin{aligned} 0 = I[\tilde{B}_j] &= \int_{\theta_a}^{\theta_b} \ell_j^2(t)(t - z_j) \frac{p_j(t)}{p_j(z_j)} \left(\frac{z_j}{t}\right)^n \mathrm{d}\mu(\theta) \\ &= \frac{z_j}{p_j(z_j)} \int_{\theta_a}^{\theta_b} (t - z_j) \ell_j(t) \left(\frac{z_j}{t}\right)^{n-1} \ell_j(t) \frac{1}{t} p_j(t) \, \mathrm{d}\mu(\theta). \end{aligned}$$

Now use $W_n(z) = (z - z_j)\ell_j(z)W_n'(z_j)$, and $z_j^{n-1}\ell_j(t) = t^{n-1}\overline{\ell_j(t)}$ for $t \in \mathbb{T}$, and because $p_j(0) = \frac{z_j \omega_j}{W_n^2(0)} \in \mathbb{T}$, we also have $p_j(t) = t p_j(0) \overline{p_j(t)}$, for $t \in \mathbb{T}$, and then

$$0 = \frac{z_j p_j(0)}{p_j(z_j) W_n'(z_j)} \int_{\theta_a}^{\theta_b} W_n(t) \overline{\ell_j(t) p_j(t)} \mathrm{d}\mu(\theta) = \frac{z_j p_j(0)}{p_j(z_j) W_n'(z_j)} \langle W_n, \ell_j p_j \rangle.$$

We can conclude

$$\langle W_n, \tilde{\ell}_j \rangle = 0, \quad j = 1, \ldots, n,$$

with $\tilde{\ell}_j(z) = \ell_j(z) p_j(z)$. We show that $\{\tilde{\ell}_j\}_{j=1}^{n}$ is a basis for $\hat{\mathbb{P}}_n = \operatorname{span}\left\{z, \ldots, z^{n-1}, z^n + \frac{\omega_n}{W_n(0)}\right\}$.

First of all, observe that $\tilde{\ell}_j$ has degree n with leading coefficient $\frac{1}{W_n'(z_j)}$ and the independent term is

$$\tilde{\ell}_j(0) = \ell_j(0)p_j(0) = \frac{W_n(0)}{(-z_j)W_n'(z_j)}\left(-\frac{z_j\omega_n}{W_n^2(0)}\right) = \frac{1}{W_n'(z_j)}\frac{\omega_n}{W_n(0)},$$

and therefore, $\tilde{\ell}_j \in \hat{\mathbb{P}}_n$, $j = 1, \ldots, n$.

Finally, let's see that the system $\{\tilde{\ell}_j\}_{j=1}^n$ is linearly independent. Take a linear combination $\sum_{j=1}^n \alpha_j\tilde{\ell}_j(z) = 0$, $\alpha_j \in \mathbb{C}$, then for $z = z_k$, we deduce $\alpha_k z_k\left(1 - \frac{\omega_n}{W_n^2(0)}\right) = 0$, so that $\alpha_k = 0$, $k = 1, \ldots, n$ since $W_n^2(0) \neq \omega_n$ and the proof follows. □

Therefore, we have proved that the nodal polynomial W_n is a monic paraorthogonal polynomial with the additional orthogonality condition.

$$\left\langle W_n, z^n + \frac{\omega_n}{W_n(0)} \right\rangle = 0. \tag{14}$$

We know that monic paraorthogonal polynomials have the expression

$$W_n(z) = z\rho_{n-1}(z) + W_n(0)\rho_{n-1}^*(z) = c_n(\rho_n(z) + \tau_n\rho_n^*(z)), \quad W_n(0) = \frac{\delta_n + \tau_n}{1 + \tau_n\overline{\delta}_n}, \quad \tau_n \in \mathbb{T},$$

where ρ_k denotes the kth Szegő polynomial with respect to μ and $\delta_k = \rho_k(0)$ and c_n is a normalizing constant. If we now impose (14), we deduce $\tau_n = -\frac{\omega_n}{W_n(0)}$. We conclude, the nodal polynomial W_n must verify

$$W_n(0) = \varphi_{\delta_n}\left(-\frac{\omega_n}{W_n(0)}\right),$$

with $W_n^2(0) \neq \omega_n$ and $\varphi_{\delta_n}(z) = \frac{z+\delta_n}{1+z\overline{\delta}_n}$. Equivalently,

$$-W_n(0)\varphi_{-\delta_n}(W_n(0)) = \omega_n. \tag{15}$$

The equation (15) has two solutions on $\mathbb{T}$, see [8], $B_n^1(0)$ and $B_n^2(0)$. That is we have two options

$$B_n^1(z) = z\rho_{n-1} + B_n^1(0)\rho_{n-1}^*(z) \quad \text{and} \quad B_n^2(z) = z\rho_{n-1} + B_n^2(0)\rho_{n-1}^*(z).$$

For our purposes, and from a constructive and effective point of view, it is useful to know under what conditions we can ensure that by fixing ω_n any of the monic polynomials B_n^i has its simple zeros in $[a, b]$. We know that all its zeros are simple, lie on $\mathbb{T}$, and at least $n-1$ zeros are in (a, b) and that all weights are positive, see [1] and Theorem 2.3, respectively. Note that the solutions in (15) implicitly verify that $W_n(0)^2 \neq \omega_n$. We can affirm that

Theorem 2.5. *Let μ be a positive Borel measure supported on an arc $\Gamma \subset \mathbb{T}$, such that the corresponding paraorthogonal polynomial verifying (15) has all simple zeros on Γ. Then there exists a quadrature formula $I_n[f]$ of the form (2) exact in $\mathcal{L}_{2n-1}(\omega_n)$. Its nodal polynomial is $W_n(z) = z\rho_{n-1}(z) + W_n(0)\rho^*_{n-1}(z)$.*

The corresponding quadrature formula is a Szegő quadrature formula and we shall denote it by $I_n^S[f]$. Its weights, denoted as $\{\lambda_k^S\}_{k=1}^n$, are positive and its nodes, denoted as $\{z_k^S\}_{k=1}^n$, are in the arc Γ.

It is well known that, in the real case, the nodes of Gauss quadrature are the zeros of the orthogonal polynomials, which coincide with those of minimum norm polynomials. In the search for a counterpart on the unit circle, some of the authors took into account the freedom provided by $W_n(0) \in \mathbb{T}$ to choose as nodal polynomials the paraorthogonal polynomials of minimum norm. They showed in [3, Theorem 3.2] that minimum norm paraorthogonal polynomials have all their zeros in the open arc (a, b), guaranteeing the existence of a Szegő quadrature formula when the measure μ belongs to the Geronimus class with support on the arc $\Gamma = [a, b] \subset \mathbb{T}$, with $b = \overline{a}$. Their nodal polynomial W_n is given by

$$W_n(z) = z\rho_{n-1}(z) + d\rho^*_{n-1}(z)$$

where $d = \frac{\delta}{|\delta|}$, and $\delta = \rho_n(0)$ for all n, and this quadrature is exact in $\mathcal{L}_{2n-1}(-d^2)$ as follows from (15) because $W_n(0) = d$. The fact that the support is symmetrical is not a problem, since it is sufficient to perform a rotation beforehand in (1). In general, they have shown that in the case a zero is outside the arc (there can be at most one zero outside), it must accumulate at one of the arc's endpoints as n increases (see [2]). But a general proof that these minimal norm paraorthogonal polynomials have all their zeros in the arc remains open. However, in the next theorem we provide a partial answer.

Theorem 2.6. *Let μ be a finite positive Borel measure on $\Gamma \subset \mathbb{T}$, with $|\Gamma| \leq \pi$. Then the zeros of the minimum norm monic paraorthogonal polynomial B_n of degree n are in the arc (a, b).*

Proof. Let us first prove by contradiction that $\rho_n(0) \neq 0$, where ρ_n is the nth monic orthogonal polynomial with respect to μ. If for $n \geq 0$ we assume that $\rho_{n+1}(0) = 0$ then $\rho_{n+1}(z) = z\rho_n(z)$, and since $\rho_{n+1} \perp \rho_n$, we have $\int_\Gamma t|\rho_n(t)|^2 \mathrm{d}\mu = 0$, and that is impossible because $|\rho_n(t)|^2 \mathrm{d}\mu$ is a positive measure in Γ and as a Stieltjes-Riemann sum, the integral is the limit of a weighted sum of values $t = e^{i\theta} \in \Gamma$ and hence, it is a convex linear combination of points on the arc and that must give a result in the convex hull of Γ. However, if $|\Gamma| \leq 1$, then 0 will be outside that convex hull, unless $|\Gamma| = \pi$, but then the integral can only be zero if there is no mass in Γ, except for 2 atoms with equal mass in the endpoints of Γ.

As $B_n(z) = z\rho_{n-1}(z) + d_n\rho^*_n(z))$, let $\{x_{j,n}\}_{j=1}^n \subset \mathbb{T}$ be its zeros, i.e., $B_n(x_{j,n}) = 0$, $j = 1, \ldots, n$, with $d_n = \frac{\rho_n(0)}{|\rho_n(0)|}$, we define

$$Q_{n-1}(z) = \frac{B_n(z)}{z - x_{j,n}} = z^{n-1} + zP_{n-3}(z) - \overline{x}_{j,n}d_n, \quad P_{n-3} \in \mathbb{P}_{n-3},$$

and since $B_n \perp \text{span}\{z, \ldots, z^{n-1}\}$, we have

$$\langle B_n, Q_{n-1}\rangle = -x_{j,n}\overline{d}_n\langle B_n, 1\rangle = -x_{j,n}(1-|\delta_n|)\|\rho_{n-1}\|^2 \tag{16}$$

where we have used

$$\langle B_n, 1\rangle = \langle z\rho_{n-1}, 1\rangle + d_n\langle \rho_n^*, 1\rangle = -\delta_n\|\rho_{n-1}\|^2 + d_n\|\rho_{n-1}\|^2 = (d_n-\delta_n)\|\rho_{n-1}\|^2,$$

since $\rho_n(z) = z\rho_{n-1}(z) + \delta_n\rho_{n-1}^*(z)$. On the other hand, we also have

$$\begin{aligned}\langle B_n, Q_{n-1}\rangle &= \int_{\theta_a}^{\theta_b}(t-x_{j,n})\,|Q_{n-1}(t)|^2\,\mathrm{d}\mu(\theta), \quad t = e^{\mathrm{i}\theta}\\ &= \int_{\theta_a}^{\theta_b} t\mathrm{d}\tilde{\mu}(\theta) - x_{j,n}\int_{\theta_a}^{\theta_b}\mathrm{d}\tilde{\mu}(\theta), \quad \mathrm{d}\tilde{\mu} = |Q_{n-1}(t)|^2\,\mathrm{d}\mu.\end{aligned} \tag{17}$$

By comparing (16) and (17), we get

$$x_{j,n}\left[\int_{\theta_a}^{\theta_b}\mathrm{d}\tilde{\mu} - (1-|\delta_n|)\|\rho_{n-1}\|^2\right] = \int_{\theta_a}^{\theta_b} t\mathrm{d}\tilde{\mu},$$

and since the expression between square brackets is positive because $Q_{n-1} = \rho_{n-1} + P_{n-2}$, $P_{n-2} \in \mathbb{P}_{n-2}$, so that

$$\int_{\theta_a}^{\theta_b}\mathrm{d}\tilde{\mu} = \int_{\theta_a}^{\theta_b}|Q_{n-1}(t)|^2\mathrm{d}\mu(\theta) = \|Q_{n-1}\|^2 \geq \|\rho_{n-1}\|^2,$$

we have

$$\arg(x_{j,n}) = \arg\left(\int_{\theta_a}^{\theta_b} t\frac{|Q_{n-1}(t)|^2}{\|Q_{n-1}\|}\,\mathrm{d}\mu(\theta)\right) = \arg\left(\int_{\theta_a}^{\theta_b} t\mathrm{d}\hat{\mu}(\theta)\right), \quad j = 1, \ldots, n, \tag{18}$$

where $\mathrm{d}\hat{\mu}$ is a probability measure. Now,

$$\int_{\theta_a}^{\theta_b} t\mathrm{d}\hat{\mu}(\theta)$$

is a Riemann-Stieljes integral of $t = e^{\mathrm{i}\theta}$ with respect to $\hat{\mu}(\theta)$, that is, it is the limit of convex combinations of points on the arc Γ (Riemann-Stieljes sums of $e^{\mathrm{i}\theta}$ with $\theta \in [\theta_a, \theta_b]$, $\theta_b - \theta_a \leq \pi$). Therefore

$$\theta_a < \arg\left(\int_{\theta_a}^{\theta_b} e^{\mathrm{i}\theta}\frac{|Q_{n-1}(t)|^2}{\|Q_{n-1}\|^2}\mathrm{d}\mu\right) < \theta_b,$$

and by (18) this means that $x_{j,n} \in (a, b)$, $j = 1, \ldots, n$. □

By a combination of Theorems 2.5 and 2.6 we can now formulate

Theorem 2.7. *Let μ be a positive Borel measure supported on an arc $\Gamma \subset \mathbb{T}$ with $|\Gamma| \leq \pi$. Then there exists a Szegő quadrature formula $I_n[f]$ of the form (2) exact in $\mathcal{L}_{2n-1}(-d_n^2)$ where $d_n = \rho_n(0)/|\rho_n(0)|$ and its nodal polynomial $W_n(z) = z\rho_{n-1}(z) + d_n\rho_{n-1}^*(z)$ is a monic paraorthogonal polynomial of degree n with minimum norm.*

All this points to the fact that the role played by the space of polynomials $\mathbb{P}_{2n-1}$ in the context of Gauss quadrature formulas for a measure supported on a finite interval, is in the case of Szegő formulas for measures supported on an arc of the unit circle played by the space $\mathcal{L}_{2n-1}(-d_n^2)$.

2.2. Error estimate and convergence

Let us now investigate the error of the Szegő quadrature formulas when the integrand is meromorphic in a neighborhood of Γ.

From now on, we shall assume that W_n is paraorthogonal verifying (15), has all its single zeros in Γ. When the regularity of the integrand reduces to a certain order of differentiation, the estimates obtained by some of the authors in [3] can be used. However, an interesting consequence of the *Markov approach* is the following expression for the remainder term $E_n^S[f] = I[f] - I_n^S[f]$ when the integrand is analytic in a simply connected domain $\Omega \supset \Gamma$ up to a finite number of isolated poles in $\Omega\backslash\Gamma$.

For this purpose, we denote by $G(z) = G_\Gamma(z;\infty)$ the unique conformal mapping of $\hat{\mathbb{C}}\backslash\Gamma$ onto $\hat{\mathbb{C}}\backslash(\mathbb{D}\cup\mathbb{T})$ such that $G(\infty) = \infty$ and $G'(\infty) > 0$. We define $H(z) = G_*(z)G(z)$, which is an analytic function on $\mathbb{C}\backslash\Gamma$ satisfying $|H(z)| > 1$ for all $z \in \mathbb{C}\backslash\Gamma$ and $|H(z)| = 1$, $\forall z \in \Gamma$ and the level sets $\gamma(r) = \{u \in \hat{\mathbb{C}}\backslash\Gamma : |H(u)| = r\}$ for H, then $\mathfrak{L} = \{\gamma(r) : r > 1\}$ is a family of Jordan curves encircling Γ for all $r > 1$. They are nested in the sense that the interior of curve $\gamma(r_1)$ is contained in the interior of $\gamma(r_2)$ if and only if $1 < r_1 < r_2$.

First, we consider analytic integrands in Ω. As a consequence of the interpolation conditions and given that f is analytic in Ω, we know that $\frac{u^n(f(u)-H_{2n-1}^f(u))}{W_n^2(u)}$ is analytic in Ω. Let $\gamma_\Omega = \gamma(s)$ be the curve in the $\mathfrak{L}$ with the largest interior contained in $\hat{\Omega}$. If $\frac{1}{s} < \rho < 1$, then the interior of $\gamma_\rho = \gamma(\rho s)$ is contained in the interior of γ_Ω. By Cauchy's theorem

$$\frac{u^n\left(f(u) - H_{2n-1}^f(u)\right)}{W_n^2(u)} = \frac{1}{2\pi\mathrm{i}}\int_{\gamma_\rho} \frac{z^n\left(f(z) - H_{2n-1}^f(z)\right)}{(z-u)W_n^2(z)}\mathrm{d}z, \quad u \in \Gamma,$$

we obtain the classical integral expression for the interpolation error

$$f(u) - H_{2n-1}^f(u) = \frac{W_n^2(u)}{2\pi\mathrm{i}u^n}\int_{\gamma_\rho} \frac{z^n f(z)}{(z-u)W_n^2(z)}\mathrm{d}z, \quad u \in \Gamma.$$

Since

$$\frac{1}{2\pi\mathrm{i}}\int_{\gamma_\rho} S(z)\mathrm{d}z = 0, \quad \text{for} \quad S(z) = \frac{z^n H_{2n-1}^f(z)}{(z-u)W_n^2(z)}, \quad u \in \Gamma,$$

because S is holomorphic in $\hat{\mathbb{C}}\backslash\Gamma$, and as $S(z) = \mathcal{O}(z^{-2})$ for $z \to \infty$. It follows that the quadrature error verifies,

$$E_n^S[f] = I[f] - I_n^S[f] = \int_{\theta_a}^{\theta_b} \frac{W_n^2(t)}{t^n} \frac{1}{2\pi \mathrm{i}} \int_{\gamma_\rho} \frac{z^n f(z)}{(z-t)W_n^2(z)} \mathrm{d}z \, \mathrm{d}\mu(\theta), \quad t = e^{\mathrm{i}\theta}.$$

By Fubini's theorem

$$E_n^S[f] = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_\rho} \int_{\theta_a}^{\theta_b} \frac{W_n^2(t)}{t^n(z-t)} \mathrm{d}\mu(\theta) \frac{z^n f(z)}{W_n^2(z)} \mathrm{d}z.$$

As W_n is invariant, $\overline{W_n(0)}W_n(z) = W_n^*(z) = z^n W_{n*}(z)$, we deduce the following error expression for the quadrature formula

$$E_n^S[f] = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_\rho} \int_{\theta_a}^{\theta_b} \frac{|W_n(t)|^2}{z-t} \mathrm{d}\mu(\theta) \frac{f(z)}{W_n(z)W_{n*}(z)} \mathrm{d}z,$$

and it follows that

$$\left|E_n^S[f]\right| \leq \frac{\|f\|_{\infty,\gamma_\rho} |\gamma_\rho|}{2\pi d(\Gamma, \gamma_\rho)} \max_{z \in \gamma_\rho} \frac{\|W_n\|^2}{|W_n(z)| \, |W_{n*}(z)|}, \quad \rho < 1,$$

where $\|f\|_{\infty,\gamma_\rho} = \max_{v \in \gamma_\rho} |f(v)|$, $|\gamma_\rho|$ is the length of the curve γ_ρ and $d(\Gamma, \gamma_\rho)$ is the distance from Γ to γ_ρ. Now we shall be able to obtain an estimation for the rate of convergence for Szegő quadrature formula. Indeed, if the measure μ is regular, see [2, Theorems 4.3 and Remark 5], we have

$$\lim_{n\to\infty} \|W_n\|^{\frac{1}{n}} = \mathrm{cap}(\Gamma),$$

where $\mathrm{cap}(\Gamma)$ is the logarithmic capacity of Γ and

$$\lim_{n\to\infty} |W_n(z)|^{\frac{1}{n}} = \mathrm{cap}(\Gamma)|G(z)|,$$

uniformly on compact subsets of $\mathbb{C}\backslash\Gamma$. Then

$$\limsup_{n\to\infty} \left|E_n^S[f]\right|^{\frac{1}{n}} \leq \frac{1}{\rho s}, \quad \rho < 1$$

and we conclude

$$\limsup_{n\to\infty} \left|E_n^S[f]\right|^{\frac{1}{n}} \leq \frac{1}{s}.$$

Therefore, we have proved the following theorem.

Theorem 2.8. *Let μ be a regular positive Borel measure supported on $\Gamma = [a, b] \subset \mathbb{T}$. Assume that the corresponding paraorthogonal polynomials verify (15), having its n simple*

zeros in Γ, $\forall n \in \mathbb{N}$. *Let* $\{I_n^S[f]\}_{n\in\mathbb{N}}$ *be the corresponding sequence of Szegő quadrature formulas and* f *an analytic function on a simply connected domain* $\Omega \supset \Gamma$. *Then,*

$$\limsup_{n\to\infty} \left|I[f] - I_n^S[f]\right|^{\frac{1}{n}} \leq \frac{1}{s},$$

where s *is such that* $\{z : |G_*(z)G(z)| < s\}$ *is the largest possible sublevel set contained in* $\hat{\Omega}$ *and* $G(z) = G_\Gamma(z, \infty)$ *the conformal mapping of* $\hat{\mathbb{C}}\backslash\Gamma$ *onto* $\hat{\mathbb{C}}\backslash(\mathbb{D}\cup\mathbb{T})$. *Moreover, if* f *is an entire function, then*

$$\lim_{n\to\infty} \left|I[f] - I_n^S[f]\right|^{\frac{1}{n}} = 0.$$

Corollary 2.9. *Let* μ *be a regular positive Borel measure on* $\Gamma \subset \mathbb{T}$ *with* $|\Gamma| \leq \pi$. *Let* $\{I_n^S[f]\}_{n\in\mathbb{N}}$ *the corresponding sequence of Szegő quadrature formulas and* f *be an analytic function on a simply connected domain* $\Omega \supset \Gamma$. *Then,*

$$\limsup_{n\to\infty} \left|I[f] - I_n^S[f]\right|^{\frac{1}{n}} \leq \frac{1}{s},$$

where s *is such that* $\{z : |G_*(z)G(z)| < s\}$ *is the largest possible sublevel set contained in* $\hat{\Omega}$ *and* $G(z) = G_\Gamma(z, \infty)$ *the conformal mapping of* $\hat{\mathbb{C}}\backslash\Gamma$ *onto* $\hat{\mathbb{C}}\backslash(\mathbb{D}\cup\mathbb{T})$. *Moreover, if* f *is an entire function, then*

$$\lim_{n\to\infty} \left|I[f] - I_n^S[f]\right|^{\frac{1}{n}} = 0.$$

Corollary 2.10. *Let* $\mu \in \mathcal{G}$ *and* $\{I_n^S[f]\}_{n\in\mathbb{N}}$ *the corresponding sequence of Szegő quadrature formulas. Let* f *be an analytic function on a simply connected domain* $\Omega \supset \Gamma$. *Then,*

$$\limsup_{n\to\infty} \left|I[f] - I_n^S[f]\right|^{\frac{1}{n}} \leq \frac{1}{s},$$

where s *is such that* $\{z : |G_*(z)G(z)| < s\}$ *is the largest possible sublevel set contained in* $\hat{\Omega}$ *and* $G(z) = G_\Gamma(z, \infty)$ *the conformal mapping of* $\hat{\mathbb{C}}\backslash\Gamma$ *onto* $\hat{\mathbb{C}}\backslash(\mathbb{D}\cup\mathbb{T})$. *Moreover, if* f *is an entire function, then*

$$\lim_{n\to\infty} \left|I[f] - I_n^S[f]\right|^{\frac{1}{n}} = 0.$$

Now, we consider integrand f analytic on an simply connected domain $\Omega \supset \Gamma$ except for a set of isolated poles $\{p_k\}_{k=1}^d \subset \Omega\backslash\Gamma$. Since f admits the expression

$$f(z) = \frac{g(z)}{(z-p_1)^{m_1}(z-p_2)^{m_2}\cdots(z-p_d)^{m_d}} = \frac{g(z)}{P(z)}, \quad g \in \mathcal{H}(\Omega),$$

then, the quadrature error for f is

$$E_n^S[f] = I[f] - I_n^S[f] = \int_{\theta_a}^{\theta_b} \frac{g(t)}{P(t)} \mathrm{d}\mu(\theta) - \sum_{k=1}^{n} \lambda_{k,n}^S \frac{g(x_{k,n}^S)}{P(x_{k,n}^S)}.$$

As g is analytic in Ω, by Cauchy's theorem $g(t)=\frac{1}{2\pi i}\int_{\gamma_\rho}\frac{g(v)}{v-t}dv$, $\forall t\in\Gamma$, so that

$$\begin{aligned}E_n^S[f]&=\int_{\theta_a}^{\theta_b}\frac{1}{2\pi\mathrm{i}}\int_{\gamma_\rho}\frac{g(v)}{v-t}\mathrm{d}v\frac{1}{P(t)}\mathrm{d}\mu(\theta)-\sum_{k=1}^{n}\lambda_{k,n}^S\frac{1}{P(x_{k,n}^S)}\frac{1}{2\pi i}\int_{\gamma_\rho}\frac{g(v)}{v-x_{k,n}^S}\mathrm{d}v\\&=\frac{1}{2\pi\mathrm{i}}\int_{\gamma_\rho}\left(\int_{\theta_a}^{\theta_b}\frac{\mathrm{d}\mu(\theta)}{(v-t)P(t)}-\sum_{k=1}^{n}\lambda_{k,n}^S\frac{1}{(v-x_{k,n}^S)P(x_{k,n}^S)}\right)g(v)\mathrm{d}v\\&=\frac{-1}{2\pi\mathrm{i}}\left(\int_{\gamma_\rho}I[F_v]-I_n^S[F_v]\right)g(v)\mathrm{d}v=\frac{-1}{2\pi\mathrm{i}}\int_{\gamma_\rho}E_n^S[F_v]\,g(v)\mathrm{d}v,\end{aligned}$$

where we have used Fubini's theorem and $F_v(t)=\frac{1}{(t-v)P(t)}$. We deduce,

$$\left|E_n^S[f]\right|\leq\frac{\|g\|_{\infty,\gamma_\rho}\,|\gamma_\rho|}{2\pi}\max_{v\in\gamma_\rho}\left|E_n^S[F_v]\right|$$

where $\|g\|_{\infty,\gamma_\rho}=\sup_{v\in\gamma_\rho}|g(v)|$ and $|\gamma_\rho|$ is the length of γ_ρ. Therefore, the control of $\left|E_n^S[f]\right|$ depends on the control of $\left|E_n^S[F_v]\right|$, $v\in\gamma_\rho$, $\rho<1$.

$$F_v(z):=\frac{1}{(z-v)(z-p_1)^{m_1}(z-p_2)^{m_2}\cdots(z-p_d)^{m_d}}=\frac{a_0}{z-v}+\sum_{k=1}^{d}\sum_{i=1}^{m_k}\frac{a_{k,i}}{(z-p_k)^i},$$

then

$$E_n^S[F_v]=a_0E_n[e_v]+\sum_{k=1}^{d}\sum_{i=1}^{m_k}a_{k,i}E_n^S[(e_{p_k})^i].$$

Essentially, it comes down to studying the error for

$$e_\zeta(t)=\frac{1}{t-\zeta}\in\mathcal{H}(\mathbb{C}\backslash\{\zeta\}).$$

By the above Theorem 2.8,

$$\lim_{n\to\infty}\left|E_n^S[e_\zeta]\right|^{\frac{1}{n}}\leq\frac{1}{|H(\zeta)|},$$

and we conclude

$$\limsup_{n\to\infty}\left|E_n^S[f]\right|^{\frac{1}{n}}\leq\max\left\{\frac{1}{|H(v)|},\frac{1}{|H(p_1)|},\ldots,\frac{1}{|H(p_d)|}\right\},\quad\forall v\in\gamma_\rho,\quad\rho<1.$$

Thus we have the following theorem.

Theorem 2.11. *Let μ be a regular positive Borel measure supported on Γ, such that the corresponding paraorthogonal polynomial verifying (15) has its n simple zeros in Γ, $\forall n\in\mathbb{N}$. Let $\{I_n^S[f]\}_{n\in\mathbb{N}}$ be the sequence of Szegő quadrature formula and f an analytic function on a simply connected domain $\Omega\supset\Gamma$ except at the poles $\{p_i\}_{i=1}^d\subset\Omega\backslash\Gamma$. Then,*

$$\limsup_{n\to\infty}\left|I[f]-I_n^S[f]\right|^{\frac{1}{n}}\leq\max\left\{\frac{1}{|H(p_1)|},\ldots,\frac{1}{|H(p_d)|}\right\},$$

where $H(z) = G_*(z)G(z)$ *with* $G(z) = G_\Gamma(z,\infty)$ *the conformal mapping of* $\hat{\mathbb{C}}\backslash\Gamma$ *onto* $\hat{\mathbb{C}}\backslash(\mathbb{D}\cup\mathbb{T})$.

As expected, the quality of convergence depends on the location of the poles.

Corollary 2.12. *Let* μ *be a regular positive Borel measure on* $\Gamma\subset\mathbb{T}$ *with* $|\Gamma|\leq\pi$. *Let* $\{I_n^S[f]\}_{n\in\mathbb{N}}$ *the corresponding sequence of Szegő quadrature formulas and* f *be an analytic function on a simply connected domain* $\Omega\supset\Gamma$ *except at the poles* $\{p_i\}_{i=1}^d\subset\Omega\backslash\Gamma$. *Let* $\{I_n^S[f]\}_{k\in\mathbb{N}}$ *be the sequence of Szegő quadrature formulas, then,*

$$\limsup_{n\to\infty}\left|I[f]-I_n^S[f]\right|^{\frac{1}{n}}\leq\max\left\{\frac{1}{|H(p_1)|},\ldots,\frac{1}{|H(p_d)|}\right\},$$

where $H(z) = G_*(z)G(z)$ *with* $G(z) = G_\Gamma(z,\infty)$ *the conformal mapping of* $\hat{\mathbb{C}}\backslash\Gamma$ *onto* $\hat{\mathbb{C}}\backslash(\mathbb{D}\cup\mathbb{T})$.

Corollary 2.13. *Let* $\mu\in\mathcal{G}$ *and* f *be an analytic function on a simply connected domain* $\Omega\supset\Gamma$ *except at the poles* $\{p_i\}_{i=1}^d\subset\Omega\backslash\Gamma$. *Let* $\{I_n^S[f]\}_{k\in\mathbb{N}}$ *be the sequence of Szegő quadrature formulas, then,*

$$\limsup_{n\to\infty}\left|I[f]-I_n^S[f]\right|^{\frac{1}{n}}\leq\max\left\{\frac{1}{|H(p_1)|},\ldots,\frac{1}{|H(p_d)|}\right\},$$

where $H(z) = G_*(z)G(z)$ *with* $G(z) = G_\Gamma(z,\infty)$ *the conformal mapping of* $\hat{\mathbb{C}}\backslash\Gamma$ *onto* $\hat{\mathbb{C}}\backslash(\mathbb{D}\cup\mathbb{T})$.

3. Markov's approach to Szegő-Radau quadrature formula

We fix as node one of the endpoints a or b of the arc $[a,b]$. Whichever choice we make, we label it z_1 to make the writing easier.

3.1. Existence and uniqueness

In this subsection, we will recover the Szegő-Radau quadrature formula exact in $\mathcal{L}_{2n-2}$ by via a new Hermite interpolation problem. Specifically, given $z_1,z_2,\ldots,z_n$ distinct numbers on $\mathbb{T}$, we fix z_1 and we take the unique Hermite Laurent polynomial $H_{2n-2}^f\in\mathcal{L}_{2n-2}$ such that

$$H_{2n-2}^f(z_j)=f(z_j),\quad j=1,\ldots,n,\quad\text{and}\quad(H_{2n-2}^f)'(z_j)=f'(z_j),\quad j=2,\ldots,n.$$

Then, we can write

$$H_{2n-2}^f(z)=\sum_{j=1}^n\tilde{A}_j(z)f(z_j)+\sum_{j=2}^n\tilde{B}_j(z)f'(z_j)\tag{19}$$

with $\tilde{A}_j,\ \tilde{B}_j\in\mathcal{L}_{2n-2}$, verifying the interpolation conditions

$$\begin{array}{llllll}\tilde{A}_j(z_k)=\delta_{jk}, & k=1,\ldots,n, & \text{and} & \tilde{A}_j'(z_k)=0, & k=2,\ldots,n, & j=1,\ldots,n\\ \tilde{B}_j(z_k)=0, & k=1,\ldots,n, & \text{and} & \tilde{B}_j'(z_k)=\delta_{jk}, & k=2,\ldots,n, & j=2,\ldots,n.\end{array}$$

Theorem 3.1. *Under the above interpolation conditions, we have the following expressions for* $\tilde{A}_1, \tilde{A}_j,\ \tilde{B}_j \in \mathcal{L}_{2n-2},\ j = 2, \ldots, n,$

1. $\tilde{B}_j(z) = \left(\frac{z_j}{z}\right)^{n-1} \frac{\ell_j^2(z)}{z - z_1}(z_j - z_1)(z - z_j), \quad j = 2, \ldots, n.$

2a. $\tilde{A}_1(z) = \left(\frac{z_1}{z}\right)^{n-1} \ell_1^2(z),$

2b. $\tilde{A}_j(z) = \left(\frac{z_j}{z}\right)^{n-1} \ell_j^2(z) + \left(\frac{n-1}{z_j} - 2\ell_j'(z_j)\right) \tilde{B}_j(z), \quad j = 2, \ldots, n.$

Proof. We begin by obtaining the expression of $\tilde{A}_1 \in \mathcal{L}_{2n-2}$. As $\tilde{A}_1(z) = \frac{A_1(z)}{z^{n-1}}$, with $A_1 \in \mathbb{P}_{2n-2}$. Since $\tilde{A}_1(z_k) = \tilde{A}_1'(z_k) = 0$, $k = 2, \ldots, n$ and $\tilde{A}_1(z_1) = 1$, we can deduce that all z_k are double zeros for A_1 except z_1 and $A_1(z_1) = z_1^{n-1}$. Therefore,

$$\tilde{A}_1(z) = \left(\frac{z_1}{z}\right)^{n-1} \ell_1^2(z).$$

Now, as $\tilde{B}_j \in \mathcal{L}_{2n-2}$ for $j = 2, \ldots, n$,

$$\tilde{B}_j(z) = \frac{B_j(z)}{z^{n-1}}, \quad B_j \in \mathbb{P}_{2n-2},$$

we have $\tilde{B}_j(z_k) = 0$, hence also $B_j(z_k) = 0$, $k = 1, \ldots, n$. It only remains to impose $\tilde{B}_j'(z_k) = \delta_{jk}$, $k = 2, \ldots, n$. Therefore we note

$$\tilde{B}_j'(z) = \frac{B_j'(z)z - B_j(z)(n-1)}{z^n},$$

so that $B_j'(z_k) = z_k^{n-1}\delta_{jk}$. Thus, all the nodes are double zeros of B_j except z_1 and z_j wich are simple zeros. So, $B_j(z) = \tilde{b}\frac{\ell_j^2(z)}{z - z_1}(z - z_j)$, $j = 2, \ldots, n$. It remains to impose $B_j'(z_j) = z_j^{n-1}$. Because

$$B_j'(z) = \tilde{b}\ell_j(z)\left(\left(\frac{\ell_j(z)}{(z - z_1)}\right)'(z - z_j) + \frac{\ell_j(z)}{z - z_1}\right),$$

we deduce $\tilde{b} = z_j^{n-1}(z_j - z_1)$. Finally, we obtain

$$B_j(z) = z_j^{n-1}\frac{\ell_j^2(z)}{z - z_1}(z_j - z_1)(z - z_j),$$

and conclude that

$$\tilde{B}_j(z) = \left(\frac{z_j}{z}\right)^{n-1} \frac{\ell_j^2(z)}{z - z_1}(z_j - z_1)(z - z_j), \quad j = 2, \ldots, n.$$

Since $\tilde{A}_j \in \mathcal{L}_{2n-2}$, for $j = 2, \ldots, n$, it holds that $\tilde{A}_j(z) = \frac{A_j(z)}{z^{n-1}}$, with $A_j \in \mathbb{P}_{2n-2}$. We have on the one hand $\tilde{A}_j(z_k) = \delta_{jk}$, hence $A_j(z_k) = z_j^{n-1}\delta_{jk}$. On the other hand, as $\tilde{A}_j'(z_k) = 0$, $k = 2, \ldots, n$ with

$$\tilde{A}_j'(z) = \frac{A_j'(z)z - A_j(z)(n-1)}{z^n},$$

we deduce that $A_j'(z_k) = z_j^{n-2}(n-1)\delta_{jk}$, $k = 2, \ldots, n$. Then, all nodes, except z_1 and z_j, are double zeros of A_j and z_1 is a simple zero. Therefore, taking into account that $A_j \in \mathbb{P}_{2n-2}$. We can write A_j in the following way

$$A_j(z) = \tilde{a}\frac{\ell_j^2(z)}{z - z_1}\big(\tilde{c} + (z - z_j)\big), \quad \tilde{a}, \tilde{c} \in \mathbb{C}.$$

Thus, since $A_j(z_j) = z_j^{n-1}$, we get that $\tilde{a}\tilde{c} = z_j^{n-1}(z_j - z_1)$, so that

$$A_j(z) = z_j^{n-1}(z_j - z_1)\frac{\ell_j^2(z)}{z - z_1}\Big(1 + \frac{1}{\tilde{c}}(z - z_j)\Big)$$

On the other hand, since $A_j'(z_j) = z_j^{n-2}(n-1)$ and

$$A_j'(z) = z_j^{n-1}(z_j - z_1)\ell_j(z)\left(\frac{2\ell_j'(z)(z - z_1) - \ell_j(z)}{(z - z_1)^2}\Big(1 + \frac{1}{\tilde{c}}(z - z_j)\Big) + \frac{1}{\tilde{c}}\frac{\ell_j(z)}{z - z_1}\right),$$

we deduce

$$(n-1) = z_j\frac{2\ell_j'(z_j)(z_j - z_1) - 1}{z_j - z_1} + \frac{z_j}{\tilde{c}} \quad \text{or} \quad \frac{1}{\tilde{c}} = \frac{n-1}{z_j} - \frac{2\ell_j'(z_j)(z_j - z_1) - 1}{z_j - z_1}.$$

It follows that

$$\begin{aligned}
A_j(z) &= z_j^{n-1}(z_j - z_1)\frac{\ell_j^2(z)}{z - z_1}\Big(1 + \frac{1}{\tilde{c}}(z - z_j)\Big) \\
&= z_j^{n-1}(z_j - z_1)\frac{\ell_j^2(z)}{z - z_1}\left[1 + \left(\frac{n-1}{z_j} - 2\ell_j'(z_j) + \frac{1}{z_j - z_1}\right)(z - z_j)\right] \\
&= z_j^{n-1}\frac{\ell_j^2(z)}{z - z_1}[(z_j - z_1) + (z - z_j)] + \left(\frac{n-1}{z_j} - 2\ell_j'(z_j)\right) z_j^{n-1}(z_j - z_1)\frac{\ell_j^2(z)}{z - z_1}(z - z_j) \\
&= z_j^{n-1}\ell_j^2(z) + \left(\frac{n-1}{z_j} - 2\ell_j'(z_j)\right) B_j(z)
\end{aligned}$$

and the proof follows. □

If we integrate both sides of formula (19), we obtain the following

$$I[H_{2n-2}^f] = I[\tilde{A}_1)]f(z_1) + \sum_{j=2}^{n} I[\tilde{A}_j]f(z_j) + \sum_{j=2}^{n} I[\tilde{B}_j]f'(z_j).$$

Since our objective is to construct a quadrature formula not involving derivatives, we impose that

$$I[\tilde{B}_j] = 0, \quad j = 2, \ldots, n, \tag{20}$$

so that we get

$$I_n[f] = \lambda_1 f(z_1) + \sum_{j=2}^{n} \lambda_j f(z_j),$$

where $\lambda_j = I[\tilde{A}_j]$, $j = 1, \ldots, n$.

Similarly as we have done in Section 2, we can recover with the same arguments, the positivity of the weights $\{\lambda_j\}_{j=1}^n$ and obtain a computable expression for them.

Theorem 3.2. *Let $\lambda_j = I[\tilde{A}_j]$, $j = 1, \ldots, n$. If (20) holds, then*

$$\lambda_j = I\big[|\ell_j|^2\big] > 0, \quad j = 1, \ldots, n.$$

Once a node has been fixed at one of the endpoints of the arc $[a, b]$, it remains to guarantee that the rest of the nodes are simple and lie on the open arc (a, b). A first step is to verify that (20) provides the appropriate orthogonality conditions, i.e., to check if the nodal polynomial W_n is paraorthogonal with respect to the measure μ. We have the following result.

Theorem 3.3. *If (20) holds, then the nodal polynomial W_n satisfies the following orthogonality conditions*

$$\big\langle W_n, z^k \big\rangle = 0, \quad k = 1, \ldots, n-1.$$

Proof. For $j = 2, \ldots, n$, we have

$$\begin{aligned}\tilde{B}_j(z) &= \Big(\frac{z_j}{z}\Big)^{n-1} \frac{\ell_j^2(z)}{z - z_1}(z_j - z_1)(z - z_j) = \frac{z_j^{n-1}}{W_n'(z_j)} W_n(z) \frac{(z_j - z_1)\ell_j(z)}{z - z_1} \frac{1}{z^{n-1}} \\ &= \frac{z_j^{n-1}}{W_n'(z_j)} W_n(z) \tilde{\ell}_j(z) e_0^{n-1}(z), \quad \tilde{\ell}_j(z) = \frac{(z_j - z_1)\ell_j(z)}{z - z_1}, \quad e_0(z) = \frac{1}{z}\end{aligned}$$

Since $I[\tilde{B}_j] = 0$, then

$$I\left[W_n \tilde{\ell}_j e_0^{n-1}\right] = 0, \quad j = 2, \ldots, n.$$

Observe that $\left\{\tilde{\ell}_j\right\}_{j=2}^n$ are the fundamental Lagrange polynomials for the nodes $z_2, \ldots, z_n$, hence, it is a basis for $\mathbb{P}_{n-2}$. Therefore,

$$\langle W_n, z^{n-1-k} \rangle = 0, \quad k = 0, \ldots, n-2,$$

and the proof follows. □

By [3, Theorem 5.2], we know that for any measure μ, W_n is the paraorthogonal polynomial given by

$$W_n(z) = z\rho_{n-1}(z) - \frac{z_1\rho_{n-1}(z_1)}{\rho^*_{n-1}(z_1)}\rho^*_{n-1}(z) \tag{21}$$

and it has exactly $n-1$ distinct nodes on (a,b), in addition to z_1.

Theorem 3.4. *Let $n \geq 2$ and μ be a positive Borel measure supported on an $\Gamma \subset \mathbb{T}$. Then there exists a unique quadrature formula $I_n[f]$ of the form (2) that is exact in $\mathcal{L}_{2n-2}$. Its nodal polynomial is the paraorthogonal polynomial (21)*

The corresponding quadrature formula is called Szegő-Radau quadrature formula and we shall denote it by $I_n^R[f]$ with positive weights $\{\lambda_k^R\}_{k=1}^n$ and nodes $\{z_k^R\}_{k=1}^n$ on the arc Γ.

3.2. Error estimate and convergence

When the regularity of the integrand reduces to a certain order of differentiation, the estimates obtained by the authors in [3] can be used. Now, we consider analytic integrands in simply connected domain $\Omega \supset \Gamma$ up to a finite number of isolated poles $\{p_k\}_{k=1}^m \subset \Omega\backslash\Gamma$. We will obtain a bound on the quadrature error.

For clarity, as in the previous case, we consider first analytic integrands in Ω. As a consequence of the interpolation conditions and given that f is analytic in Ω, we know that $\frac{u^{n-1}(u-z_1)(f(u)-H^f_{2n-2}(u))}{W_n^2(u)}$ is analytic in Ω. Analogous to the Szegő quadrature error

$$I[f] - I_n^R[f] = -\frac{1}{2\pi \mathrm{i}}\int_{\gamma_\rho}\int_{\theta_a}^{\theta_b} \frac{t|W_n(t)|^2}{(t-z_1)(t-z)}\mathrm{d}\mu(\theta)\frac{(z-z_1)f(z)}{zW_n(z)W_{n*}(z)}\mathrm{d}z,$$

it follows that

$$\left|I[f] - I_n^R[f]\right| \leq \frac{\|f\|_{\infty,\gamma_\rho}|\gamma_\rho|}{2\pi d(\Gamma,\gamma_\rho)}\max_{z\in\gamma_\rho}\frac{\|W_n\|\,\|W_n e_{z_1}\|}{|W_n(z)|\,|W_{n*}(z)|}, \quad \rho < 1,$$

where we have applied Schwarz's inequality and $\|f\|_{\infty,\gamma_\rho} = \max_{z\in\gamma_\rho}|\frac{z-z_1}{z}f(z)|$, $|\gamma_\rho|$ is the length of the curve γ_ρ and $d(\Gamma,\gamma_\rho)$ is the distance from Γ to γ_ρ and $\gamma_\rho = \gamma(\rho s)$ a level curve contained in the interior of $\gamma_\Omega = \gamma(s)$, which is the Jordan curve of the family $\mathfrak{L}$ with the largest interior contained in $\hat{\Omega}$, i.e., $\frac{1}{s} < \rho < 1$. For the same reason as in the previous case

$$\limsup_{n\to\infty}\left|I[f] - I_n^R[f]\right|^{\frac{1}{n}} \leq \frac{1}{s},$$

for any measure μ regular on Γ.

Theorem 3.5. *Let μ be a regular positive Borel measure supported on an Γ. Let $\{I_n^R[f]\}_{n\in\mathbb{N}}$ be the sequence of Szegő-Radau quadrature formulas and let f be an analytic function on a simply connected domain $\Omega \supset \Gamma$. Then,*

$$\limsup_{n\to\infty}\left|I[f] - I_n^R[f]\right|^{\frac{1}{n}} \leq \frac{1}{s},$$

where s is defined by the condition that $\gamma(s) = \gamma_\Omega = \{v : |G_*(v)G(v)| = s\} \in \mathfrak{L}$ is the Jordan curve with the largest interior contained in $\hat{\Omega}$ with $G(z) = G_\Gamma(z, \infty)$ the conformal mapping of $\hat{\mathbb{C}}\backslash\Gamma$ onto $\hat{\mathbb{C}}\backslash(\mathbb{D} \cup \mathbb{T})$. Moreover, if f is an entire function, then

$$\lim_{n\to\infty} \left|I[f] - I_n^R[f]\right|^{\frac{1}{n}} = 0.$$

Now, we consider integrand f analytic on a simply connected domain $\Omega \supset \Gamma$ except for a set of isolated poles $\{p_k\}_{k=1}^d \subset \Omega\backslash\Gamma$. By the same procedure as in the previous section, we obtain

Theorem 3.6. *Let μ be a regular positive Borel measure supported on Γ. Let $\{I_n^R[f]\}_{n\in\mathbb{N}}$ be the sequence of Szegő-Radau quadrature formulas and f an analytic function on a simply connected domain $\Omega \supset \Gamma$ except at the poles $\{p_i\}_{i=1}^d \subset \Omega\backslash\Gamma$. Then*

$$\limsup_{n\to\infty} \left|I(f) - I_n^R(f)\right|^{\frac{1}{n}} \leq \max\left\{\frac{1}{|H(p_1)|}, \ldots, \frac{1}{|H(p_d)|}\right\},$$

where $H(z) = G_*(z)G(z)$ with $G(z) = G_\Gamma(z, \infty)$ is the conformal mapping of $\hat{\mathbb{C}}\backslash\Gamma$ onto $\hat{\mathbb{C}}\backslash(\mathbb{D} \cup \mathbb{T})$.

4. Markov's approach to Szegő-Lobatto quadrature formula

4.1. Existence and uniqueness

To obtain the Szegő-Lobatto quadrature formulas, let's fix both endpoints of the arc $[a, b]$ as nodes of the quadrature. We shall find the unique Hermite Laurent polynomial $H_{2n-3}^f \in \mathcal{L}_{2n-3}(\omega_{n-1})$ such that

$$H_{2n-3}^f(a) = f(a), \quad H_{2n-3}^f(z_j) = f(z_j), \quad j = 2, \ldots, n-1, \quad H_{2n-3}^f(b) = f(b),$$
$$(H_{2n-3}^f)'(z_j) = f'(z_j), \ j = 2, \ldots, n-1.$$

Then, we can write

$$H_{2n-3}^f(z) = \tilde{A}_1(z)f(a) + \sum_{j=2}^{n-1} \tilde{A}_j(z)f(z_j) + \tilde{A}_n(z)f(b) + \sum_{j=2}^{n-1} \tilde{B}_j(z)f'(z_j) \tag{22}$$

with $\tilde{A}_j, \ \tilde{B}_j \in \mathcal{L}_{2n-3}(\omega_{n-1})$, verifying the interpolation conditions,

$$\tilde{A}_j(z_k) = \delta_{jk}, \quad k = 1, \ldots, n, \qquad \tilde{A}_j'(z_k) = 0, \quad k = 2, \ldots, n-1, \quad j = 1, \ldots, n,$$
$$\tilde{B}_j(z_k) = 0, \quad k = 1, \ldots, n, \qquad \tilde{B}_j'(z_k) = \delta_{jk}, \quad k = 2, \ldots, n-1, \quad j = 2, \ldots, n-1$$

where $z_1, z_2, \ldots, z_n$ are distinct numbers on $\mathbb{T}$, we have labeled $z_1 = a$ and $z_n = b$ to make writing easier.

Theorem 4.1. *Let $\overline{ab}W_n^2(0) \neq \omega_{n-1}$. Under the above interpolation conditions, we have the following expressions for $\tilde{A}_1$, $\tilde{A}_n, \tilde{A}_j$, $\tilde{B}_j \in \mathcal{L}_{2n-3}(\omega_{n-1})$, $j = 2, \ldots, n-1$, where we used the abbreviations $v_{n-1} = z_1 z_n \omega_{n-1}/W_n^2(0) \neq 1$, and $p_j(z) = z - z_j v_{n-1}$, $j = 1, \ldots, n$,*

$$1. \quad \tilde{B}_j(z) = \frac{(z_j - z_1)(z_j - z_n)}{(z - z_1)(z - z_n)} \frac{p_j(z)}{p_j(z_j)} \left(\frac{z_j}{z}\right)^{n-1} \ell_j^2(z)(z - z_j).$$

$$2a. \quad \tilde{A}_1(z) = \frac{z_1 - z_n}{z - z_n} \frac{p_1(z)}{p_1(z_1)} \left(\frac{z_1}{z}\right)^{n-1} \ell_1^2(z),$$

$$\tilde{A}_n(z) = \frac{z_n - z_1}{z - z_1} \frac{p_n(z)}{p_n(z_n)} \left(\frac{z_n}{z}\right)^{n-1} \ell_n^2(z).$$

$$2b. \quad \tilde{A}_j(z) = \frac{(z_j - z_1)(z_j - z_n)}{(z - z_1)(z - z_n)} \left(\frac{z_j}{z}\right)^{n-2} \ell_j^2(z) + \left(\frac{n-2}{z_j} - \left(2\ell_j'(z_j) - \frac{2z_j - z_1 - z_n}{(z_j - z_1)(z_j - z_n)}\right)\right) \tilde{B}_j(z).$$

Proof. Let $j = 2, \ldots, n-1$. Since $\tilde{B}_j \in \mathcal{L}_{2n-3}(\omega_{n-1})$, then $\tilde{B}_j(z) = \frac{B_j(z)}{z^{n-1}}$, where $B_j \in \mathbb{P}_{2n-2}$ and

$$B_j(z) = \tilde{b} z^{2n-2} + z q(z) + \tilde{b}\omega_{n-1}, \quad q \in \mathbb{P}_{2n-4}, \quad \tilde{b} \in \mathbb{C}. \tag{23}$$

From the interpolation conditions for $\tilde{B}_j$, we have that $B_j(z_k) = 0$, $k = 1, \ldots, n$ and since $\tilde{B}_j'(z) = \frac{B_j'(z)z - B_j(z)(n-1)}{z^n}$ and $\tilde{B}_j'(z_k) = \delta_{jk}$, $k = 2, \ldots, n-1$, we deduce

$$B_j'(z_k) = z_k^{n-1}\delta_{jk}, \quad k = 2, \ldots, n-1. \tag{24}$$

Thus, all nodes are double zeros of B_j, except z_1, z_n and z_j which are simple zeros. Therefore, we can write

$$B_j(z) = \tilde{b}\left(W_n'(z_j)\right)^2 \frac{\ell_j^2(z)}{(z - z_1)(z - z_n)}(z - z_j)(z + d), \quad d \in \mathbb{C}. \tag{25}$$

At $z = 0$, from (23) and (25), we deduce

$$\tilde{b}\omega_{n-1} = \tilde{b}\frac{W_n^2(0)}{(-z_1 z_n z_j)} d,$$

and it follows that

$$d = -z_1 z_n z_j \frac{\omega_{n-1}}{W_n^2(0)} = -z_j v_{n-1}, \qquad v_{n-1} = z_1 z_n \frac{\omega_{n-1}}{W_n^2(0)}.$$

Recapitulating

$$B_j(z) = \tilde{b}\big(W_n'(z_j)\big)^2 \frac{\ell_j^2(z) p_j(z)(z - z_j)}{(z - z_1)(z - z_n)}.$$

From (24) we have that $B_j'(z_j) = z_j^{n-1}$ and as

$$B_j'(z) = \tilde{b}\,(W_n'(z_j))^2 \left\{ \left(\frac{\ell_j^2(z)p_j(z)}{(z-z_1)(z-z_n)} \right)' (z-z_j) + \frac{\ell_j^2(z)p_j(z)}{(z-z_1)(z-z_n)} \right\}.$$

By evaluating at $z = z_j$, we deduce

$$z_j^{n-1} = \tilde{b}\big(W_n'(z_j)\big)^2 \frac{p_j(z_j)}{(z_j-z_1)(z_j-z_n)} \quad \text{or} \quad \tilde{b}\big(W_n'(z_j)\big)^2 = \frac{z_j^{n-1}(z_j-z_1)(z_j-z_n)}{p_j(z_j)},$$

so that B_j takes the form

$$B_j(z) = \frac{(z_j-z_1)(z_j-z_n)}{(z-z_1)(z-z_n)} \frac{p_j(z)}{p_j(z_j)} z_j^{n-1} \ell_j^2(z)(z-z_j).$$

Since $\tilde{A}_1 \in \mathcal{L}_{2n-3}(\omega_{n-1})$, we have $\tilde{A}_1(z) = \frac{A_1(z)}{z^{n-1}}$, where $A_1 \in \mathbb{P}_{2n-2}$ satisfies

$$A_1(z) = \tilde{a}z^{2n-2} + zq(z) + \tilde{a}\omega_{n-1}, \quad q \in \mathbb{P}_{2n-4}, \quad \tilde{a} \in \mathbb{C}. \tag{26}$$

From the interpolation conditions for $\tilde{A}_1$, we have on the one hand that $A_1(z_k) = z_k^{n-1}\delta_{1k}$, $k = 1, \ldots, n$ and the other hand, since $\tilde{A}_1'(z) = \frac{A_1'(z)z - A_1(z)(n-1)}{z^n}$ and $\tilde{A}_1'(z_k) = 0$, $k = 2, \ldots, n-1$, we deduce

$$A_1'(z_k) = 0, \quad k = 2, \ldots, n-1.$$

Thus, all nodes are double zeros of A_1 except z_1 and z_n, where z_n is a simple zero. Therefore, we can write, in terms of the fundamental Lagrange polynomials,

$$A_1(z) = \tilde{a}\big(W_n'(z_1)\big)^2 \frac{\ell_1^2(z)}{z-z_n}(z+c), \quad c \in \mathbb{C}. \tag{27}$$

For $z = 0$, from (26) and (27), we obtain $\tilde{a}\omega_{n-1} = \tilde{a}\frac{W_n^2(0)}{-z_n z_1^2}c$, i.e., $c = -\frac{z_1^2 z_n \omega_{n-1}}{W_n^2(0)} = -z_1 v_{n-1}$. Recapitulating,

$$A_1(z) = \tilde{a}\big(W_n'(z_1)\big)^2 \frac{\ell_1^2(z)}{z-z_n} p_1(z).$$

In addition, since $A_1(z_1) = z_1^{n-1}$, we have $z_1^{n-1} = \tilde{a}\big(W_n'(z_j)\big)^2 \frac{p_1(z_1)}{z_1-z_n}$ and so

$$\tilde{a}\big(W_n'(z_j)\big)^2 = z_1^{n-1} \frac{z_1-z_n}{p_1(z_1)}.$$

Therefore

$$A_1(z) = z_1^{n-1} \frac{z_1-z_n}{p_1(z_1)} \frac{\ell_1^2(z)}{z-z_n} p_1(z),$$

and thus

$$\tilde{A}_1(z) = \frac{(z_1-z_n)\ell_1^2(z)}{z-z_n} \frac{p_1(z)}{p_1(z_1)} \left(\frac{z_1}{z}\right)^{n-1}.$$

Similarly, if we replace z_1 by z_n we can obtain the expression for $\tilde{A}_n$.

For $j = 2, \ldots, n-1$, $\tilde{A}_j \in \mathcal{L}_{2n-3}(\omega_{n-1})$, implying that $\tilde{A}_j(z) = \frac{A_j(z)}{z^{n-1}}$, where $A_j \in \mathbb{P}_{2n-2}$ satisfies

$$A_j(z) = \tilde{a} z^{2n-2} + z q(z) + \tilde{a}\omega_{n-1}, \quad q \in \mathbb{P}_{2n-4}, \quad \tilde{a} \in \mathbb{C}. \tag{28}$$

and $\tilde{A}_j'(z) = \frac{A_j'(z)z - A_j(z)(n-1)}{z^n}$. From the interpolation conditions for $\tilde{A}_j$, we have that $A_j(z_k) = z_k^{n-1}\delta_{jk}$, $k = 1, \ldots, n$ and $A_j'(z_k) = \delta_{jk}(n-1)z_k^{n-2}$, $k = 2, \ldots, n-1$. Thus, the fixed nodes z_1 and z_n are simple zeros and the rest of the nodes except z_j are double zeros of A_j. Therefore, we can write

$$A_j(z) = \tilde{a}\big(W_n'(z_j)\big)^2 \frac{\ell_j^2(z)}{(z-z_1)(z-z_n)} q_2(z) \tag{29}$$

where $q_2 \in \mathbb{P}_2^1$. Since we want to write A_j in terms of B_j, we will take in $\mathbb{P}_2$ the basis $\{1, z-z_j, (z-z_j)p_j(z)\}$. Then

$$q_2(z) = \hat{c} + \hat{b}(z-z_j) + (z-z_j)p_j(z), \quad \hat{c}, \hat{b} \in \mathbb{C}.$$

Now, we evaluate both (28) and (29) at $z = 0$, we can obtain

$$\tilde{a}\omega_{n-1} = \frac{\tilde{a}\omega_{n-1}}{z_j^2 v_{n-1}}(\hat{c} - \hat{b}z_j + z_j^2 v_{n-1}),$$

we deduce $\hat{c} = \hat{b}z_j$, then

$$q_2(z) = \hat{b}z + (z-z_j)p_j(z).$$

On the one hand, since $A_j(z_j) = z_j^{n-1}$, by (29), we have that

$$z_j^{n-1} = \tilde{a}\left(W_n'(z_j)\right)^2 \frac{1}{(z_j-z_1)(z_j-z_n)} \hat{b} z_j.$$

Then, the polynomial A_j takes the form

$$\begin{aligned} A_j(z) &= \tilde{a}\left(W_n'(z_j)\right)^2 \frac{\ell_j^2(z)}{(z-z_1)(z-z_n)} q_2(z) \\ &= z_j^{n-1} \frac{(z_j-z_1)(z_j-z_n)}{(z-z_1)(z-z_n)} \ell_j^2(z) \frac{1}{\hat{b}z_j} q_2(z) \\ &= z_j^{n-2} \frac{(z_j-z_1)(z_j-z_n)}{(z-z_1)(z-z_n)} \ell_j^2(z) \Big(z + \frac{1}{\hat{b}}(z-z_j)p_j(z) \Big). \end{aligned}$$

On the other hand,

$$A_j'(z) = z_j^{n-2}(z_j-z_1)(z_j-z_n) \left\{ \left(\frac{\ell_j^2(z)}{(z-z_1)(z-z_n)} \right)' \hat{q}_2(z) + \frac{\ell_j^2(z)}{(z-z_1)(z-z_n)} \hat{q}_2'(z) \right\},$$

where $\hat{q}_2(z) = z + \frac{1}{\hat{b}}(z - z_j)p_j(z)$, since

$$\left(\frac{\ell_j^2(z)}{(z-z_1)(z-z_n)}\right)' = \ell_j(z)\frac{2\ell_j'(z)(z-z_1)(z-z_n) - \ell_j(z)(2z-z_1-z_n)}{(z-z_1)^2(z-z_n)^2}.$$

As $A_j'(z_j) = (n-1)z_j^{n-2}$, $\hat{q}_2(z_j) = z_j$ and $\hat{q}_2'(z_j) = 1 + \frac{1}{\hat{b}}p_j(z_j)$, we deduce

$$(n-1)z_j^{n-2} = z_j^{n-2}\left\{\left(2\ell_j'(z_j) - \frac{2z_j - z_1 - z_n}{(z_j-z_1)(z_j-z_n)}\right)z_j + 1 + \frac{1}{\hat{b}}p_j(z_j)\right\},$$

or equivalently,

$$\frac{1}{\hat{b}} = \frac{(n-2) - z_j\left(2\ell_j'(z_j) - \frac{2z_j-z_1-z_n}{(z_j-z_1)(z_j-z_n)}\right)}{p_j(z_j)}.$$

Then, the polynomial A_j admits the expression

$$\begin{aligned} A_j(z) = {} & z_j^{n-2} z\frac{(z_j-z_1)(z_j-z_n)}{(z-z_1)(z-z_n)}\ell_j^2(z) + \\ & + z_j^{n-2}\tfrac{(z_j-z_1)(z_j-z_n)}{(z-z_1)(z-z_n)}\ell_j^2(z)\left[(n-2) - z_j\left(2\ell_j'(z_j) - \tfrac{2z_j-z_1-z_n}{(z_j-z_1)(z_j-z_n)}\right)\right](z-z_j)\tfrac{p_j(z)}{p_j(z_j)}. \end{aligned}$$

Equivalently,

$$A_j(z) = z_j^{n-2}\frac{z(z_j-z_1)(z_j-z_n)\ell_j^2(z)}{(z-z_1)(z-z_n)} + \left(\frac{n-2}{z_j} - \left(2\ell_j'(z_j) - \frac{2z_j-z_1-z_n}{(z_j-z_1)(z_j-z_n)}\right)\right)B_j(z).$$

□

Now, by integrating both sides of the formula (22), we obtain

$$I[H_{2n-3}^f] = f(a)I[\tilde{A}_1] + \sum_{j=2}^{n-1} I[\tilde{A}_j]f(z_j) + f(b)I[\tilde{A}_n] + \sum_{j=2}^{n-1} I[\tilde{B}_j]f'(z_j).$$

Since our goal is to construct a quadrature formula, we should impose that

$$I[\tilde{B}_j] = 0, \quad j = 2, \ldots, n-1, \tag{30}$$

and we get

$$I_n[f] = \lambda_1 f(a) + \sum_{j=2}^{n-1}\lambda_j f(z_j) + \lambda_n f(b),$$

where $\lambda_j = I_\omega[\tilde{A}_j]$, $j = 1, \ldots, n$. We know that if this quadrature formula exists, it has at least $n-1$ weights that are positive, see [4, Theorem 2.3]. We shall show that all the weights are positive is equivalent to all the nodes laying on the arc.

Theorem 4.2. *Let $\overline{ab}(W_n(0))^2 \neq \omega_{n-1}$ and assume (30) holds. The weights $\lambda_1 = I[\tilde{A}_1], \lambda_n = I[\tilde{A}_n]$ are positive. Moreover, all nodes are on $[a,b]$ iff all the weights $\lambda_j = I[\tilde{A}_j]$, $j = 1,\dots,n$ are positive.*

Proof. As we noted above, by [4, Theorem 2.3], all the weights are real and at least $n-1$ of the weights are positive. We first determine that weights λ_1 and λ_n are both positive.

$$\tilde{A}_1(z) = \ell_1^2(z) M(z, z_1, z_n) \left(\frac{z_1}{z}\right)^{n-1}, \quad M(z, z_1, z_n) = \frac{(z_1 - z_n)p_1(z)}{(z - z_n)p_1(z_1)}.$$

Note that $M(z, z_1, z_n)$ is a rational function of degree 1, therefore, it is an injective function such that $M(t, z_1, z_n) \in \mathbb{R}$, $\forall t \in \mathbb{T}$ since $z_1, z_n, v_{n-1} \in \mathbb{T}$. Clearly $M(z_1, z_1, z_n) = 1$, $M(z_n, z_1, z_n) = \infty$, and $M(t_0, z_1, z_n) = 0$, for $t_0 = z_1 v_{n-1} = a^2 b \frac{\omega_{n-1}}{(W_n(0))^2}$. Then $M(t, z_1, z_n) > 0$, $\forall t \in \Gamma$ iff $t_0 \notin \Gamma$ iff $ab\frac{\omega_{n-1}}{(W_n(0))^2} = e^{-\mathrm{i}\theta_0}$, with $\theta_0 \in (0, 2\pi - |\Gamma|)$ (note that $b\overline{a} = e^{\mathrm{i}(\theta_b - \theta_a)}$). Hence, under these conditions

$$\tilde{A}_1(t) = \ell_1^2(t) \left(\frac{z_1}{t}\right)^{n-1} M(t, z_1, z_n) = |\ell_1(t)|^2 M(t, z_1, z_n) > 0, \quad \forall t \in \Gamma,$$

and we can conclude that $\lambda_1 > 0$.

With a similar argument but using $M(z, z_n, z_1)$, we also deduce that $\lambda_n > 0$ under the same condition.

Therefore, if there is a negative weight, it must be associated with a node different from the boundary points a and b. So we now show that the weights for the other nodes are positive iff they are in the open arc (a, b).

For $j = 2, \dots, n-1$, taking into account (30), we have here $\lambda_j = I[|\ell_j|^2 b_j]$ where $b_j(z) = \frac{z}{z_j}\frac{(z_j - z_1)(z_j - z_n)}{(z - z_1)(z - z_n)}$ is a rational function of degree 2. Since, $\overline{b_j(t)} = b_j(t)$, $t \in \mathbb{T}$ and $b_j(z_j) = 1$, $b_j(z_1) = b_j(z_n) = \infty$ and $b_j(0) = b_j(\infty) = 0$, we conclude that $b_j(t)$ is either positive for $t \in \Gamma$ and negative for $t \notin \Gamma$, or the other way around. And since $b_j(z_j) = 1$, it holds that $z_j \in (z_1, z_n) \Leftrightarrow b_j(t) > 0$, $\forall t \in (z_1, z_n) \Leftrightarrow \lambda_j = I[|\ell_j|^2 b_j] > 0$. □

We must check if requirement (30) is consistent with our objectives. That is, if the nodal polynomial, in addition to having the endpoints of the arc as zeros, has the rest of its zeros simple and lying in the interior of arc.

Theorem 4.3. *If $\overline{ab}(W_n(0))^2 \neq \omega_{n-1}$ and (30) holds, then the nodal polynomial W_n satisfies the following orthogonality conditions*

$$\langle W_n, z^k \rangle = 0, \quad k = 2, \dots, n-2, \quad \textit{and} \quad \langle W_n, z^{n-1} + \tfrac{\omega_{n-1}}{W_n(0)} z \rangle = 0. \tag{31}$$

Proof. Observe that

$$\begin{aligned}\tilde{B}_j(z) &= \frac{z_j^{n-1}}{W_n'(z_j)p_j(z_j)} W_n(z) \frac{(z_j - z_1)(z_j - z_n)\ell_j(z)}{(z - z_1)(z - z_n)} p_j(z) \frac{1}{z^{n-1}} \\ &= \frac{z_j^{n-1}}{W_n'(z_j)p_j(z_j)} W_n(z) \tilde{\ell}_j(z) p_j(z) e_0^{n-1}(z), \quad e_0(z) = z^{-1},\end{aligned}$$

where $\tilde{\ell}_j(z) = \frac{(z_j - z_1)(z_j - z_n)\ell_j(z)}{(z - z_1)(z - z_n)} \in \mathbb{P}_{n-3}$, $j = 2, \ldots, n-1$.

Hence, if $I[\tilde{B}_j] = 0$, $j = 2, \ldots, n-1$, then

$$0 = I\left[W_n \tilde{\ell}_j p_j e_0^{n-1}\right] = \left\langle W_n, z^{n-1}\overline{\tilde{\ell}_j p_j}\right\rangle.$$

Note that $\left\{\tilde{\ell}_j\right\}_{j=2}^{n-1}$ are the fundamental Lagrange polynomials for the nodes $z_2, \ldots, z_{n-1}$. Let's prove that the system $\left\{\tilde{\ell}_j p_j\right\}_{j=2}^{n-1}$ is a basis for $\hat{\mathbb{P}}_{n-1} = \operatorname{span}\left\{z, \ldots, z^{n-3}, z^{n-2} + \frac{\omega_{n-1}}{W_n(0)}\right\}$. Observe that the leading coefficient of $\tilde{\ell}_j(z)p_j(z)$ is given by $\frac{(z_j - z_1)(z_j - z_n)}{W_n'(z_j)}$ and the independent coefficient by

$$\begin{aligned}
-\tilde{\ell}_j(0) z_j v_{n-1} &= -(z_j - z_1)(z_j - z_n)\frac{\ell_j(0)}{z_1 z_n} z_1 z_n z_j \frac{\omega_{n-1}}{W_n^2(0)} \\
&= (z_j - z_1)(z_j - z_n)\frac{W_n(0)}{W_n'(z_j)}\frac{\omega_{n-1}}{W_n^2(0)} \\
&= \frac{(z_j - z_1)(z_j - z_n)}{W_n'(z_j)}\frac{\omega_{n-1}}{W_n(0)}.
\end{aligned}$$

Therefore, we have that $\tilde{\ell}_j(z)p_j(z) \in \hat{\mathbb{P}}_{n-1}$, $j = 2, \ldots, n-1$. We shall prove now that the system $\left\{\tilde{\ell}_j(z)p_j(z)\right\}_{j=2}^{n-1}$ is linearly independent. In fact, let

$$\sum_{j=2}^{n-1} \alpha_j \tilde{\ell}_j(z) p_j(z) = 0, \quad \alpha_j \in \mathbb{C}.$$

Then for $z = z_k$ and taking into account that $\tilde{\ell}_j(z_k) = \delta_{jk}$ and $p_k(z_k) \neq 0$, we get $\alpha_k = 0$, for $k = 2, \ldots, n-1$. □

In other words, the nodal polynomial W_n is a quasi-paraorthogonal of order 3 with the additional orthogonality condition.

$$\left\langle W_n, z^{n-1} + \frac{\omega_{n-1}}{W_n(0)} z \right\rangle = 0. \tag{32}$$

We know that every quasi-paraorthogonal of order 3 has $n-3$ simple zeros in (a, b), see [1, Theorem 3.2]. Therefore, our candidate has $n-1$ zeros in $[a, b]$, it remains to check if condition (32) provides a way to locate the other zero in (a, b). If we analyze the orthogonality conditions (31), we can verify that if $W_n(z) = Q_{n-2}(z)(z-a)(z-b)$, then Q_{n-2} is a monic paraorthogonal polynomial with respect to the new measure on Γ

$$\mathrm{d}\hat{\mu}(\theta) = \frac{\sqrt{\overline{ab}}}{4}(t-a)(b-t)\bar{t}\mathrm{d}\mu(\theta) = \sin\Big(\frac{\theta - \theta_a}{2}\Big)\sin\Big(\frac{\theta_b - \theta}{2}\Big)\mathrm{d}\mu(\theta),$$

with the additional condition

$$\int_{\theta_a}^{\theta_b} Q_{n-2}(t)\left(\overline{t^{n-2}+\frac{\omega_{n-1}}{abQ_{n-2}(0)}}\right)\mathrm{d}\hat{\mu}(\theta)=0,$$

as happened in the case of Szegő quadrature formula. It is equivalent to

$$\varphi_{-\tilde{\delta}_{n-2}}(Q_{n-2}(0))=-\frac{\omega_{n-1}}{abQ_{n-2}(0)},$$

or

$$-Q_{n-2}(0)\varphi_{-\tilde{\delta}_{n-2}}(Q_{n-2}(0))=\overline{ab}\omega_{n-1}, \tag{33}$$

where $\hat{\delta}_{n-2}=\hat{\rho}_{n-2}(0)$ and $\hat{\rho}_{n-2}$ is the monic orthogonal polynomial of degree $n-2$ with respect to $\mathrm{d}\hat{\mu}$. Therefore, we arrive at a problem analogous to Szegő quadrature formulas, since equation (33) has two solutions on $\mathbb{T}$. The question is, do either of the two solutions $Q^1_{n-2}(0), Q^2_{n-2}(0)\in\mathbb{T}$ guarantee that their $n-2$ nodes are in (a,b)? That is, of the following two polynomials

$$W^1_n(z)=z\rho_{n-1}(z)+abQ^1_{n-2}(0)\rho^*_{n-1}(z)\ \text{ or }\ W^2_n(z)=z\rho_{n-1}(z)+abQ^2_{n-2}(0)\rho^*_{n-1}(z),$$

that both have zeros at both endpoints of the arc, will all their other zeros be simple and in the interior of the arc? Note that the solutions to the equation (33) implicitly verify that $(Q^i_{n-2}(0))^2\neq\overline{ab}\omega_{n-1}$, $i=1,2$ and therefore $\overline{ab}W_n(0)^2\neq\omega_{n-1}$.

Theorem 4.4. *Let μ be a positive Borel measure supported on an arc $\Gamma\subset\mathbb{T}$, such that the zeros of the corresponding quasi-paraorthogonal polynomial W_n of order 3 verifying (32), has all its simple zeros in Γ. Then there exists a quadrature formula $I_n[f]$ of the form (2), exact in $\mathcal{L}_{2n-3}(\omega_{n-1})$.*

The corresponding quadrature formula is called a Szegő-Lobatto quadrature formula and we shall denote it by $I^L_n[f]$. Its nodes $\{z^L_k\}_{k=1}^n$, ($z^L_1=a$ and $z^L_n=b$) are all in the arc Γ and the corresponding weights $\{\lambda^L_k\}_{k=1}^n$, are positive.

We must point out that in [3, Corollary 6.1] the existence of the Szegő-Lobatto quadrature formula with $\omega_{n-1}=-d^2\frac{|\delta|+\overline{\eta}_n}{|\delta|+\eta_n}$, $\eta_n\in\mathbb{D}$ and $d=\frac{\delta}{|\delta|}$ is guaranteed when the measure μ belongs to Geronimus class for $|\delta|\geq\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$. Its nodal polynomial W_n is given by

$$W_n(z)=z(z-\eta_n)\rho_{n-2}(z)+d(1-\overline{\eta}_nz)\rho^*_{n-2}(z).$$

Furthermore, if $|\delta|<\frac{\sqrt{2}}{2}$ there exist $n_0=n_0(\delta)$ such that $\forall n\geq n_0$, W_n retains the properties. Note that if $\eta_n\in\mathbb{R}$, then $\omega_{n-1}=-d^2$.

Note that $|\delta|\geq\frac{\sqrt{2}}{2}$ for Geronimus measures means that the support is an arc smaller than or equal to a semicircle. By using Theorem 2.6 we can now generalize this for any measure satisfying this limitation about its support. As it was noted above, Q_{n-2} is a monic paraorthogonal polynomial for the measure $\mathrm{d}\hat{\mu}$, whose support is still Γ, it follows immediately that if we choose $Q_{n-2}(0)=\hat{d}_{n-2}=\frac{\hat{\delta}_{n-2}}{|\hat{\delta}_{n-2}|}$ then $W_n(0)=ab\,\hat{d}_{n-2}$ and $\omega_{n-1}=-ab\,\hat{d}^2_{n-2}$ so that indeed $\overline{ab}(W_n(0))^2\neq\omega_{n-1}$. So we can formulate:

Theorem 4.5. *Let μ be a positive Borel measure supported on an arc $\Gamma = [a,b] \subset \mathbb{T}$ such that $|\Gamma| \leq \pi$. Then there exists a Szegő-Lobatto quadrature formula exact in $\mathcal{L}_{2n-3}(-ab\,\hat{d}_{n-2}^2)$ whose nodal polynomial is given by $W_n(z) = Q_{n-2}(z)(z-a)(z-b)$ where Q_{n-2} is the monic paraorthogonal polynomial of degree $n-2$ of minimal norm for the auxiliary measure $\mathrm{d}\hat{\mu}(\theta) = \frac{\sqrt{\overline{ab}}}{4}(e^{i\theta}-a)(be^{-i\theta}-1)\mathrm{d}\mu(\theta)$ and $Q_{n-2}(0) = \hat{d}_{n-2}$.*

4.2. Error estimate and convergence

Similarly as we did in previous cases, we can obtain the same result about the approximation error for Szegő-Lobatto formulas, assuming the integrand is analytic on an simply connected domain $\Omega \supset \Gamma$ possibly except for a set of isolated poles $\{p_k\}_{k=1}^m \subset \Omega\backslash\Gamma$. We shall assume that W_n is a quasi-paraorthogonal polynomial of order 3 verifying (32), that has all its n simple zeros in Γ. First, we consider integrands f analytic in Ω. Taking into account interpolation conditions and analyticity of f in Ω, we know that

$$\frac{u^{n-1}(u-z_1)(u-z_2)\big(f(u)-H_{2n-3}^f(u)\big)}{W_n^2(u)}$$

is analytic in Ω. Proceeding in analogy with the previous cases we deduce an expression for the error of the quadrature:

$$I[f]-I_n^L[f] = -\frac{1}{2\pi\mathrm{i}}\int_{\gamma_\rho}\int_{\theta_a}^{\theta_b}\frac{t|W_n(t)|^2}{(t-z_1)(t-z_n)(t-z)}\mathrm{d}\mu(\theta)\frac{(z-z_1)(z-z_n)f(z)}{zW_n(z)\overline{W_n(1/\overline{z})}}\mathrm{d}z,$$

it follows

$$|I[f]-I_n^L[f]| \leq \frac{\|f\|_{\infty,\gamma_\rho}|\gamma_\rho|}{2\pi d(\Gamma,\gamma_\rho)}\max_{z\in\gamma_\rho}\frac{\|W_n e_{z_1}\|\,\|W_n e_{z_n}\|}{|W_n(z)|\,|W_{n*}(z)|}, \quad \rho<1,$$

where we have applied Schwarz's inequality and $\|f\|_{\infty,\gamma_\rho} = \max_{v\in\gamma_\rho}|\frac{(v-z_1)(v-z_n)}{v}f(v)|$, $|\gamma_\rho|$ is the length of the curve γ_ρ and $d(\Gamma,\gamma_\rho)$ is the distance from Γ to γ_ρ with $\gamma_\rho = \gamma(\rho s)$, $\frac{1}{s} < \rho < 1$ and $\mathrm{Int}(\gamma(s)) = \{z : |H(z)| < s\}$ is the largest sublevel set contained in $\hat{\Omega}$. For the same reason as in the previous sections, if μ is a regular measure, we obtain

$$\limsup_{n\to\infty}\big|I[f]-I_n^L[f]\big|^{\frac{1}{n}} \leq \frac{1}{s}.$$

Theorem 4.6. *Let μ be a regular positive Borel measure supported on an $\Gamma = [a,b]$ and we shall assume that W_n is a quasi-paraorthogonal polynomial of order 3 verifying (32), has all its single zeros in Γ for $n \geq 3$. Furthermore, let $\{I_n^L[f]\}_{n\geq 3}$ be the sequence of Szegő-Lobatto quadrature formulas with nodal polynomial W_n and let f be an analytic function on a simply connected domain $\Omega \supset \Gamma$. Then,*

$$\limsup_{n\to\infty}\big|I[f]-I_n^L[f]\big|^{\frac{1}{n}} \leq \frac{1}{s},$$

where $\{z : |H(z)| < s\}$ is the largest sublevel set contained in $\hat{\Omega}$ with $H(z) = G_*(z)G(z)$ where $G(z) = G_\Gamma(z,\infty)$ the conformal mapping of $\hat{\mathbb{C}}\backslash\Gamma$ onto $\hat{\mathbb{C}}\backslash(\mathbb{D}\cup\mathbb{T})$. Moreover, if f is an entire function, then

$$\lim_{n\to\infty}\big|I[f]-I_n^L[f]\big|^{\frac{1}{n}} = 0.$$

Corollary 4.7. *Let $\mu \in \mathcal{G}$ or μ be a regular positive Borel measure supported on an $\Gamma \subset \mathbb{T}$ with $|\Gamma| \leq \pi$ and $W_n(0)$ satisfying the hypothesis of Theorem 4.5. Let $\{I_n^L[f]\}_{n\geq 3}$ be the sequence of Szegő-Lobatto quadrature formulas with nodal polynomial W_n and let f be an analytic function on a simply connected domain $\Omega \supset \Gamma$. Then,*

$$\limsup_{n\to\infty} \left|I[f] - I_n^L[f]\right|^{\frac{1}{n}} \leq \frac{1}{s},$$

where $\{z : |H(z)| < s\}$ is the largest sublevel set contained in $\hat{\Omega}$ with $H(z) = G_*(z)G(z)$ where $G(z) = G_\Gamma(z, \infty)$ the conformal mapping of $\hat{\mathbb{C}}\backslash\Gamma$ onto $\hat{\mathbb{C}}\backslash(\mathbb{D} \cup \mathbb{T})$. Moreover, if f is an entire function, then

$$\lim_{n\to\infty} \left|I[f] - I_n^L[f]\right|^{\frac{1}{n}} = 0.$$

Now, we consider integrand f analytic on an simply connected domain $\Omega \supset \Gamma$ except for a set of isolated poles $\{p_k\}_{k=1}^d \subset \Omega\backslash\Gamma$. We obtain, following the same argument as in previous sections

Theorem 4.8. *Let μ be a regular positive Borel measure supported on Γ and we will assume that for $n > 3$, W_n is a quasi-paraorthogonal polynomial of order 3 verifying (32), has all its n single zeros in Γ. Furthermore, let $\{I_n^L[f]\}_{n\in\mathbb{N}}$ be the sequence of Szegő-Lobatto quadrature formulas with nodal polynomials W_n and f an analytic function on a simply connected domain $\Omega \supset \Gamma$ except at the poles $\{p_i\}_{i=1}^d \subset \Omega\backslash\Gamma$. Then,*

$$\limsup_{n\to\infty} \left|I[f] - I_n^L[f]\right|^{\frac{1}{n}} \leq \max\left\{\frac{1}{|H(p_1)|}, \ldots, \frac{1}{|H(p_d)|}\right\},$$

where $H(z) = G_*(z)G(z)$ where $G(z) = G_\Gamma(z, \infty)$ the conformal mapping of $\hat{\mathbb{C}}\backslash\Gamma$ onto $\hat{\mathbb{C}}\backslash(\mathbb{D} \cup \mathbb{T})$.

Corollary 4.9. *Let $\mu \in \mathcal{G}$ or μ be a regular positive Borel measure supported on an $\Gamma \subset \mathbb{T}$ with $|\Gamma| \leq \pi$ and $W_n(0)$ satisfying the hypothesis of Theorem 4.5. Let $\{I_n^L[f]\}_{n\geq 3}$ be the sequence of Szegő-Lobatto quadrature formulas with nodal polynomial W_n and let f be an analytic function on a simply connected domain $\Omega \supset \Gamma$ except at the poles $\{p_i\}_{i=1}^d \subset \Omega\backslash\Gamma$. Then,*

$$\limsup_{n\to\infty} \left|I[f] - I_n^L[f]\right|^{\frac{1}{n}} \leq \max\left\{\frac{1}{|H(p_1)|}, \ldots, \frac{1}{|H(p_d)|}\right\},$$

where $H(z) = G_*(z)G(z)$ where $G(z) = G_\Gamma(z, \infty)$ the conformal mapping of $\hat{\mathbb{C}}\backslash\Gamma$ onto $\hat{\mathbb{C}}\backslash(\mathbb{D} \cup \mathbb{T})$.

5. Conclusion

In this paper we have proposed the Markov approach for the construction of quadrature formulas for a positive measure supported on an arc of the unit circle. The idea to construct quadrature formulas for intervals on the real line as the integral of an Hermite interpolating

polynomial for the integrand was introduced by Markov [7]. The optimal interpolation points, i.e., the n nodes of the quadrature, are the zeros of a polynomial of degree n that has minimal norm, which is the orthogonal polynomial. For an arc on the unit circle, the role of the interpolating polynomial is naturally played by an interpolating Laurent polynomial, and the n nodes are the zeros of the paraorthogonal polynomial that has minimum norm.

Unlike for an interval of the real line where the zeros of an onthogonal polynomial are always in the interval, it is still an open problem to show that the zeros of the paraorthogonal polynomial of minimum norm lie on the arc for any (positive) measure. However, we proved that if the support of the measure is a semicircle or less, these zeros are inside the arc. We develop this approach not only for the Szegő quadrature formulas, which are the natural analogues of the Gauss quadrature formulas (Section 2), but also for their Radau (Section 3) and Lobatto (Section 4) variants and we obtain the speed of convergence for these formulas when the integrand is analytic in a simply connected neighbourhood of the arc, with the possible exception of a number of poles.

References

[1] A. Bultheel, R. Cruz-Barroso, and C. Díaz Mendoza. Zeros of quasi-paraorthogonal polynomials and positive quadrature. *J. Comput. Appl. Math.*, 407:14039, 2022.

[2] A. Bultheel and C. Díaz Mendoza. Padé approximation to Herglotz-Riesz transforms on an arc. *Numer. Algorithms*, pages 1–43, 2025.

[3] A. Bultheel and C. Díaz Mendoza. Positive quadrature formula on an arc of the unit circle. *Numer. Algorithms*, 100:769–802, 2025.

[4] R. Cruz-Barroso, C. Díaz Mendoza, and F. Perdomo-Pío. Szegő-type quadrature formulas. *J. Math. Anal. Appl.*, 455(1):592–605, 2017.

[5] W. Gautschi. *Numerical analysis, an introduction.* Birkhäuser, 1997.

[6] Y. Geronimus. Sur les polynômes orthogonaux relatifs à une suite des nombres donnée et sur le théorème de W. Hahn. *Izv. Akad. Nauk SSSR Ser. Mat.*, 4(2):215–228, 1940.

[7] A. Markov. Sur la méthode de Gauss pour le calcul approché de intégrales. *Math. Anal.*, 25:427–432, 1885.

[8] N.Y. Özgür. Finite Blaschke products and circles that pass through the origin. *Bull. Math. Anal. Appl.*, 3(3):64–72, 2011.