The para-meter of para-orthogonality

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Congrats Francisco

Congratulations



Dr. Francisco José Perdomo Pío

Survey

POP $\phi_n + \tau_n \phi_n^*$ (Jones-Njåstad-Thron 1989)

- Szegő QF on $\mathbb{T}$
- Szegő-Radau QF on $\mathbb{T}$
- Anti-Szegő QF
- Christoffel-Darboux
- Kronrod-Szegő QF and Stieltjes POP

Interpolatory quadrature formula

- Consider $I_\mu(f) = \int f(z) d\mu(z)$, $\mu > 0$, $\int d\mu(z) = 1$.
- Quadrature: $I_n(f) = \sum_{i=1}^n A_{ni} f(z_{ni})$, $\{A_{ni}\}_{i=1}^n \subset \mathbb{R}_+$, $\{z_{ni}\}_{i=1}^n \subset \operatorname{supp} \mu$
- $\mathcal{L}_0 = \mathbb{C}$, $\mathcal{L}_n \subset \mathcal{L}_{n+1}$, $\dim \mathcal{L}_n = n+1$
- $\mathcal{L}_n = \operatorname{span}\{e_0, e_1, \ldots, e_n\}$ Chebyshev system
- $\exists! F_n \in \mathcal{L}_{n-1}$ s.t. $F_n(z_{ni}) = f_{ni}$, $i = 1, \ldots, n$
  $\operatorname{Int}_{\Pi_{n-1}}(f)(z) = F_n(z) = \sum_{i=1}^n \ell_{ni}(z) f(z_{ni})$, $\ell_{ni}(z_{nj}) = \delta_{ij}$
- Interpolatory qf

$$I_\mu(f) \approx I_{\mu_n}(f) = \sum_{k=1}^n A_{nk} f(x_{nk}), \quad A_{nk} = \int \ell_{ni}(z) d\mu(z).$$

- Equality for all $f \in \mathcal{L}_{n-1}$.

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Reflexive & para-orthogonal

- Special choice of nodes. $\operatorname{supp}\mu = \mathbb{T}$, $\mathcal{L}_n = \Pi_n$.
  Node polynomial $N_n(z) = \prod_{k=1}^{n}(z - z_{nk})$
- $f_*(z) = \overline{f(1/\overline{z})}$, $p_n \in \Pi_n$: $p_n^*(z) = z^n p_{n*}(z)$
- $\{z_{nk}\} \subset \mathbb{T} \Rightarrow N_n^*(z) = c_n N_n(z)$, $c_n \in \mathbb{T}$ (reflexive)
- IQF $I_\mu(f) = I_{\mu_n}(f) \quad \forall f \in \Pi_{n-1}$.
- Exact in $\Lambda_{-(n-1),n-1} \Rightarrow$
  $I_\mu(N_n(z)z^{-k}) - \sum_i A_{ni} N_n(z_{ni}) z_{ni}^{-k} = 0,\ k = 1, \ldots, n-1.$
- $\langle f, g \rangle_\mu = \int \overline{f(z)} g(z) d\mu(z) \Rightarrow$
  $\langle N_n, z^k \rangle = 0, \quad k = 1, \ldots, n-1 \ \&\ \langle N_n, 1 \rangle \langle N_n, z^n \rangle \neq 0$
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Szegő quadrature formula

- N_n Reflexive & para-orthogonal $\Leftrightarrow$ nodes are zeros of $N_n(z) = c_n[\phi_n(z) + \tau_n \phi_n^*(z)]$, $c_n \neq 0$, $\tau_n \in \mathbb{T}$, $\{\phi_k\}$ OPS for $\langle \cdot, \cdot \rangle_\mu$.

Theorem (Szegő QF)

POP $P_n = \phi_n + \tau_n \phi_n^*$ has n simple zeros $Z_n = \{z_{ni}\}_{i=1}^n \subset \mathbb{T}$.
IQF$[\Pi_{n-1}; Z_n]$: $\sum_{i=1}^n A_{ni} f(z_{ni})$ is exact in $\Lambda_{-(n-1),(n-1)}$.
There is no such an IQF that is exact in $\Lambda_{-n,n-1}$ or $\Lambda_{-(n-1),n}$

- More general: IQF$[z^r \Pi_{n-1}, Z_n]$ has max domain $\Lambda_{r-(n-1),r+n-1}$ of max dimension $2n-1$.
- However, it is exact in a subspace of dimension $2n$ (not of the form $\Lambda_{-r,s}$).

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Max domain of exactness

- Change of basis: $s_n^{\pm} = \frac{\sigma_n \pm \tau_n}{\sigma_n \pm \tau_n} = \frac{N_n^{\pm}(0)}{N_n^{\pm}(0)}$

$$[z^k \quad z^{-k}] \begin{bmatrix} 1 & 1 \\ s_k^{\pm} & -s_k^{\pm} \end{bmatrix} = [e_k^{\pm}(z) \quad e_{-k}^{\pm}(z)], \quad k = 0, 1, \ldots, n$$

- $\operatorname{span}\{e_i^{\pm} : |i| \leq k\} = \Lambda_{-k,k} = \operatorname{span}\{z^i : |i| \leq k\}$

Theorem (Santos-León-Njåstad 2007)

- Z_n zeros of $N_n^+ = \phi_n + \tau_n \phi_n^* \Rightarrow$ IQF $I_{\mu_n}^+$ exact in $\mathcal{R}_n^+ = \operatorname{span}\{e_i^+ : i = -(n-1), \ldots, n-1, n\}$
- Z_n zeros of $N_n^- = \phi_n - \tau_n \phi_n^* \Rightarrow$ IQF $I_{\mu_n}^-$ exact in $\mathcal{R}_n^- = \operatorname{span}\{e_i^- : i = -n, -(n-1), \ldots, n-1\}$
- $\Lambda_{-(n-1),n-1} \subset \mathcal{R}_n^{\pm} \subset \Lambda_{-n,n}$

- zeros of N_n^+ and N_n^- interlace

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- zeros of N_n^+ and N_n^- interlace

Max domain of exactness

- Change of basis: $s_n^{\pm} = \frac{\sigma_n \pm \tau_n}{\sigma_n \pm \tau_n} = \frac{N_n^{\pm}(0)}{N_n^{\pm}(0)}$

$$[z^k \quad z^{-k}] \begin{bmatrix} 1 & 1 \\ s_k^{\pm} & -s_k^{\pm} \end{bmatrix} = [e_k^{\pm}(z) \quad e_{-k}^{\pm}(z)], \quad k = 0, 1, \ldots, n$$

- $\operatorname{span}\{e_i^{\pm} : |i| \leq k\} = \Lambda_{-k,k} = \operatorname{span}\{z^i : |i| \leq k\}$

Theorem (Santos-León-Njåstad 2007)

- *Z_n zeros of $N_n^+ = \phi_n + \tau_n \phi_n^* \Rightarrow$ IQF $I_{\mu_n}^+$*

exact in $\mathcal{R}_n^+ = \operatorname{span}\{e_i^+ : i = -(n-1), \ldots, n-1, n\}$

- *Z_n zeros of $N_n^- = \phi_n - \tau_n \phi_n^* \Rightarrow$ IQF $I_{\mu_n}^-$*

exact in $\mathcal{R}_n^- = \operatorname{span}\{e_i^- : i = -n, -(n-1), \ldots, n-1\}$

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- zeros of N_n^+ and N_n^- interlace

$\phi_n + \tau_n \phi_n^*$

First role of para-meter

The parameter τ_n defines the subspace of dimension $2n$ in which the QF is exact.

Orthogonal Rational Functions

What has been said for polynomials also holds for rational functions

- Introduce poles: define $\mathcal{L}_n = \{\frac{p_n}{\pi_n} : p_n \in \Pi_n\}$
  $\pi_n(x) = \prod_{i=0}^{n}(1 - \overline{\alpha}_i z)$, $\alpha_0 = 0$, $\{\alpha_i\}_{i=0}^{n} \subset \mathbb{D}$
- $\mathcal{L}_n = \text{span}\{B_k = \frac{\pi_k^*}{\pi_k}\}_{k=0}^{n}$, $B_k = \prod_{i=0}^{k} \zeta_i$, $\zeta_k = \frac{z-\alpha_k}{1-\overline{\alpha}_k z}$
- Orthogonalize $\{B_k\} \Rightarrow \mathcal{L}_n \ni \varphi_n \perp \mathcal{L}_{n-1}$ (OP = all α's at 0)
- $\varphi_n^* = B_n \varphi_{n*} = B_n(\frac{p_n}{\pi_n})_* = \frac{p_n^*}{\pi_n}$
- Zeros $Z_n \subset \mathbb{T}$ of PORF $P_n = \varphi_n + \tau_n \varphi_n^* = \frac{p_n \pm \tau_n p_n^*}{\pi_n}$, $\tau_n \in \mathbb{T}$
- para-orthogonality: $\langle 1, P_n \rangle \langle B_n, P_n \rangle \neq 0$, & $\langle P_n, f \rangle = 0$
  $\forall f \in \mathcal{L}_{n-1}(\alpha_n) = \{f \in \mathcal{L}_{n-1} : f(\alpha_n) = 0\}$
- IQF for $Z_n \Rightarrow I_\mu(f) \approx I_{\mu_n}(f) = \sum_{k=1}^{n} A_{nk} f(z_{nk})$
  (equality for $f \in \mathcal{L}_{-(n-1),n-1} \doteq \mathcal{L}_{(n-1)*} \cdot \mathcal{L}_{n-1}$)

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Max domain of exactness

- Change of basis: $B_{-k} = B_{k*} = 1/B_k$, $s_n^{\pm} = \dfrac{N_n^{\pm}(\alpha_n)}{N_n^{\pm}(\alpha_n)}$

$$[B_k \;\; B_{-k}] \begin{bmatrix} 1 & 1 \\ s_k^{\pm} & -s_k^{\pm} \end{bmatrix} = [e_k^{\pm}(z) \;\; e_{-k}^{\pm}(z)], \quad k = 0, 1, \ldots, n$$

- $\operatorname{span}\{e_i^{\pm} : |i| \leq k\} = \mathcal{L}_{-k,k} = \operatorname{span}\{B_i : |i| \leq k\}$

Theorem (BGHN 2007)

- Z_n zeros of $N_n^{+} = \phi_n + \tau_n \phi_n^* \Rightarrow$ IQF $I_{\mu_n}^{+}$ exact in $\mathcal{R}_n^{+} = \operatorname{span}\{e_i^{+} : i = -(n-1), \ldots, n-1, n\}$
- Z_n zeros of $N_n^{-} = \phi_n - \tau_n \phi_n^* \Rightarrow$ IQF $I_{\mu_n}^{-}$ exact in $\mathcal{R}_n^{-} = \operatorname{span}\{e_i^{-} : i = -n, -(n-1), \ldots, n-1\}$
- $\mathcal{L}_{-(n-1),n-1} \subset \mathcal{R}_n^{\pm} \subset \mathcal{L}_{-n,n}$

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- Z_n zeros of $N_n^{-} = \phi_n - \tau_n \phi_n^{*} \Rightarrow$ IQF $I_{\mu_n}^{-}$

exact in $\mathcal{R}_n^{-} = \operatorname{span}\{e_i^{-} : i = -n, -(n-1), \ldots, n-1\}$

- $\mathcal{L}_{-(n-1),n-1} \subset \mathcal{R}_n^{\pm} \subset \mathcal{L}_{-n,n}$

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- Z_n *zeros of* $N_n^- = \phi_n - \tau_n \phi_n^* \Rightarrow$ *IQF* $I_{\mu_n}^-$

exact in $\mathcal{R}_n^- = \operatorname{span}\{e_i^- : i = -n, -(n-1), \ldots, n-1\}$

- $\mathcal{L}_{-(n-1),n-1} \subset \mathcal{R}_n^{\pm} \subset \mathcal{L}_{-n,n}$

Szegő polynomials recurrence

- $\phi_0 = 1$, $\phi_n = z\phi_{n-1} + \sigma_n \phi_{n-1}^*$ (MOP)
  $\sigma_n = \phi_n(0) \in \mathbb{D}$
- $P_n = \phi_n + \tau_n \phi_n^*$, $\tau_n \in \mathbb{T} \Rightarrow$
  $P_n = (1 + \overline{\sigma}_n \tau_n)[z\phi_{n-1} + s_n \phi_{n-1}^*]$, $s_n = \frac{\tau_n + \sigma_n}{1 + \overline{\sigma}_n \tau_n} \in \mathbb{T}$
- Thus compute P_n as ϕ_n but replace $\sigma_n \in \mathbb{D}$ by $s_n \in \mathbb{T}$

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ORF recurrence

- $\phi_0 = 1$, $\phi_n = \frac{z-\alpha_{n-1}}{1-\overline{\alpha}_n z}\phi_{n-1} + \sigma_n \frac{1-\overline{\alpha}_{n-1} z}{1-\overline{\alpha}_n z}\phi^*_{n-1}$
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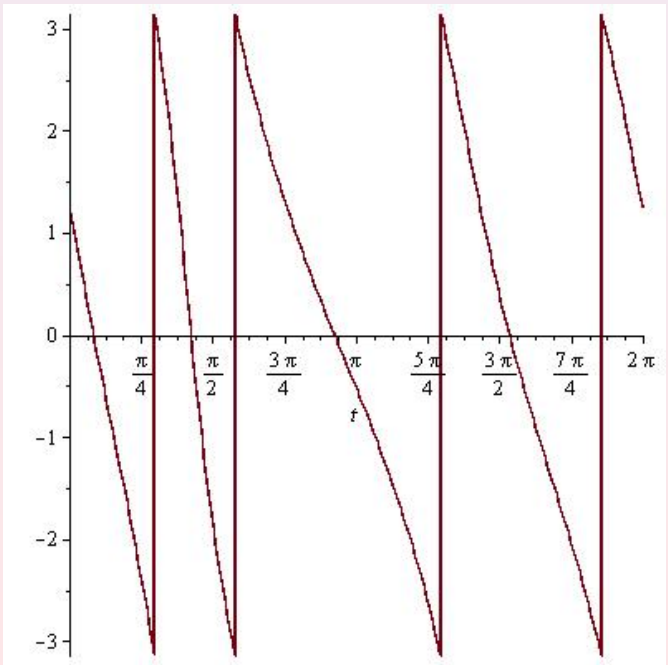
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Prefixed node

(B-Daruis-González-Vera 2009)

- Zero of $P_n = \phi_n + \tau_n \phi_n^*$ iff $\frac{\phi_n}{\phi_n^*} = -\tau_n$
- $b_n = \frac{\phi_n}{\phi_n^*}$ is Blaschke product, winding number $= n$
  $\Rightarrow b_n(e^{i\theta})$ takes all values on $\mathbb{T}$ n times as θ varies in $[-\pi, \pi]$.



$e^{it} = b_n(e^{i\theta})$; t as fct of θ

$\phi_n + \tau_n \phi_n^*$

Role 2 of para-meter

The parameter τ_n allows to fix one node on $\mathbb{T}$
and still have max domain of validity

Prefixed node

(BGHN 2010)

- Zero of $P_n = \frac{z-\alpha_{n-1}}{1-\overline{\alpha}_n z}\phi_{n-1} + s_n \frac{1-\overline{\alpha}_{n-1} z}{1-\overline{\alpha}_n z}\phi^*_{n-1}$ iff $\zeta_{n-1}(z)\frac{\phi_n}{\phi^*_n} = -s_n \in \mathbb{T}$
- $b_n = \zeta_{n-1}(z)\frac{\phi_{n-1}}{\phi^*_{n-1}}$ is Blaschke product, winding number $= n$

  $\Rightarrow b_n(e^{i\theta})$ takes all values on $\mathbb{T}$ $n+1$ times as θ varies in $[-\pi, \pi]$.

Anti-Szegő QF

(Kim-Reichel 2007)

- $I_n(f) = \sum_{i=1}^{n} A_{ni} f(z_{ni})$ Szegő QF (zeros of $P_n = z\phi_{n-1} + s_n \phi_{n-1}^*$)
- $\hat{I}_n(f) = \sum_{i=1}^{n} \hat{A}_{ni} f(\hat{z}_{ni})$ Anti-Szegő QF s.t. $I(p) - \hat{I}_n(p)) = -c(I(p) - I_n(p)),\ \forall p \in \Lambda_{-n,n},\ c > 0$
- $\hat{I} = (1+c)I - cI_n \Rightarrow \hat{\mu} = (1+c)\mu - c\mu_n$
- recall error $E_n(p) = (I - I_n)(p) = 0\ \forall p \in \Lambda_{-(n-1),n-1} \Rightarrow$ $\hat{I}(p) = (I - cE_n)(p) = I(p)\ \forall p \in \Lambda_{-(n-1),n-1} \Rightarrow$ moments $\hat{m}_k = m_k\ \forall |k| \leq n-1 \Rightarrow$ $\hat{\sigma}_k = \sigma_k,\ \forall k \leq n-1;\ \hat{\sigma}_n = (1+c)\sigma_n - cs_n \Rightarrow$ $\hat{\phi}_k = \phi_k,\ \forall k \leq n-1;\ \hat{\phi}_n = z\phi_{n-1} + \hat{\sigma}_n \phi_{n-1}^*$.
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(Kim-Reichel 2007)

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Theorem (Kim-Reichel 2007)

With the choice of $c > 0$ such that $\hat{\sigma}_n \in \mathbb{T}$, then the Szegő QF $\hat{I}_n$ for $\hat{\mu}$ whose nodes are zeros of the POP $\hat{P}_n = z\phi_{n-1} + \hat{\sigma}_n \phi_{n-1}^*$ has positive weights and is exact in $\Lambda_{-n,n}$.
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The "average QF formula" is "more accurate" than $\hat{I}_n$ or I_n.
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$\phi_n + \tau_n \phi_n^*$

Role 3 of para-meter

The parameter $\hat{\sigma}_n \in \mathbb{T}$ defines an anti-Szegő QF
with nodes the zeros of $\hat{P}_n = z\phi_{n-1} + \hat{\sigma}_n \phi_{n-1}^*$
which is useful to estimate error of I_n or find more accurate QF.

R-Anti-Szegő QF

- $I_n(f) = \sum_{i=1}^{n} A_{ni} f(z_{ni})$ R-Szegő QF
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Christoffel-Darboux

- $k_n(z,w) = \sum_{k=0}^{n} \frac{\phi_k(z)\overline{\phi_k(w)}}{\|\phi_k\|^2} = \frac{\phi^*_{n+1}(z)\overline{\phi^*_{n+1}(w)} - \phi_{n+1}(z)\overline{\phi_{n+1}(w)}}{\|\phi_{n+1}\|^2(1-z\overline{w})}$

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para-orthogonal recursion

Theorem (Costa-Felix-Sri Ranga 2012)

$P_0(z,w) = 1,\ P_1(z,w) = z + b_1(w)$

$$P_{n+1}(z,w) = [z + b_{n+1}(w)]P_n(z,w) - a_{n+1}(w)zP_{n-1}(z,w)$$

$b_{n+1}(w) = \frac{\tau_{n+1}(w)}{\tau_n(w)},\ a_{n+1}(w) = [1 - \tau_n(w)\overline{\sigma}_n][1 + \overline{w\tau_n(w)}\sigma_{n+1}]w$

$$P_n(z,w) = \frac{z\phi_n(z) + \tau_n(w) w \phi_n^*(z)}{z - w}$$

Role 4 of para-meter

With a specific relation $\tau_{n+1}(w) = \frac{\sigma_n + w\tau_n(w)}{1 + w\tau_n(w)\overline{\sigma}_n}$
A recursion between the P_n becomes possible.

However: $P_n(z,w) \sim k_n(z,w)$ reflexive but not POP !!
$(z-w)k_n(z,w)$ is POP $\Rightarrow$ $(n+1)$-point Radau=Szegő exact in $\Lambda_{-n,n}$ with prefixed node w.

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- Recurrence ???

Kronrod and Stieltjes

- Radau etc $=$ n point IQF s.t. r nodes are fixed $n-r$ max degree of exactness
- Kronrod $=$ $n+r$ point IDF s.t. n are Szegő node, r max degree of exactness
- nodes on $\mathbb{T} \Rightarrow N_{n+r}(z) = P_n(z)F_r(z)$, $P_n = \phi_n + \tau_n \phi_n^*$, F_r reflexive
- Optimal extension: $2n$-point KSQF exact in $\Lambda_{-t,t}$ with t maximal
- Stieltjes polynomials $S_{n,k} \in \Pi_n$ are s.t.
  $\langle P_n S_{n,k}, z^j \rangle = 0, \quad j = k+1, \ldots, n+k$
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Para-orthogonal Stieltjes polynomials (POSP)

(De La Calle Ysern 2012)

$n = 2m + \eta$, $\eta \in \{0, 1\}$, $\tau_n \in \mathbb{T}$, $\lambda_n \in \mathbb{T} \setminus \{\tau_n\}$
POP $P_n = \phi_n + \tau_n \phi_n^*$,
Stieltjes Polynomials: S_{nm}: $\langle P_n S_{nm}, z^k \rangle = 0$, $k = m+1, ..., n+m$

$$Q_n = \frac{S_{nm} + \lambda_n S_{nm}^*}{1 + \overline{\tau}_n \lambda_n} \in \Pi_n$$

Q_n is a para-orthogonal Stieltjes polynomial

Para-orthogonal Stieltjes polynomials (POSP)

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Theorem (De La Calle Ysern, 2012)

$Q_n \in \Pi_n$ is reflexive
SKQF = $2n$-point IQP in zeros of $P_n Q_n$ is optimal extension of SQF and there are no other optimal extensions.

$\eta = 0$, $n = 2m$, $4m$-point SKQF exact at least in $\Lambda_{-(3m-1),3m-1}$
$\eta = 1$, $n = 2m+1$, $4m+2$-point SKQF exact in $\Lambda_{-(3m+1),3m+1}$
zeros Q_n on $\mathbb{T}$?

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$n = 2m$ then SKQF is exact for the Laurent polynomial

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Note

$$\frac{\lambda_n z^{3m}}{\tau_n + \sigma_n} + \frac{\tau_n^2 z^{-3m}}{\overline{\tau_n + \sigma_n}} = \frac{\lambda_n}{\tau_n + \sigma_n}\left[z^{3m} + t_n z^{-3m}\right],$$

$$t_n = \overline{\lambda}_n \tau_n^2 \nu_n \in \mathbb{T},\ \nu_n = \frac{\tau_n + \sigma_n}{\overline{\tau_n + \sigma_n}}$$

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Define (for $\mathbb{T} \ni \lambda_n \neq \tau_n \in \mathbb{T}$)

$$[z^k \quad z^{-k}] \begin{bmatrix} 1 & 1 \\ t_n^{\pm} & -t_n^{\pm} \end{bmatrix} = [e_k^{\pm} \quad e_{-k}^{\pm}], \quad t_n^{\pm} = \overline{\lambda}_n \tau_n^2 \frac{\sigma_n \pm \tau_n}{\sigma_n \pm \tau_n},$$

then SKQF (zeros of $N_n^{\pm}$) with $n = 2m$ exact in $\mathcal{R}_{2m}^{+}$

$\Lambda_{-(3m-1),3m-1} \subset \mathcal{R}_{2m}^{+} = \operatorname{span}\{e_{-(3m-1)}, \ldots, e_{3m-1}, e_{3m}\} \subset \Lambda_{-3m,3m}$.

Note: replacing τ_n by $-\tau_n$, similarly SKQF exact in $\mathcal{R}_{2m}^{-}$

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We can play with $\pm\tau_n$ of POP and with $\pm\lambda_n$ of POSP.

A special choice of τ_n so that $t_n = t_n^{+} = t_n^{-}$, can lead to combination of SKQF$^{\pm}$ exact in $\Lambda_{-3m,3m}$.

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Para-orthogonal Stieltjes polynomials (POSP)

Theorem (De La Calle Ysern 2012)

Suppose $d\mu(z) = h(z)d\theta$, $0 < h \in \mathrm{Lip}_\alpha$, $\alpha \in (0,1)$
$\lambda_n, \tau_n \in \mathbb{T}$, $\lambda_n \neq \tau_n$ & $\exists m > 0$ *s.t.* $|1 - \overline{\tau}_n \lambda_n| \geq m$, *for all* n.
Then for n *large enough, the zeros of* P_n *and* Q_n *interlace and are all on* $\mathbb{T}$.
The SKQF has $2n$ *simple nodes on* $\mathbb{T}$ *and positive weights.*

Proof based on asymptotics in Peherstorfer style.

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