Computation of Gauss-type quadrature formulas with some preassigned nodes

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Dedicated to Manuel Calvo Pinilla on the occasion of his 65-th birthday

Contents

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General aim of the talk

- Approximate calculation of a weighted integral

$$I_\sigma(f) = \int_a^b f(x)\sigma(x)dx, \quad -\infty \leq a < b \leq +\infty,$$

$\sigma(x)$: weight function on $[a, b]$.

- n-point quadrature rule:

$$I_n(f) = \sum_{j=1}^{n} A_j f(x_j), \quad x_j \neq x_k \text{ if } j \neq k, \quad \{x_j\}_{j=1}^n \subset [a, b].$$

- Selection of nodes and weights: $I_n(P) = I_\sigma(P), \ \forall P \in \mathbb{P}_N$,

$\mathbb{P}_k = span\{x^j : 0 \leq j \leq k\}$, $\mathbb{P} = span\{x^j : j \geq 0\}$,

N: as large as possible $\rightsquigarrow$ $N \leq 2n - 1$.

General aim of the talk

- $N = 2n - 1$: Gauss-Christoffel, Gauss or Gaussian quadrature formulas (highest algebraic degree of precision)

Theorem

$I_n(f) = \sum_{j=1}^{n} A_j f(x_j)$ is the n-point Gauss formula for $\sigma(x)$ (or for $I_\sigma(f)$), if and only if,

1. *The nodes $\{x_j\}_{j=1}^{n}$ are the zeros of any orthogonal polynomial of degree n w.r.t. $\sigma(x)$.*
2. *The Christoffel numberes for $\sigma(x)$ of order n are given by*

$$A_j = \frac{1}{\sum_{k=0}^{n-1} [Q_k(x_j)]^2} > 0,$$

$\{Q_k(x)\}_{k\geq 0}$ being the sequence of orthonormal polynomials for σ.

- Gaussian formulas can be efficiently computed in terms of Jacobi matrices.

General aim of the talk

$$P_{n+1}(x) = (x - b_n)P_n(x) - a_n^2 P_{n-1}(x), \quad n \geq 1$$

$$\mathcal{J} = \begin{pmatrix} b_0 & 1 & 0 & 0 & \cdots \\ a_1^2 & b_1 & 1 & 0 & \cdots \\ 0 & a_2^2 & b_2 & 1 & \ddots \\ 0 & 0 & a_3^2 & b_3 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

Quadrature formulas with prescribed nodes

- Typical examples:
  1. Gauss-Radau formulas.
  2. Gauss-Lobatto formulas.
  3. Gauss-Kronrod formulas.

$$I_{2n+1}(f) = \sum_{j=1}^{n} A_j f(\xi_j) + \sum_{j=1}^{n+1} B_j f(x_j),$$

$\{\xi_j\}_{j=1}^{n}$: the zeros of the n-th orthogonal polynomial w.r.t. σ.

$\{A_j\}_{j=1}^{n}$, $\{B_j\}_{j=1}^{n+1}$, $\{x_j\}_{j=1}^{n+1}$ optimally chosen:

$$I_{2n+1}(P) = I_\sigma(P), \quad \forall P \in \mathbb{P}_{3n+1}.$$

General situation

- Given $a_1, \ldots, a_m$ ($a_i \neq a_j$ if $i \neq j$) on $[a, b]$, find $\{x_j\}_{j=1}^n \subset [a, b]$ and $\{A_i\}_{i=1}^m$, $\{B_j\}_{j=1}^n$ such that

$$I_{n+m}(f) = \sum_{i=1}^{m} A_i f(a_i) + \sum_{j=1}^{n} B_j f(x_j) = I_\sigma(f), \quad \forall f \in \mathbb{P}_{2n+m-1}.$$

- Set
$E_m(x) = (x - a_1) \cdots (x - a_m), \quad P_n(x) = (x - x_1) \cdots (x - x_n).$

Theorem

$I_{n+m}(f) = I_\sigma(f)$ for all $f \in \mathbb{P}_{2n+m-1}$, if and only if,

1. *$I_{n+m}(f)$ is of interpolatory type, i.e. $I_{n+m}(P) = I_\sigma(P)$ for all $P \in \mathbb{P}_{n+m-1}$,*
2. $\int_a^b P_n(x) x^k E_m(x) \sigma(x) dx = 0, \quad \forall k = 0, 1, \ldots, n-1.$

General situation

- Suppose $[a, b]$ finite, $[a, b] = [-1, 1]$. Take $m = 1, 2$ and $a_i \in \{\pm 1\} \Rightarrow$ Gauss Radau and Gauss-Lobatto formulas.

General situation

Theorem

Given $r, s \in \{0, 1\}$, consider the n-point quadrature rule

$$I_n^{(r,s)}(f) = rA_+ f(1) + sA_- f(-1) + \sum_{j=1}^{n-r-s} A_j f(x_j).$$

Then, $I_n^{(r,s)}(P) = I_\sigma(P)$ for all $P \in \mathbb{P}_{2n-1-r-s}$, if and only if,

1. $I_n^{(r,s)}(f)$ *is of interpolatory type.*
2. $\{x_j\}_{j=1}^{n-r-s}$ *are the zeros of the $(n-r-s)$-th orthogonal polynomial w.r.t. $\sigma_{r,s}(x) = (1-x)^r(1+x)^s\sigma(x)$, $x \in [-1, 1]$.*

   Furthermore,

   $$A_j = \frac{\tilde{A}_j}{(1-x_j)^r(1+x_j)^s}, \quad j = 1, \ldots, n-r-s,$$

   where $\{\tilde{A}_j\}_{j=1}^{n-r-s}$ are the Christoffel numbers for $\sigma_{r,s}(x)$ of order $(n-r-s)$.

Gauss-type formulas with a prescribed node on (-1,1)

- Problem: given $r, s \in \{0, 1\}$ and $x_\alpha \in (-1, 1)$, find weights A_+, A_-, A_α and $\{A_j\}_{j=1}^{n-r-s-1}$ along with nodes $\{x_j\}_{j=1}^{n-r-s-1}$ $(x_j \neq x_\alpha)$ such that

$$\begin{aligned} I_n^{(r,s)}(f) &= rA_+f(1) + sA_-f(-1) + A_\alpha f(x_\alpha) \\ &\quad + \sum_{j=1}^{n-1-r-s} A_j f(x_j) \\ &= I_\sigma(f), \quad \forall f \in \mathbb{P}_{2(n-1)-r-s}. \end{aligned}$$

Gauss-type formulas with a prescribed node on (-1,1)

- Recent contributions:

1. S. González-Pinto, D. Hernández-Abreu and J.I. Montijano.- **An efficient family of strongly A-stable Runge-Kutta collocation methods for stiff systems and DAEs. Part I: stability and order results**, *Journal of Computational and Applied Mathematics, 2009 (to appear).*

2. *A. Bultheel, R. Cruz-Barroso and M. Van Barel.*- **On Gauss-type quadrature formulas with prescribed nodes anywhere on the real line**, Calcolo, 2009 (to appear).

Alternative approach: passing to the unit circle

- $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.
- Joukowski Transformation: $x = \frac{1}{2}\left(z + \frac{1}{z}\right)$
- $\omega(\theta) = \sigma(\cos\theta)|\sin\theta|$, $\theta \in [-\pi, \pi]$.
- $\sigma(x)$ a weight function on $[-1, 1]$.

1. Considerations on the existence of $I_n^{(r,s)}(f)$.
2. Positivity of the weights.
3. Efficient computation of the weights and nodes.

Periodic integrands: Szegő-type formulas

- $I_\omega(g) = \int_{-\pi}^{\pi} g(\theta)\omega(\theta)d\theta,$

  g and ω: 2π-periodic functions, ω being a weight function.

- Preliminary example: elliptic integrals:

$$\begin{aligned} G(a,b) &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} = \\ &= \frac{1}{4}\int_{-\pi}^{\pi} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}, \quad (\omega(\theta) \equiv 1). \end{aligned}$$

- A comparision between Gauss-Legendre and Trapezoidal formulas with the same number of nodes.

Periodic integrands: Szegő-type formulas

$n = 10$

$$\int_0^{\pi/2} \frac{f(x)}{\sqrt{a^2 \sin^2(x) + b^2 \cos^2(x)}} dx$$

a	b	$Gauss - Legendre$	$Trapezoidal$
2	3	$.4162331047e-10$	$.4437619e-13$
0.3	0.2	$.4162331045e-9$	$.4437620e-12$
4	5	$.21149802e-12$	$.62e-18$
1	2	$.64079926085e-8$	$.7559892789e-9$
0.9	1	$.477908e-13$	$.1e-18$
0.4	0.8	$.160199815210e-7$	$.18899731972e-8$
0.3	0.6	$.213599753614e-7$	$.25199642629e-8$
0.9	1.6	$.18351021946e-8$	$.386458062e-10$
0.9	1.8	$.71199917871e-8$	$.8399880877e-9$

Periodic integrands: Szegő-type formulas

- $I_\omega(g) = \int_{-\pi}^{\pi} g(\theta)\omega(\theta)d\theta$,
- $I_n(g) = \sum_{j=1}^{n} \lambda_j g(\theta_j)$, $\{\theta_j\}_{j=1}^{n} \subset [-\pi, \pi)$, $\theta_j \neq \theta_k$ if $j \neq k$.
- $I_n(T) = I_\omega(T)$ for any trigonometric polynomial with as high degree N as possible $\rightsquigarrow$ $N \leq n-1$.
- $N = n-1$: quadrature formulas with the maximum trigonometric precision $\rightsquigarrow$

  bi-orthogonal systems of trigonometric polynomials.

Periodic integrands: Szegő-type formulas

- $I_\omega(g) = \int_{-\pi}^{\pi} g(e^{i\theta})\omega(\theta)d\theta$,
- Alternatively:
  $I_n^\omega(g) = \sum_{j=1}^n \lambda_j g(z_j)$, $\{z_j\}_{j=1}^n \subset \mathbb{T}$, $z_j \neq z_k$ if $j \neq k$.
- $I_n^\omega(L) = I_\omega(L)$, for all $L \in \Lambda_{-(n-1),n-1}$.
- For $p, q \in \mathbb{Z}$, $p \leq q$, $\Lambda_{p,q} = \operatorname{span}\{z^k : p \leq k \leq q\}$ (Laurent polynomials), $\Lambda = \operatorname{span}\{z^k : k \in \mathbb{Z}\}$.
  If $T(\theta)$ is a trigonometric polynomial of degree $n-1$, then $T(\theta) = L(e^{i\theta})$ with $L \in \Lambda_{-(n-1),n-1}$
- Let $\{\rho_n(z)\}_{n=0}^\infty$ be the sequence of monic Szegő polynomials w.r.t $\omega(\theta)$, $\rho_0(z) \equiv 1$, $\rho_n(z) = z^n + \cdots$ and for all $n \geq 1$ $(z = e^{i\theta})$,

  $$\langle \rho_n(z), z^k \rangle_\omega = \int_{-\pi}^{\pi} \rho_n(z)\overline{z^k}\omega(\theta)d\theta = 0, \quad k = 0, 1, \ldots, n-1,$$

  $$\rho_n^*(z) := z^n \overline{\rho_n\left(1/\bar{z}\right)} \text{ (reversed polynomial)}.$$

Periodic integrands: Szegő-type formulas

Theorem

Given $I_\omega(g) = \int_{-\pi}^{\pi} g(e^{i\theta})\omega(\theta)d\theta$ *and* $I_n^\omega(g) = \sum_{j=1}^{n} \lambda_j g(z_j)$, $z_j \in \mathbb{T}$ *for all* $j = 1, \ldots, n$.
Then, $I_n^\omega(L) = I_\omega(L)$ *for all* $L \in \Lambda_{-(n-1),n-1}$, *if and only if,*

1. $\{z_j\}_{j=1}^n$ *are the zeros of* $B_n(z, \tau_n) = z\rho_{n-1}(z) + \tau_n \rho_{n-1}^*(z)$ *for some* $\tau_n \in \mathbb{T}$.
2. $\lambda_j^{-1} = \sum_{k=0}^{n-1} \frac{|\rho_k(z_j)|^2}{\langle \rho_k, \rho_k \rangle} > 0, \quad j = 1, \ldots, n.$

$I_n^\omega(g)$: *an n-point Szegő quadrature rule for* $\omega(\theta)$ *(or for* $I_\omega(g)$*).*

- The "freedom" of parameter τ_n enables us to fix a node z_α in advance, by taking

$$\tau_n = -\frac{z_\alpha \rho_{n-1}(z_\alpha)}{\rho_{n-1}^*(z_\alpha)} \in \mathbb{T}$$

"Szegő-Radau formulas".

Periodic integrands: Szegő-type formulas

Example

Suppose $\omega(\theta) \equiv 1 \quad \rightsquigarrow \quad I_\omega(g) = \int_{-\pi}^{\pi} g(e^{i\theta}) d\theta.$

$\rho_n(z) = z^n, \forall n = 0, 1, 2, \ldots \quad \Rightarrow$

$B_n(z, \tau_n) = z\rho_{n-1}(z) + \tau_n \rho_{n-1}^*(z) = z^n + \tau_n.$

The nodes of any n-point Szegő formula: $z_j = \sqrt[n]{\tau_n}$, $|\tau_n| = 1$.

Weights: $\lambda_j = \dfrac{1}{\sum_{k=0}^{n-1} \frac{|\rho_k(z_j)|^2}{\langle \rho_k, \rho_k \rangle}} = \dfrac{2\pi}{n}, \quad j = 1, \ldots, n.$

Thus,

$I_n(g) = \frac{2\pi}{n} \sum_{j=1}^{n} g(e^{i\theta_j}),$

$\{\theta_j\}_{j=1}^n$ *uniformly distributed on* $[\pi, \pi]$ *i.e.,*
$I_n(g)$ *coincides with the n-point Trapezoidal rule.*

Computation of Szegő formulas

- $\{\rho_k(z)\}_{k=0}^{\infty}$: monic Szegő polynomials w.r.t. $\omega(\theta)$.
- Verblunsky parameters, reflection coefficients or Schur parameters:
  $\delta_k = \rho_k(0)$, $\delta_0 = 1$, $|\delta_k| < 1$ for all $k = 1, 2, \ldots$.
- Set
  $H_n(\tau) =$

$$D_n^{-1/2} \begin{pmatrix} -\delta_1 & -\delta_2 & \cdots & -\delta_{n-1} & -\tau \\ \sigma_1^2 & -\overline{\delta_1}\delta_2 & \cdots & -\overline{\delta_1}\delta_{n-1} & -\overline{\delta_1}\tau \\ 0 & \sigma_2^2 & \cdots & -\overline{\delta_2}\delta_{n-1} & -\overline{\delta_2}\tau \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_{n-1}^2 & -\overline{\delta_{n-1}}\tau \end{pmatrix} D_n^{1/2},$$

where $\sigma_k = \sqrt{1 - |\delta_k|^2}$, $D_n = \operatorname{diag}[\gamma_0, \gamma_1, \ldots, \gamma_{n-1}]$,

$\gamma_0 = 1$, $\gamma_k = \gamma_{k-1}\sigma_k^2$, $k = 1, \ldots, n-1$.

Computation of Szegő formulas

Theorem

For all $\tau \in \mathbb{T}$, $H_n(\tau)$ is an unreduced unitary upper Hessenberg matrix so that its eigenvalues $\{z_j\}_{j=1}^n$ which are distinct and of unit magnitude are the nodes of the n-th Szegő rule for the parameter

$$\tilde{\tau} = \frac{\tau + \delta_n}{1 + \tau\overline{\delta_n}} \in \mathbb{T}.$$

Furthermore, the square of the first component of the eigenvector of unit lenght associated with z_j yields the weight λ_j ($\mu_0 = \int_{-\pi}^{\pi} \omega(\theta) d\theta = 1$).

Szegő-Lobatto quadrature formulas (Jagels and Reichel, JCAM 2007)

- $I_\omega(g) = \int_{-\pi}^{\pi} g(e^{i\theta})\omega(\theta)d\theta.$
- Fix z_α and z_β on $\mathbb{T}$, $z_\alpha \neq z_\beta$ and take $n > 2$.
- Then there exist nodes $\{z_j\}_{j=1}^{n-2} \subset \mathbb{T}$, $z_j \neq z_k$ if $j \neq k$ and $z_j \notin \{z_\alpha, z_\beta\}$ along with positive weights A, B and $\{\lambda_j\}_{j=1}^{n-2}$ such that

$$\tilde{I}_n^\omega(g) = Ag(z_\alpha)+Bg(z_\beta)+\sum_{j=1}^{n-2}\lambda_j g(z_j) = I_\omega(g), \quad \forall g \in \Lambda_{-(n-2),n-2}.$$

Szegő-Lobatto quadrature formulas (Jagels and Reichel, JCAM 2007)

$$\tilde{I}_n^{\omega}(g) = Ag(z_\alpha)+Bg(z_\beta)+\sum_{j=1}^{n-2}\lambda_j g(z_j) = I_\omega(g), \quad \forall g \in \Lambda_{-(n-2),n-2}.$$

- The nodes z_α, z_β and z_j for all $j = 1, \ldots, n-2$ are the zeros of

$$\tilde{B}_n(z) = z\tilde{\rho}_{n-1}(z) + \tilde{\tau}_n\tilde{\rho}^*_{n-1}(z)$$

where

$$\tilde{\rho}_{n-1}(z) = z\rho_{n-2}(z) + \tilde{\delta}_{n-1}\rho^*_{n-2}(z),$$

$\left|\tilde{\delta}_{n-1}\right| < 1$ and $|\tilde{\tau}_n| = 1$ computed in terms of $\rho_0(z), \ldots, \rho_{n-1}(z)$, z_α and z_β.

The symmetric case

- $\omega(-\theta) = \omega(\theta)$, $\theta \in [-\pi, \pi]$.
- Trigonometric moments: $\mu_k = \int_{-\pi}^{\pi} e^{-ik\theta}\omega(\theta)d\theta$, $k = 0, \pm 1, \pm 2, \ldots$ are real.
- For all $k = 0, 1, \ldots$, $\delta_k = \rho_k(0)$ are real too.
- $I_n^{\omega}(g) = \sum_{j=1}^{n} \lambda_j g(z_j)$ is said to be symmetric, if and only if, the nodes $\{z_j\}_{j=1}^{n}$ are real or appear in complex conjugate pairs on $\mathbb{T}$.

Theorem

Let $I_n^{\omega}(g) = \sum_{j=1}^{n} \lambda_j g(z_j)$ be an n-point Szegő formula for $\omega(\theta)$ symmetric and generated by $B_n(z, \tau_n) = z\rho_{n-1}(z) + \tau_n \rho_{n-1}^*(z)$. Then, $I_n^{\omega}(g)$ is symmetric, if and only if, $\tau_n \in \{\pm 1\}$. Furthermore, if for some j and k, $1 \leq j, k \leq n$, $z_j = \overline{z_k}$, then $\lambda_j = \lambda_k$.

Symmetric Szegő-Lobatto formulas

- Fix z_α and $\overline{z_\alpha}$ on $\mathbb{T}$, let $\omega(\theta)$ be symmetric and set $n > 2$.

$$\tilde{I}_n^\omega(g) = Ag(z_\alpha)+Bg(\overline{z_\alpha})+\sum_{j=1}^{n-2}\lambda_j g(z_j) = I_\omega(g), \quad \forall g \in \Lambda_{-(n-2),n-2}.$$

- Recall: the nodes $z_\alpha, \overline{z_\alpha}$ and z_j for all $j = 1, \ldots, n-2$ are the zeros of
  $\tilde{B}_n(z) = z\tilde{\rho}_{n-1}(z) + \tilde{\tau}_n\tilde{\rho}^*_{n-1}(z)$ with
  $\tilde{\rho}_{n-1}(z) = z\rho_{n-2}(z) + \tilde{\delta}_{n-1}\rho^*_{n-2}(z)$,

  $\left|\tilde{\delta}_{n-1}\right| < 1$ and $|\tilde{\tau}_n| = 1$.

- Now, $\tilde{\delta}_{n-1}$ can be taken real: $\tilde{\delta}_{n-1} \in (-1, 1)$ and $\tilde{\tau}_n \in \{\pm 1\}$.

- Hence, $\tilde{I}_n^\omega(g)$ is symmetric and its computation reduces one half.

Construction of Gauss-type quadrature formulas

- $I_\sigma(f) = \int_{-1}^{1} f(x)\sigma(x)dx$.
- $\omega(\theta) = \sigma(\cos\theta)|\sin\theta|$, $\theta \in [-\pi, \pi]$, ω symmetric.
- $I_\sigma(f) = \int_{-1}^{1} f(x)\sigma(x)dx = \frac{1}{2}\int_{-\pi}^{\pi} g(e^{i\theta})\omega(\theta)d\theta = \frac{1}{2}I_\omega(g)$,

  $g(e^{i\theta}) = f(\cos\theta) = f\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)$.
- Connection between Gauss-type quadrature formulas for $\sigma(x)$ and Szegő-type formulas for $\omega(\theta)$.
- A. Bultheel, L. Daruis and P. G.-V..- JCAM, 2001.
- A. Bultheel, L. Daruis and P. G.-V..- JCAM, 2005.

Construction of Gauss-type quadrature formulas

Theorem

Take $r, s \in \{0, 1\}$ *and consider* $n - r - s$ *distinct nodes* $\{x_j\}_{j=1}^{n-r-s}$ *on* $(-1, 1)$ *along with the real numbers* A_+, A_- *and* $\{A_j\}_{j=1}^{n-r-s}$.

Suppose that $x_j = \cos\theta_j$, $\theta_j \in (0, \pi)$ *and define* $z_j = e^{i\theta_j}$, $z_{n+j-r-s} = \overline{z_j}$ *and* $\lambda_j = \lambda_{n+j-r-s} = A_j$ *for* $1 \leq j \leq n - r - s$.

Then for all $N \geq 0$,

$$\begin{aligned} I_n^\sigma(f) &= rA_+ f(1) + sA_- f(-1) + \sum_{j=1}^{n-r-s} A_j f(x_j) \\ &= I_\sigma(f), \quad \forall f \in \mathbb{P}_N, \end{aligned}$$

if and only if,

$$\begin{aligned} I_{2n-r-s}^\omega(g) &= 2\left[rA_+ g(1) + sA_- g(-1)\right] + \sum_{j=1}^{2(n-r-s)} \lambda_j g(z_j) \\ &= I_\omega(g), \quad \forall g \in \Lambda_{-N,N}. \end{aligned}$$

As usual $\sigma(x)$ a weight function on [-1,1] and $\omega(\theta) = \sigma(\cos(\theta))|\sin(\theta)|$.

Construction of Gauss-type quadrature formulas

Problem: given x_α, $r, s, \in \{0, 1\}$ and $n > 1 + r + s$, find weights A_+, A_-, A_α and $\{A_j\}_{j=1}^{n-r-s-1}$ along with distinct nodes $\{x_j\}_{j=1}^{n-r-s-1}$ such that $x_j \neq x_\alpha$ and

$$\begin{aligned} I_n^{(r,s)}(f) &= rA_+f(1) + sA_-f(-1) + A_\alpha f(x_\alpha) + \sum_{j=1}^{n-r-s-1} A_j f(x_j) \\ &= I_\sigma(f), \quad \forall f \in \mathbb{P}_{2(n-1)-r-s}. \end{aligned}$$

Construction of Gauss-type quadrature formulas

Theorem

Given $r, s \in \{0, 1\}$ *and* $x_\alpha \in (-1, 1)$, *take* $z_\alpha \in \mathbb{T}$ *such that* $x_\alpha = \Re(z_\alpha)$. *For* $n > 1 + r + s$, *consider the* $(2n - r - s)$*-point symmetric Szegő-Lobatto rule to* $I_\omega(f)$ *with prescribed nodes at* z_α *and* $\overline{z_\alpha}$, *where* $\omega(\theta) = \sigma(\cos\theta)|\sin\theta|$, $\sigma(x)$ *being a weight function on* $[-1, 1]$.

Let $\tilde{\delta}_n^{(r,s)} \in (-1, 1)$ *and* $\tilde{\tau}_n^{(r,s)} \in \{\pm 1\}$ *the parameters characterizing this Szegő-Lobatto rule. Then, a quadrature formula* $I_n^{(r,s)}(f)$ *as in the Problem above there exists, if and only if,*

$$\tilde{\tau}_n^{(r,s)} = \begin{cases} 1 & as\ (r,s) = (0,0)\ and\ (0,1) \\ -1 & as\ (r,s) = (1,1)\ and\ (1,0) \end{cases}$$

About the parameter $\tilde{\delta}_n^{(r,s)}$

- According to the work by Jagels and Reichel, the parameter $\tilde{\delta}_n^{(r,s)}$ characterizing the $(2n-r-s)$-point symmetric Szegő-Lobatto rule for $I_\omega(f)$, arises from the non-empty intersection of a circle with the segment $(-1,1)$.
  Hence, there exist two values at most.
- Thus, if $\tilde{\delta}_n^{(0,0)}$ is unique, then either formulas $I_n^{(0,0)}(f)$ or $I_{n+1}^{(1,1)}(f)$ there exist but not both simultaneously. On the contrary, in case of two distinct values of $\tilde{\delta}_n^{(0,0)}$, both formulas $I_n^{(0,0)}(f)$ and $I_{n+1}^{(1,1)}(f)$ exist.
- On the other hand, if $\tilde{\delta}_n^{(1,0)} = \tilde{\delta}_n^{(0,1)} \in (-1,1)$ is unique, then either $I_n^{(1,0)}(f)$ or $I_n^{(0,1)}(f)$ there exist but not both simultaneously.
- Finally, if two distinct values of $\tilde{\delta}_n^{(1,0)}$ there exist, then both formulas $I_n^{(1,0)}(f)$ or $I_n^{(0,1)}(f)$ can be constructed.

Computation

$$I_\sigma(g) = \int_{-1}^{1} f(x)\sigma(x)dx$$

$r, s \in \{0, 1\}$, $x_\alpha \in (-1, 1)$

$$\begin{aligned} I_n^{(r,s)} &= rA_+f(1) + sA_-f(-1) + A_\alpha f(x_\alpha) + \\ &+ \sum_{j=1}^{n-1-r-s} A_j f(x_j) = I_\sigma(f),\ \forall f \in \mathbb{P}_{2(n-1)-r-s} \end{aligned}$$

Computation of the $(2n - r - s)$-point symmetric Szegő-Lobatto quadrature formula $I^{\omega}_{2n-r-s}(g)$ with prescribed nodes z_α and $\overline{z}_\alpha$ on the unit circle such that $x_\alpha = \Re(z_\alpha)$ for $\omega(\theta) = \sigma(\cos(\theta))|\sin(\theta)|$

Characterizing parameters

$$\tilde{\delta}_{n-1}^{(r,s)} \in (-1, 1), \quad \tilde{\tau}_n^{(r,s)} \in \{\pm 1\}$$

Computation

Let $\tilde{\omega}(\theta)$ be the weight function with Verblunsky parameters:

$$\delta_0, \delta_1, \ldots, \delta_{n-2}, \tilde{\delta}_{n-1}^{(r,s)}, \delta_n, \delta_{n+1}, \ldots \quad (\delta_{n-1} \rightsquigarrow \tilde{\delta}_{n-1}^{(r,s)})$$

Then $I_{2n-r-s}^{\omega}(g) = I_{\tilde{\omega}}(g)$, $\forall g \in \Lambda_{-(2n-r-s-1),(2n-r-s-1)}$
(Symmetric Szegő formula for $I_{\tilde{\omega}}(g)$)
$I_n^{(r,s)}(f)$: n-point Gauss-type formula (Gauss, Gauss-Radau or Gauss-Lobatto) for $\tilde{\sigma}(x)$ such that

$$\tilde{\omega}(\theta) = \tilde{\sigma}(\cos(\theta))|\sin(\theta)|$$

$\{a_k\}_1^{\infty}$, $\{b_k\}_0^{\infty}$ Jacobi parameters for $\sigma(x)$.
We need to compute the Jacobi parameters $\{\tilde{a}_k\}_1^{\infty}$ and $\{\tilde{b}_k\}_0^{\infty}$ for $\tilde{\sigma}(x)$.

Computation

Let $\omega(\theta)$ be a symmetric weight function on $[-1,1]$ and $\sigma(x)$ a weight function on $[-1,1]$ such that

$$\omega(\theta) = \sigma(\cos(\theta))|\sin(\theta)|$$

Let $\{\delta_k\}_0^\infty$ be the Verblunsky parameters for $\omega(\theta)$ and $\{a_k\}_1^\infty$ and $\{b_k\}_0^\infty$ the Jacobi parameters for $\sigma(x)$. Then, the following holds:

Geronimus Relations

$b_0 = \frac{\mu_1}{2}$ and for $k \geq 1$,

$$2a_k = \sqrt{(1-\delta_{2k})(1-\delta_{2k-1}^2)(1+\delta_{2k-2})},$$

$$2b_k = \delta_{2k-1}(1-\delta_{2k}) - \delta_{2k+1}(1+\delta_{2k}).$$

Initial available information on $\sigma(x)$. $\{a_k\}_1^\infty$, $\{b_k\}_0^\infty !\Rightarrow \{\delta_k\}_0^\infty$?

Computation

$$\text{Set,} \quad \mathcal{J} = \begin{pmatrix} b_0 & 1 & 0 & 0 & \cdots \\ a_1^2 & b_1 & 1 & 0 & \cdots \\ 0 & a_2^2 & b_2 & 1 & \ddots \\ 0 & 0 & a_3^2 & b_3 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix};$$

$$\mathcal{L} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ u_2 & 1 & 0 & 0 & \cdots \\ 0 & u_4 & 1 & 0 & \ddots \\ 0 & 0 & u_6 & 1 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}; \mathcal{U} = \begin{pmatrix} u_1 & 1 & 0 & 0 & \cdots \\ 0 & u_3 & 1 & 0 & \cdots \\ 0 & 0 & u_5 & 1 & \ddots \\ 0 & 0 & 0 & u_7 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

Then:

$$\mathcal{J} + \mathcal{I} = \mathcal{L}\mathcal{U} \Rightarrow \begin{cases} a_k^2 = u_{2k} u_{2k-1}, \ k \geq 1; \\ b_k + 1 = u_{2k} + 2u_{2k+1}, \ k \geq 0, \ (k_0 = 0) \end{cases}$$

From the Geronimus relations:

$$\delta_k = 1 - \frac{2u_k}{1 + \delta_{k-1}}, \ (\delta_0 = 1), \ k \geq 1$$

Computation

To fix ideas, $r = s = 0$, $x_\alpha \in (-1, 1)$

$$I_n^\sigma(f) = A_\alpha f(x_\alpha) + \sum_{j=1}^{n-1} A_j f(x_j) = I_\sigma(f),\ \forall f \in \mathbb{P}_{2n-2}$$

$$\Updownarrow$$

$$I_{2n}^\omega(g) = A_\alpha[g(z_\alpha) + g(\overline{z}_\alpha)] + \sum_{j=1}^{n-1} A_j[g(z_j) + g(\overline{z}_j)]$$

$X_\alpha = \Re(z_\alpha)$, $x_j = \Re(z_j)$ (n-point Symmetric Szegő-Lobatto formula for $I_\omega(g)$, $\omega(\theta) = \sigma(\cos(\theta))|\sin(\theta)|$)

Characteristic parameters

$$\tilde{\delta}_{2n-1} \in (-1, 1), \quad \tilde{\tau}_{2n} \in \{\pm 1\}$$

$I_n^\sigma(f)$ exists $\iff$ $\tilde{\tau}_{2n} = 1$

Computation

Set $\tilde{a}_k = a_k$, $1 \leq k \leq n-1$
$\tilde{b}_k = b_k$, $0 \leq k \leq n-1$
$2\tilde{b}_{n-1} = \delta_{2n-3}[1 - \delta_{2n-2}] - \tilde{\delta}_{2n-1}(1 - \delta_2 n - 2)$

$$\tilde{\mathcal{J}}_n = \begin{pmatrix} b_0 & a_1 & 0 & 0 & \cdots \\ a_1 & b_1 & a_2 & 0 & \cdots \\ 0 & a_2 & b_2 & a_3 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ 0 & 0 & 0 & a_{n-1} & \tilde{b}_{n-1} \end{pmatrix}$$

Theorem

Suppose $I_n^\sigma(f) = A_\alpha f(x_\alpha) + \sum_{j=1}^{n-1} A_j f(x_j) = I_\sigma(f)$, $\forall f \in \mathbb{P}_{2n-2}$ *with* $x_\alpha \in (-1, 1)$. *Then, the eigenvalues of* $\tilde{\mathcal{J}}_n$ *are the nodes of* $I_n^\sigma(f)$ *and the square of the first component of the eigenvector of unit length associated with* x_j *yields the weights* A_j.

Numerical Experiment

$$\sigma(x) = \frac{1}{\sqrt{1-x^2}};\ T_{n+1}(x) = xT_n(x) - \frac{1}{4}T_{n-1}(x) \Rightarrow a_n = \frac{1}{2},\ b_n = 0$$

$$\omega(\theta) = \sigma(\cos(\theta))|\sin(\theta)| \equiv 1;\ \rho_n(z) = z^n,\ \{\delta_k\}_0^\infty,\ \delta_0 = 1,\ \delta_k = 0,\ k \geq 1$$

$n = 5$

$x_\alpha = 0.2836621854,\ \tilde{\delta}_{n-1} = -0.1449722751 \Rightarrow \tilde{\tau}_n = -1$ (no exist)

$x_\alpha = -0.9899924966,\ \tilde{\delta}_{n-1} = 0.8691947009 \Rightarrow \tilde{\tau}_n = 1$ (exist)

Nodes	*Weights*
−0.9899924966	0.1267664916
−0.7461525707	0.2126337289
−0.1601488664	0.2191896426
0.5057741115	0.2205156246
0.9403898530	0.2208945120