A matricial computation of rational quadrature formulas on the unit circle

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Abstract

A matricial computation of quadrature formulas for orthogonal rational functions on the unit circle, is presented in this paper. The nodes of these quadrature formulas are the zeros of the para-orthogonal rational functions with poles in the exterior of the unit circle and the weights are given by the corresponding Christoffel numbers. We show how these nodes can be obtained as the eigenvalues of the operator Möbius transformations of Hessenberg matrices and also as the eigenvalues of the operator Möbius transformations of five-diagonal matrices, recently obtained. We illustrate the preceding results with some numerical examples.

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1 Introduction

In the study of quadrature formulas on the real line as well as on the unit circle, orthogonal polynomials play an important role. Consider a positive Borel measure μ and a function f defined on a real interval $[a, b]$. When approximating the integral $\int_a^b f(x)d\mu(x)$ by an n-point Gauss-Christoffel quadrature rule it is very well-known that the estimation is exact for any polynomial f of degree up to $2n-1$. The effective computation of the nodes and weights in this approximation, has become an interesting matter of study both numerically and theoretically.

As shown by Gautschi (see [16, 17, 15]) among others, here the basic fact is the three-terms recurrence relation satisfied by the sequence of orthogonal polynomials with respect

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to the measure μ giving rise to certain tridiagonal matrices (Jacobi matrices) so that the eigenvalues of the n-th principal submatrix coincide with the nodes, i.e., with the zeros of the n-th orthogonal polynomial. Furthermore, the weights can be easily expressed in terms of the first component of the corresponding normalized eigenvector.

For an estimation of integrals of 2π-periodic functions with respect to a positive measure μ defined on the unit circle $\mathbb{T}$, an n-point quadrature rule with distinct nodes on $\mathbb{T}$ can also be proposed, but imposing exactness not for algebraic polynomials but for trigonometric polynomials or more generally, for Laurent polynomials. Quadrature formulas on the unit circle, or Szegő quadrature formulas, were introduced by Jones, Njåstad and Thron in [20]. (See also [14] for a different approach).

The recurrence relation satisfied by the orthogonal Laurent polynomials with respect to the measure μ is a five-term recurrence relation giving rise to certain unitary five-diagonal matrices. These matrices constitute the narrowest banded representation of the multiplication operator defined in the linear space of Laurent polynomials and play the same role on the unit circle as the Jacobi matrices on the real line (see [21, 22] and references therein). For instance, the eigenvalues of the n-th principal submatrix obtained by a unitary truncation of the five-diagonal matrix, are the nodes of Szegő quadrature formulas and coincide with the zeros of the so-called para-orthogonal polynomials. The weights are the first component of the normalized eigenvector. On the other hand, the nodes of Szegő quadrature formulas can also be calculated by using the Hessenberg representation for orthogonal polynomials on the unit circle. The computation of the nodes and the weights of Szegő quadrature formulas by using the five-diagonal and Hessenberg matrices, has been studied recently, (see [9]).

The orthogonal polynomials are a particular case of a more general kind of orthogonal functions with interest in many pure and applied sciences: the orthogonal rational functions with prescribed poles, (see [7] and references therein). The natural generalization of the orthogonal polynomials on the real line are the rational functions with poles on the extended real line while for the unit circle, the poles have to be in the exterior of the closed unit disk. Considered as orthogonal rational functions, the main difference between orthogonal polynomials on the real line and on the unit circle is not the location of the support of the measure, but the relative localization of the poles with respect to this support. Orthogonal rational functions on the unit circle generalizes the well known Szegő polynomials.

A more general situation can also be considered with respect to the quadrature formulas: involving orthogonal rational functions on the unit circle with prescribed poles not on $\mathbb{T}$, but inside or outside of the unit disk. This situation, already studied in [5], gives rise to the rational Szegő quadrature formulas. In this case, the nodes are the zeros of the corresponding para-orthogonal rational functions and the quadrature formula is the integral of the rational Lagrange interpolant in these nodes, so that the weights are obtained as the integral of the corresponding rational Lagrange basis functions. An alternative approach to these quadrature formulas, using the Hermite interpolation is also considered in [5].

In this paper we present a new way to calculate the nodes and the weights to the rational Szegő quadrature formulas by using the recently obtained matricial representation for orthogonal rational functions on the unit circle with prescribed poles (see [27]). As we will expose, these matrices are the result of applying the matrix Möbius transformation to the Hessenberg and the five-diagonal unitary matrices associated with the polynomial case. The operator Möbius transformation of the unitary truncations of order n of these matrices, led us to obtain the nodes and the weights of rational Szegő quadrature problem as in the polynomial case, by computing the eigenvalues and eigenvectors of such matrices.

The structure of this article is the following. First, in section 2 we give some basic definitions and notation, as well as some previous results that we will use in the rest of the paper. In section 3, we will introduce two different bases in the space of rational functions and give the corresponding matrix representations of the multiplication operator using these bases. We expose the expressions for the sequences of para-orthogonal rational functions and their zeros as eigenvectors and eigenvalues of matrix Möbius transformations of the unitary truncations of Hessenberg and five-diagonal matrices, in section 4. In section 5, we apply these results to the computation the nodes of Szegő rational quadrature formulas on the unit circle. Finally, in section 6, we illustrate the preceding results with some examples.

2 Some preliminary results

Let us first introduce some notation. In what follows, we will denote by $\mathbb{Z}$ the set of all integer numbers, and $\mathbb{N} = \{n \in \mathbb{Z} : n \geq 0\}$. Let $\mathbb{P}$ be the vector space of polynomials with complex coefficients, and $\mathbb{P}_n$ the vector subspace of polynomials whose degree is less than or equal to n, while $\mathbb{P}_{-1} := \{0\}$ is the trivial subspace. As usual, if $p \in \mathbb{P}$ has degree n, p^* denotes its reversed polynomial, that is, $p^*(z) = z^n \overline{p}(z^{-1})$ and p_* means $p_*(z) = \overline{p}(z^{-1})$. The symbol $\Lambda := \mathbb{C}[z, z^{-1}]$ represents the complex vector space of Laurent polynomials, and for $m, n \in \mathbb{Z}$, $m \leq n$, we define the vector subspace $\Lambda_{m,n} := \operatorname{span}\{z^m, z^{m+1}, \ldots, z^n\}$. Also $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{E} := \{z \in \mathbb{C} : |z| > 1\}$ represent respectively, the unit circle, the open unit disk and the exterior of the closed unit disk in the complex plane.

For any positive Borel measure μ on the unit circle $\mathbb{T}$, we consider the Hilbert space of square integrable functions with respect to the inner product

$$\langle f, g \rangle_\mu = \int f(z)\overline{g(z)} d\mu(z), \qquad f, g \in L^2_\mu$$

and we will suppose that the support of μ is an infinite set. The family $(z^n)_{n\geq 0}$ is a linearly independent subset of L^2_μ whose orthonormalization gives the corresponding sequence of orthogonal polynomials (φ_n) with respect to the measure μ. If we choose these polynomials with positive leading coefficient, they satisfy the recurrence relation

$$\rho_n \begin{pmatrix} \varphi_n(z) \\ \varphi_n^*(z) \end{pmatrix} = \begin{pmatrix} 1 & a_n \\ \overline{a}_n & 1 \end{pmatrix} \begin{pmatrix} z\varphi_{n-1}(z) \\ \varphi_{n-1}^*(z) \end{pmatrix}, \tag{1}$$

with $a_n = \varphi_n(0)/\varphi_n^*(0)$, and $\rho_n = \sqrt{1 - |a_n|^2}$.

Whenever the polynomials are dense in L^2_μ, the matricial representation for the multiplication operator $T_\mu(f) = zf$, $f \in L^2_\mu$ with respect to the OP $(\varphi_n)_{n\geq 0}$ is the irreducible Hessenberg matrix (see [19, 21])

$$\mathcal{H} = \begin{pmatrix} -a_1 & -\rho_1 a_2 & -\rho_1\rho_2 a_3 & -\rho_1\rho_2\rho_3 a_4 & \cdots \\ \rho_1 & -\overline{a}_1 a_2 & -\overline{a}_1 \rho_2 a_3 & -\overline{a}_1\rho_2\rho_3 a_4 & \cdots \\ 0 & \rho_2 & -\overline{a}_2 a_3 & -\overline{a}_2\rho_3 a_4 & \cdots \\ 0 & 0 & \rho_3 & -\overline{a}_3 a_4 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \tag{2}$$

In the general case $\mathcal{H}$ is a matrix representation of the restriction $T_\mu \restriction \mathcal{P} : \mathcal{P} \to \mathcal{P}$ of T_μ to the T_μ-invariant subspace given by the closure $\mathcal{P}$ of the linear space of the polynomials $\mathbb{P}$, in L^2_μ. Notice that as a restriction of a unitary operator, $T_\mu \restriction \mathcal{P}$ is isometric but not necessarily unitary.

On the other hand, if we use as a basis of L^2_μ the orthogonal Laurent polynomials (χ_n) that arise from the orthonormalization of $(1, z, z^{-1}, z^2, z^{-2}, \ldots)$ which are given by (see [10, 21, 24, 28])

$$\chi_{2n}(z) = z^{-n}\varphi^*_{2n}(z), \qquad \chi_{2n+1}(z) = z^{-n}\varphi_{2n+1}(z), \qquad n \geq 0,$$

the representation of T_μ is the five-diagonal matrix (see [10], [21, 28])

$$\mathcal{C} = \begin{pmatrix} -a_1 & -\rho_1 a_2 & \rho_1\rho_2 & 0 & 0 & 0 & 0 & \cdots \\ \rho_1 & -\overline{a}_1 a_2 & \overline{a}_1\rho_2 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -\rho_2 a_3 & -\overline{a}_2 a_3 & -\rho_3 a_4 & \rho_3\rho_4 & 0 & 0 & \cdots \\ 0 & \rho_2\rho_3 & \overline{a}_2\rho_3 & -\overline{a}_3 a_4 & \overline{a}_3\rho_4 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -\rho_4 a_5 & -\overline{a}_4 a_5 & -\rho_5 a_6 & \rho_5\rho_6 & \cdots \\ 0 & 0 & 0 & \rho_4\rho_5 & \overline{a}_4\rho_5 & -\overline{a}_5 a_6 & \overline{a}_5\rho_6 & \cdots \\ 0 & 0 & 0 & 0 & 0 & -\rho_6 a_7 & -\overline{a}_6 a_7 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \tag{3}$$

We want it to stand out that the Laurent polynomials Λ are always dense in the space L^2_μ. Hence, this matrix is always the representation of the full multiplication operator T_μ. Notice that it is a banded matrix instead of a Hessenberg matrix. Also, the dependence of its elements on the parameters $(a_n)_{n\geq 0}$ is simpler than in the Hessenberg case. This five-diagonal representation has a factorization $\mathcal{C} = \mathcal{C}_o\mathcal{C}_e$ as a product of 2×2 block-diagonal symmetric unitary matrices

$$\mathcal{C}_o = \begin{pmatrix} \Theta_1 & & & \\ & \Theta_3 & & \\ & & \Theta_5 & \\ & & & \ddots \end{pmatrix}, \quad \mathcal{C}_e = \begin{pmatrix} I_1 & & & \\ & \Theta_2 & & \\ & & \Theta_4 & \\ & & & \ddots \end{pmatrix} \text{ with } \Theta_k = \begin{pmatrix} -a_k & \rho_k \\ \rho_k & \bar{a}_k \end{pmatrix}. \tag{4}$$

Unlike for the full infinite matrices, the principal submatrices of order n, which we denote as $\mathcal{H}_n$ and $\mathcal{C}_n$, are not unitary. But if we change the last parameter $a_n \in \mathbb{D}$ into a complex number $u \in \mathbb{T}$, we obtain unitary matrices whose characteristic polynomials are the so called para-orthogonal polynomials (POP), introduced for the first time in [20].

There it was proved that such POP are a multiple of $z\varphi_{n-1}(z) + u\varphi^*_{n-1}(z)$ and have simple zeros lying on $\mathbb{T}$, which play the role of the nodes of Szegő quadrature formulas on $\mathbb{T}$.

So, the nodes of Szegő-quadrature formulas can be obtained as eigenvalues of the Hessenberg or the five-diagonal unitary matrices. Both of these matrices can be expressed as a product of embedded Givens rotations. The difference being the order in which the Givens rotations are introduced. See for example the recent paper [11], but the original ideas are already in [1]. The Hessenberg matrix corresponds to a monotone ordering, while the five-diagonal matrix corresponds to an even-odd ordering. See also the survey paper by B. Simon [23] and the references given there.

3 The spaces $\mathcal{L}_n$ and $\mathcal{M}_n$

In general, and particularly on $\mathbb{T}$, a fundamental ingredient in the theory of orthogonal rational functions are the Möbius transforms ζ_α defined for any $\alpha \in \mathbb{D}$ by

$$\zeta_\alpha = \frac{\varpi_\alpha^*(z)}{\varpi_\alpha(z)} = \frac{z-\alpha}{1-\overline{\alpha}z},$$

where $\varpi_\alpha(z) = 1 - \overline{\alpha}z$. This ζ_α is a bijection on $\overline{\mathbb{C}}$ that leaves invariant $\mathbb{T}$, $\mathbb{D}$ and $\mathbb{E}$. We shall write ζ_i for ζ_{α_i} and ϖ_i for ϖ_{α_i} etc.

In order to obtain rational functions with fixed poles in $\mathbb{E}$, we consider a sequence $(\alpha_n)_{n\geq 1}$, $(\alpha_0 = 0)$, and define the finite Blaschke products $(B_n)_{n\geq 0}$ by

$$B_0 = 1 \text{ and } B_n = \zeta_1 \ldots \zeta_n = \frac{\pi_n^*}{\pi_n} \text{ for } n \geq 1$$

where $\pi_n(z) = \varpi_1(z) \cdots \varpi_n(z)$. The subspace

$$\mathcal{L}_n = \operatorname{span}\{B_0, B_1, \ldots, B_n\} = \left\{ \frac{p(z)}{\pi_n(z)} : p \in \mathbb{P}_n \right\},$$

is a space of rational functions whose poles are all in the prescribed set $\hat{\alpha} = \{\hat{\alpha}_i : i = 1\ldots,n\} \subset \mathbb{E}$, with $\hat{\alpha}_i = 1/\overline{\alpha}_i$. Notice that if we choose $\alpha_i = 0, \forall i \in \mathbb{N}$, then $\mathcal{L}_n = \mathbb{P}_n$. We will use the notation $\mathcal{L} = \cup_{n\geq 0}\mathcal{L}_n$, i.e., $\mathcal{L} = \operatorname{span}\{B_n\}_{n\geq 0}$. By the substar transformation we can also define

$$\mathcal{L}_{n*} = \{f : f_* \in \mathcal{L}_n\} = \operatorname{span}\{B_{1*}, \ldots, B_{n*}\} = \left\{ \frac{q(z)}{\pi_n^*(z)} : q \in \mathbb{P}_n \right\},$$

and $\mathcal{L}_* = \cup_{n\geq 0}\mathcal{L}_{n*}$. For $m, n \in \mathbb{N}$ denote $\mathcal{R}_{m,n} = \mathcal{L}_{m*} + \mathcal{L}_n$ and $\mathcal{R} = \mathcal{L}_* + \mathcal{L}$. Notice that

$$\mathcal{R}_{m,n} = \operatorname{span}\{B_{m*}, \ldots, B_{1*}, 1, B_1, \ldots, B_n\}$$

Finally we introduce for $\phi_n = p_n/\pi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ the notation

$$\phi_n^* = B_n \phi_{n*} = \frac{p_n^*}{\pi_n} \in \mathcal{L}_n.$$

The rational functions $(\phi_n)_{n\geq 0}$ that arise from the orthonormalization of the basis $(B_n)_{n\geq 0}$ with respect to the inner product induced by the measure μ on $\mathbb{T}$, are called orthogonal rational functions (ORF) with respect to μ. We can normalize then to get orthonormal functions, but these are not uniquely defined since we can always multiply with a unimodular constant. This constant can be fixed by choosing the leading coefficient $\kappa_n = \overline{\phi_n^*(\alpha_n)}$ of ϕ_n to be real and positive. For simplicity though we do not stick to this normalisation here. Instead we suppose these functions satisfy the simple recurrence relation (see [7]) which implicitly fixes these unimodular factors for each n.

$$\begin{pmatrix} \phi_n(z) \\ \phi_n^*(z) \end{pmatrix} = e_n \frac{\varpi_{n-1}(z)}{\varpi_n(z)} \begin{pmatrix} 1 & a_n \\ \overline{a}_n & 1 \end{pmatrix} \begin{pmatrix} \zeta_{n-1}(z)\phi_{n-1}(z) \\ \phi_{n-1}^*(z) \end{pmatrix}, \qquad n \geq 1, \tag{5}$$

with

$$a_n = \frac{\phi_n(\alpha_{n-1})}{\phi_n^*(\alpha_{n-1})}, \qquad e_n = \sqrt{\frac{\varpi_n(\alpha_n)}{\varpi_{n-1}(\alpha_{n-1})}\frac{1}{1-|a_n|^2}}. \tag{6}$$

In the polynomial case, corresponding to $\alpha_n = 0$ for all n, (5) turns out to be exactly (1). As in the polynomial situation the parameters $(a_n)_{n\geq 1}$ of (5) lie on $\mathbb{D}$.

The matrix representation $\mathcal{V}$ for the isometric operator $T_\mu \restriction \mathcal{L}$ with respect to a basis of ORF has been recently obtained (see [27]). Under the general assumption that the sequence $(\alpha_n)_{n\geq 1}$ is compactly included in $\mathbb{D}$, the space $\mathcal{L}$ is T_μ-invariant and the matrix of the isometric operator $T_\mu \restriction \mathcal{L}$ with respect to the corresponding ORF $(\phi_n)_{n\geq 0}$ is

$$\mathcal{V} = \tilde{\zeta}_{\mathcal{A}}(\mathcal{H}), \tag{7}$$

$\mathcal{A}$ being the diagonal matrix with the sequence $(\alpha_n)_{n\geq 0}$ in its diagonal, $\mathcal{H}$ is the Hessenberg matrix given by (2) where $\rho_n = \sqrt{1-|a_n|^2}$ with a_n like on (6) and $\tilde{\zeta}_{\mathcal{A}}(\mathcal{H})$ represents the matrix Möbius transformation of $\mathcal{H}$, (see [27] and references therein),

$$\tilde{\zeta}_{\mathcal{A}}(\mathcal{H}) = \eta_{\mathcal{A}}^{-1}\tilde{\varpi}^*(\mathcal{H})\tilde{\varpi}(\mathcal{H})^{-1}\eta_{\mathcal{A}^\dagger} \tag{8}$$

where, $\eta_A = \sqrt{1-AA^\dagger}$, $\tilde{\varpi}_A(T) = 1 + A^\dagger T$ and $\tilde{\varpi}_A^*(T) = T + A$, $A^\dagger$ being the adjoint of A. Contrary to the polynomial case, the unitary matrix $\mathcal{V}$ is not a Hessenberg matrix in general, but is the operator Möbius transformation of a Hessenberg matrix.

Like in the polynomial case, we will use another basis of ORF in order to obtain a simpler matricial representation. Instead of ORF $(\phi_n)_{n\geq 0}$ with poles in $\mathbb{E}$ we can use other ones whose poles are alternatively in $\mathbb{E}$ and $\mathbb{D}$. For this purpose we define the odd and even Blaschke products

$$B_0^o = B_0^e = 1 \text{ while } B_n^o = \zeta_1\zeta_3\ldots\zeta_{2n-1}, \quad B_n^e = \zeta_2\zeta_4\ldots\zeta_{2n} \text{ for } n \geq 1,$$

and consider the rational functions given by

$$\chi_{2n} = B_{n*}^e\phi_{2n}^*, \qquad \chi_{2n+1} = B_{n*}^e\phi_{2n+1}, \qquad n \geq 0.$$

The subspaces $\mathcal{M}_n = \text{span}\,\{\chi_0, \chi_1, \ldots, \chi_n\}$ are

$$\mathcal{M}_{2n} = B_{n*}^e\mathcal{L}_{2n} = \text{span}\left\{B_{0*}^e, B_1^o, B_{1*}^e, \ldots, B_{n-1*}^e B_n^o\right\}$$

and

$$\mathcal{M}_{2n+1} = B_{(n+1)*}^e\mathcal{L}_{2n+1} = \text{span}\,\{B_{0*}^e, B_1^o, B_{1*}^e, \ldots, B_n^o, B_{n*}^e\}$$

i.e., $\mathcal{M}_{2n}$, $\mathcal{M}_{2n+1}$ are the sets of ORF whose poles, counted with multiplicity, are in $\{\hat{\alpha}_1, \alpha_2, \ldots, \alpha_{2n-2}, \hat{\alpha}_{2n-1}\}$ and in $\{\hat{\alpha}_1, \alpha_2, \ldots, \hat{\alpha}_{2n-1}, \alpha_{2n}\}$ respectively.

The orthonormality of $(\phi)_{n\geq 0}$ is equivalent to the orthonormality of the sequence $(\chi_n)_{n\geq 0}$ (see [27]). Another way of constructing this basis is introduced in [6].

Notice that $\mathcal{M}_{2n} = \mathcal{R}_{n,n}$ and $\mathcal{M}_{2n+1} = \mathcal{R}_{n,n+1}$. In what follows we will use both notations as will be convenient.

This new basis $(\chi_n)_{n\geq 0}$ provides a simpler matricial representation for the multiplication operator under non-restrictive conditions. As in the Hessenberg case, if the sequence $(\alpha_n)_{n\geq 1}$

is compactly included in $\mathbb{D}$, the ORF $(\chi_n)_{n\geq 0}$ associated with $\{\alpha_1, \hat{\alpha}_2, \alpha_3, \hat{\alpha}_4 \ldots\}$, are a basis of L^2_μ and the matrix of T_μ with respect to $(\chi_n)_{n\geq 0}$ is

$$\mathcal{U} = \tilde{\zeta}_{\mathcal{A}}(\mathcal{C}), \tag{9}$$

where again $\mathcal{A}$ is the diagonal matrix with the poles in its diagonal, $\mathcal{C}$ given by (3) where $\rho_n = \sqrt{1-|a_n|^2}$ with a_n like on (6) and $\tilde{\zeta}_{\mathcal{A}}(\mathcal{C})$ is given by (8). The unitary matrix $\mathcal{U}$ is not a five-diagonal matrix in general, but is the operator Möbius transformation of a five-diagonal matrix.

4 Para-orthogonal rational functions

The goal of this section is to introduce the para-orthogonal rational functions and to show how their zeros can be calculated as eigenvalues of the matrix Möbius transformations considered in the previous section.

Given a sequence of orthogonal rational functions $(\phi_n)_{n\geq 0}$ with poles $\{\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n\}$ in $\mathbb{E}$, the para-orthogonal rational functions (PORF) are defined by

$$P_n^v(z) = \phi_n(z) + v\phi_n^*(z), \qquad v \in \mathbb{T}. \tag{10}$$

The PORF are the generalization of the POP to the rational case. The interest of the PORF relies in the following result, (see [7]).

Theorem 4.1. *The para-orthogonal rational function P_n^v has n simple zeros which lie on the unit circle.*

A matricial interpretation can be obtained for the zeros of PORF. Using the recurrence relation (5), we can write P_n^v as follows

$$P_n^v(z) = (1+\overline{a}_n v)e_n \frac{\varpi_{n-1}}{\varpi_n}(\zeta_{n-1}\phi_{n-1}(z) + u\phi_{n-1}^*), \qquad u = \tilde{\zeta}_{a_n}(v).$$

So, as in the polynomial case, P_n^v is obtained from n steps of the recurrence relation (5), but changing in the n-th step the parameter $a_n \in \mathbb{D}$ to $u = \tilde{\zeta}_{a_n}(v) \in \mathbb{T}$. Here $\tilde{\zeta}_a(v) = (1+\overline{a}v)/(v+a_n)$ is the scalar version of the previously introduced operator transformation (8). Note that the factors η_a^{-1} and η_a cancel in the scalar case.

Let us introduce the following notation: $\mathcal{A}_n = \operatorname{diag}(\alpha_0, \ldots, \alpha_{n-1})$, $\mathcal{H}_n^u$ stand for the unitary truncated Hessenberg matrix of dimension n based on the parameters $a_1, \ldots, a_{n-1}, u$ like in (2) and set $\mathcal{V}^{(n;u)} = \tilde{\zeta}_{\mathcal{A}_n}(\mathcal{H}_n^u)$. Also $\mathcal{C}_n^u$ and $\mathcal{U}^{(n;u)}$ are introduced in a similar way. This means that $\mathcal{V}^{(n;u)}$ and $\mathcal{U}^{(n;u)}$ represent the unitary n-th principal submatrices of $\mathcal{V}$ and $\mathcal{U}$ respectively, given by (7) and (9) after substituting $u \in \mathbb{T}$ for the last parameter $a_n \in \mathbb{D}$.

Then the following result provides a first matricial representation of the PORF as well as of their zeros in terms of matrix Möbius transform of a Hessenberg matrices, (see [27]).

Theorem 4.2. *Let $(\alpha_n)_{n\geq 1}$ be an arbitrary sequence in $\mathbb{D}$, μ a measure on $\mathbb{T}$, $(\phi_n)_{n\geq 0}$ the corresponding ORF. If $P_n^v = \phi_n + v\phi_n^*$ is the n-th PORF related to $v \in \mathbb{T}$, and if we introduce $u = \tilde{\zeta}_{a_n}(v)$, then:*

(i) The zeros of P_n^v are the eigenvalues of $\mathcal{V}^{(n;u)}$ and if λ is an eigenvalue then the related eigenvectors are spanned by $(\phi_0(\lambda),\ldots,\phi_n(\lambda))^\dagger$. Equivalently, this λ is also a generalized eigenvalue of the pencil $(\mathcal{A}_n+\mathcal{H}_n^u, \mathcal{I}_n+\mathcal{A}_n^\dagger\mathcal{H}_n^u)$ and the corresponding left eigenvectors are spanned by $(\phi_0(\lambda),\ldots,\phi_{n-1}(\lambda))\eta_{\mathcal{A}_n}^{-1/2}$.

(ii) $P_n^v = \dfrac{p_n^v}{\pi_n}$, with p_n^v proportional to the characteristic polynomial of $\mathcal{V}^{(n;u)}$.

A second matricial representation of the PORF is also obtained in [27].

Theorem 4.3. *Let $(\alpha_n)_{n\geq 1}$ be an arbitrary sequence in $\mathbb{D}$, μ a measure on $\mathbb{T}$, $(\phi_n)_{n\geq 0}$ the corresponding ORF and $(\chi_n)_{n\geq 0}$ the ORF associated with the sequence $\{\alpha_1,\hat{\alpha}_2,\alpha_3,\hat{\alpha}_4,\ldots\}$. If $P_n^v=\phi_n+v\phi_n^*$ is the n-th PORF related to $v\in\mathbb{T}$, and if we introduce $u=\tilde{\zeta}_{a_n}(v)$, then:*

(i) The zeros of P_n^v are the eigenvalues of $\mathcal{U}^{(n;u)}$ and if λ is an eigenvalue then the related eigenvectors are spanned by $(\chi_0(\lambda),\ldots,\chi_{n-1}(\lambda))^\dagger$. Equivalently, this λ is also a generalized eigenvalue of the pencil $(\mathcal{A}_n+\mathcal{C}_n^u, \mathcal{I}_n+\mathcal{A}_n^\dagger\mathcal{C}_n^u)$ and the corresponding left eigenvectors are spanned by $(\chi_0(\lambda),\ldots,\chi_n(\lambda))\eta_{\mathcal{A}_n}^{-1}$.

(ii) $P_n^v = \dfrac{p_n^v}{\pi_n}$, with p_n^v proportional to the characteristic polynomial of $\mathcal{U}^{(n;u)}$.

Taking into account (4) we have also the factorization $\mathcal{C}_n^u=\mathcal{C}_{on}^u\mathcal{C}_{en}^u$ where $\mathcal{C}_{on}^u$ and $\mathcal{C}_{en}^u$ are the result of substituting a_n by u in $\mathcal{C}_{on}$ and $\mathcal{C}_{en}$ respectively.

Using the fact that $\mathcal{U}^{(n;u)}$ is unitary, we can write $\mathcal{U}^{(n;u)}=[(\mathcal{U}^{(n;u)})^\dagger]^{-1}$ to see that

$$\mathcal{U}^{(n;u)}=\eta_{\mathcal{A}_n}(\mathcal{C}_n^{u\dagger}+\mathcal{A}_n^\dagger)^{-1}(\mathcal{I}_n+\mathcal{C}_n^{u\dagger}\mathcal{A}_n)\eta_{\mathcal{A}_n}^{-1}.$$

Hence, the eigenvalue problem for $\mathcal{U}^{(n;u)}$, viz. $\mathcal{U}^{(n;u)}X_n=X_n\Lambda_n$ can be rewritten as

$$\begin{aligned}
\eta_{\mathcal{A}_n}(\mathcal{C}_n^{u\dagger}+\mathcal{A}_n^\dagger)^{-1}(\mathcal{I}_n+\mathcal{C}_n^{u\dagger}\mathcal{A}_n)\eta_{\mathcal{A}_n}^{-1}X_n &= X_n\Lambda_n\\
(\mathcal{C}_n^{u\dagger}+\mathcal{A}_n^\dagger)^{-1}(\mathcal{I}_n+\mathcal{C}_n^{u\dagger}\mathcal{A}_n)Y_n &= Y_n\Lambda_n, \quad Y_n=\eta_{\mathcal{A}_n}^{-1}X_n\\
(\mathcal{C}_{on}^{u\dagger}+\mathcal{C}_{en}^{u}\mathcal{A}_n^\dagger)^{-1}(\mathcal{C}_{en}^{u}+\mathcal{C}_{on}^{u\dagger}\mathcal{A}_n)Y_n &= Y_n\Lambda_n.
\end{aligned}$$

In other words, we have proved the following theorem.

Theorem 4.4. *Let $(\alpha_n)_{n\geq 1}$ be an arbitrary sequence in $\mathbb{D}$, μ a measure on $\mathbb{T}$, $(\phi_n)_{n\geq 0}$ the corresponding ORF and $(\chi_n)_{n\geq 0}$ the ORF associated with the sequence $\{\alpha_1,\hat{\alpha}_2,\alpha_3,\hat{\alpha}_4,\ldots\}$. Furthermore let $P_n^v=\phi_n+v\phi_n^*$ be the n-th PORF related to $v\in\mathbb{T}$, and suppose that $\mathcal{C}_{on}^u$ and $\mathcal{C}_{en}^u$ are the unitary truncations of the matrices of (4) where a_n is replaced by $\tilde{\zeta}_{a_n}(v)$. Then the zeros of P_n^v are the generalized eigenvalues of the pencil $(\mathcal{C}_{en}^u+\mathcal{C}_{on}^{u\dagger}\mathcal{A}_n, \mathcal{C}_{on}^{u\dagger}+\mathcal{C}_{en}^u\mathcal{A}_n^\dagger)$ and if λ is such an eigenvalue, then the corresponding right eigenvectors are spanned by $[(\chi_0(\lambda),\ldots,\chi_n(\lambda))\eta_{\mathcal{A}_n}^{-1}]^\dagger$.*

5 The computation of rational quadrature formulas

In this section we are interested in the computation of the rational quadrature formulas on the unit circle. For the polynomial case, the Szegő-quadrature formulas were introduced in [14]. A different approach, for Laurent polynomials, is considered in [20].

Given a finite Borel measure μ on $\mathbb{T}$ in order to estimate the integral

$$I_\mu\{f\} = \int_{\mathbb{T}} f(z)d\mu(z)$$

for any function defined on $\mathbb{T}$, the so-called Szegő quadrature formulas, are of the form

$$I_n\{f\} = \sum_{j=1}^{n} A_{j,n} f(z_{j,n}), \tag{11}$$

with mutually distinct nodes $z_{j,n}$ on $\mathbb{T}$ and positive weights $A_{j,n}$, $j = 1, \ldots, n$, and such that $I_\mu\{f\} = I_n\{f\}$ for all $f \in \Lambda_{-(n-1),(n-1)}$.

We will consider formulas like (11) but, instead of Laurent polynomials, we consider the more general rational functions introduced in section 3. This gives rise to the class of rational Szegő quadrature formulas, (see [3, 4, 5, 6]).

The following result about interpolatory quadrature formulas in the rational case, can be found in [4].

Theorem 5.1. *Let $N = 2n+1$. Given the sequence $\{z_{j,N}\}_{j=1}^{N} \subset \mathbb{T}$ there exist unique positive numbers $A_{1,N}, \ldots, A_{N,N}$ depending of μ, such that the formula*

$$I_N\{f\} = \sum_{j=1}^{N} A_{j,N} f(z_{j,N})$$

is exact in $\mathcal{R}_{n,n}$, that is, $I_N\{f\} = I_\mu\{f\}$, $\forall f \in \mathcal{R}_{n,n}$. Furthermore, $I_N\{f\} = I_\mu\{L_N^f\}$ where L_N^f is the unique rational function on $\mathcal{R}_{n,n}$ satisfying the interpolation condition $L_N^f(z_{j,N}) = f(z_{j,N})$, $j = 1, 2 \ldots, N$.

The interpolatory quadrature formulas with $N = 2n + 1$ nodes are exact in the space $\mathcal{R}_{n,n}$ of dimension $2n + 1$. As a special choice of these nodes, one can make the formulas exact in spaces of dimension $2N - 1$. Therefore, the nodes should be chosen as the zeros of para-orthogonal rational functions.

Theorem 5.2. *Let $z_{1,N}, z_{2,N}, \ldots, z_{N,N}$ be the zeros of the N-th PORF $P_N^{u_N}$. Then, there exist positive numbers $A_{1,N}, \ldots, A_{N,N}$ such that the formula*

$$I_N\{f\} = \sum_{j=1}^{N} A_{j,N} f(z_{j,N})$$

is exact, that is, $I_N\{f\} = I_\mu\{f\}$, $\forall f \in \mathcal{R}_{N-1,N-1}$.

In this case, $\mathcal{R}_{N-1,N-1}$ is the maximal domain of validity in the sense that the formula can not be exact in $\mathcal{R}_{N,N-1}$ nor in $\mathcal{R}_{N-1,N}$. Moreover, it was also proved in [5] that the only quadrature formulas with such a maximal domain of validity are just the ones given in Theorem 5.2, i.e., with weights are given by

$$A_{j,N} = \int_{\mathbb{T}} L_{j,N}(z)d\mu(z)$$

and $L_{j,N} \in \mathcal{R}_{0,N-1}$ is defined by the interpolation conditions $L_{j,N}(z_{i,N}) = \delta_{ij}$. These quadrature formulas are called rational Szegő quadrature formulas.

A matricial computation of the nodes and weights can be also considered. With respect to the nodes, after Theorem 5.2 and Theorem 4.2, Theorem 4.3 of the previous section we can enunciate:

Corollary 5.3. *The nodes of rational quadrature formulas on the unit circle are the eigenvalues of matrix Möbius transformations $\mathcal{V}^{(n;u)}$ of unitary truncations of Hessenberg matrices.*

Corollary 5.4. *The nodes of rational quadrature formulas on the unit circle are the eigenvalues of matrix Möbius transformations $\mathcal{U}^{(n;u)}$ of unitary truncations of five-diagonal matrices.*

The weights, as in the polynomial case, can be obtained in terms of the corresponding normalized eigenvector. In [7, Theorem 5.4.2], it has been proved that the weights of rational Szegö quadrature formulas λ_j are given by

$$\lambda_j = \left(\sum_{k=0}^{n-1} |\phi_k(z_j)|^2 \right)^{-1} \tag{12}$$

(ϕ_n) being the sequence of ORF with poles on $\mathbb{E}$. Keeping in mind Theorem 4.2, If z_j is a zero of P_n^v, i.e., z_j is an eigenvalue of $\mathcal{V}^{(n;u)}$ and $V_n(z_j) = (\phi_0(z_j), \cdots \phi_{n-1}(z_j))^\dagger$ is the corresponding eigenvector, then the normalized eigenvector $W(z_j) = (w_0(z_j), \cdots, w_{n-1}(z_j))^\dagger$ is given by

$$W_n(z_j) = V_n(z_j)/\|V_n\| \quad \text{with} \quad \|V_n\| = \left(\sum_{k=0}^{n-1} |\phi_k(z_j)|^2 \right)^{1/2},$$

or equivalently, by (12)

$$(w_0(z_j), \cdots, w_{n-1}(z_j))^\dagger = \lambda_j^{1/2} \left(\phi_0(z_j), \cdots \phi_{n-1}(z_j)\right)^\dagger.$$

Equating the first component of both sides gives

$$w_0(z_j) = \lambda_j^{1/2} \phi_0(z_j).$$

If we are dealing with a probability measure, then $\phi_0(z_j) = 1$ so that $\lambda_j = w_0(z_j)^2$, showing that the weight corresponding to the node z_j is indeed the square of the first component of the corresponding normalized eigenvector.

Using the matrix representation of Theorem 4.3, if z_j is a zero of P_n^v, i.e., an eigenvalue of $\mathcal{U}^{(n;u)}$ and $(\chi_0(z_j), \cdots, \chi_{n-1}(z_j))^\dagger$ the corresponding eigenvector, it is sufficient to remind the fact that

$$\sum_{k=0}^{n-1} |\chi_k(z_j)|^2 = \sum_{k=0}^{n-1} |\phi_k(z_j)|^2$$

to obtain the same result for this situation.

In short the following Propositions have been proved:

Proposition 5.5. *If $z_0, \ldots, z_{n-1}$ are the nodes of the rational Szegö quadrature formula, i.e., the eigenvalues of $\mathcal{V}^{(n;u)}$, the weights are given by the squares of the first components of the corresponding normalized eigenvectors.*

Proposition 5.6. *If $z_0, \dots, z_{n-1}$ are the nodes of the rational Szegö quadrature formula, i.e., the eigenvalues of $\mathcal{U}^{(n;u)}$, the weights are given by the squares of the first components of the corresponding normalized eigenvectors.*

This new point of view avoids the calculation of the nodes of rational quadrature formulas on the unit circle evaluating the zeros of the corresponding PORF. It provides a method to calculate the nodes as eigenvalues of the above mentioned matrices, that requires only the knowledge of the parameters a_n (and hence also ρ_n) and the poles of the corresponding orthogonal rational functions.

6 Some numerical examples

We will illustrate the preceding results with some numerical examples. All the computations will be done by matlab.

1.- In the first one we will consider the function f given by

$$f(z) = \frac{1+z}{1-z/2} + \frac{z}{2-z} + \frac{z^2+3z-z^3}{(z-5)(1-z/6)(1-2z)} \tag{13}$$

and we will approximate $I_\mu\{f\}$, μ being the Lebesgue measure $d\theta/(2\pi)$. For a matricial computation using the operator Möbius transform for Hessenberg and five-diagonal matrices, we need the poles and the sequences (a_n) and (ρ_n) that parameterize these matrices.

We consider $n = 7$ and we choose $\alpha_n = 1/(n+1)$, $n = 1, \dots, 7$ and $\alpha_0 = 0$ as usual. For the Lebesgue measure $a_n = 0$ and hence $\rho_n = 1$, $\forall n \geq 1$.

Solving, in this case, the generalized eigenvector problem for the pencil with an appropriate $v \in \mathbb{T}$

$$\left(\mathcal{A}_n + \mathcal{H}_n^u, \mathcal{I}_n + \mathcal{A}_n^\dagger \mathcal{H}_n^u\right)$$

we obtain the nodes and weights of the Table 1. If we compute them via the eigenvalues and eigenvectors of the matrix $\mathcal{V}^{(n;u)}$, the results are only slightly less accurate since the last digit is affected by rounding errors.

Similarly one can use the generalized eigenvalues and vectors for the pencils

$$\left(\mathcal{A}_n + \mathcal{C}_n^u, \mathcal{I}_n + \mathcal{A}_n^\dagger \mathcal{C}_n^u\right), \quad \text{or} \quad (\mathcal{C}_{en}^u + \mathcal{C}_{on}^{u\dagger}\mathcal{A}_n, \mathcal{C}_{on}^{u\dagger} + \mathcal{C}_{en}^u \mathcal{A}_n^\dagger)$$

or compute the eigenvalues and eigenvectors of $\mathcal{U}^{(n;u)}$. The numerical values of the nodes and weights are the same except for some rounding which only affects one or at most two of the last digits.

Because the given function belongs to the domain of validity, the quadrature formula should give the exact integral, viz. 1, which is indeed the case within machine precision, because the relative error is approximately 10^{-16}.

For a different, nonreal, choice of $v \in \mathbb{T}$, the symmetry is lost, as can be seen in Table 2. Although the symmetry is lost, the integral is still approximated with the same accuracy as in the symmetric case.

Table 1: The nodes and weights for the Lebesgue measure $\frac{d\theta}{2\pi}$ when the poles are given by $\alpha_k = 1/(k+1)$, $k = 1, \ldots, 7$ and $v = -1$.

θ_j of nodes $z_j = \exp(i\theta_j)$	weights
0	$8.403361344537813\text{E} - 002$
$\pm 2.465407008647439\text{E} + 000$	$2.077776600510375\text{E} - 001$
$\pm 1.336738281545516\text{E} + 000$	$1.488205944472059\text{E} - 001$
$\pm 5.650413576629920\text{E} - 001$	$1.013849387790675\text{E} - 001$

Table 2: The nodes and weights for the Lebesgue measure $\frac{d\theta}{2\pi}$ when the poles are given by $\alpha_k = 1/(k+1)$, $k = 1, \ldots, 7$ and $v = i$.

θ_j of nodes $z_j = \exp(i\theta_j)$	weights
$2.798995563470065\text{E} + 000$	$2.161442192462172\text{E} - 001$
$-2.148422117836067\text{E} + 000$	$1.952619467196891\text{E} - 001$
$1.582596667215560\text{E} + 000$	$1.643566429924263\text{E} - 001$
$-1.114446080717590\text{E} + 000$	$1.344496322877105\text{E} - 001$
$7.313150857335874\text{E} - 001$	$1.106238819774966\text{E} - 001$
$-4.118403543362309\text{E} - 001$	$9.399019672023695\text{E} - 002$
$1.325975632655727\text{E} - 001$	$8.517348005622358\text{E} - 002$

Note that this quadrature formula with only two points will compute up to rounding errors the exact value of any integral of the form

$$\int_{-\pi}^{\pi} \frac{az + b + c/z}{|z - \alpha_1|^2} \frac{d\theta}{2\pi}, \quad \text{where } z = e^{i\theta}.$$

Another, less trivial example is given by the integral of the function $f(z) = z^p/(\sin(z^5) + 0.2)$. This function has singularities that are close to the fifth roots of 0.2, which can be easily computed. Let us call these roots $\alpha_1, \ldots, \alpha_5$ and set $\alpha_{i+5k} = \alpha_i$ for $i = 1, \ldots, 5$ and $k = 1, 2, \ldots$, then one may expect that a rational Szegő quadrature formula based on these poles will give a good approximation.

Indeed, Figure 1 shows the log of the relative error of the quadrature as the number n of nodes increases. On the left you see the values for $p = -1$ and $n = 5, 6, \ldots, 100$. As the approximation becomes better, an oscilatory convergence behaviour is observed which is caused by the asymmetry occurring when not all the five poles are included in another round. So we have computed only the approximants for $n = 5, 10, 15, \ldots, 200$, which is shown on the right for $p = 0$ (solid line), $p = -1$ (dashed line) and $p = 1$ (dash-dot line).

The linear convergence is obvious. Also the stability is clear since the rounding errors do not blow up when maximal precision is reached. Since the exact value of the integral is not available, the "exact value" is computed by the matlab built-in function quad to maximal precision, which is about 14 digits.

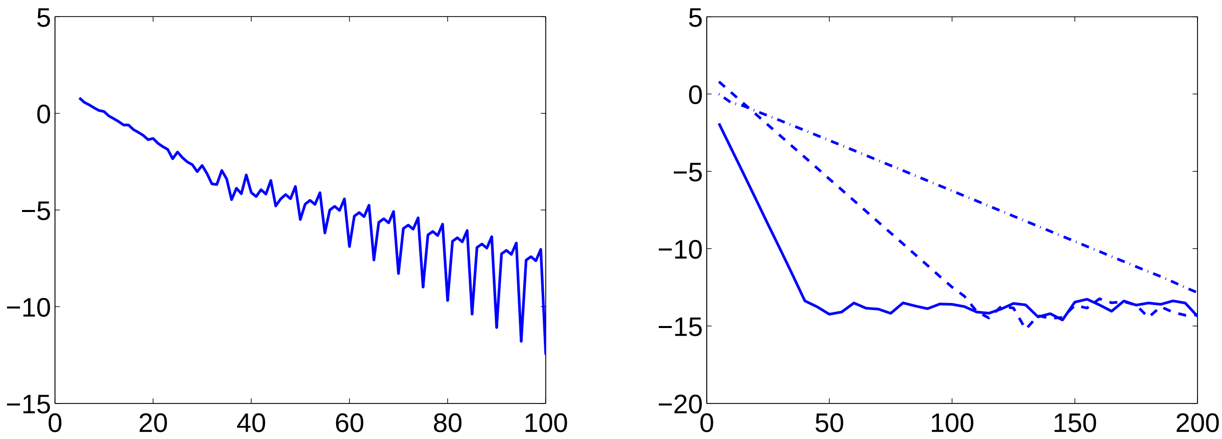

Figure 1: The relative error of the integral $\int \frac{z^p}{\sin(z^5)+0.2} d\theta$ when it is computed with an n-point formula. The log of the relative error as a function of n. On the left you see the values for $p = -1$ and $n = 5, 6, \ldots, 100$ while on the right it plots only the values for $n = 5, 10, 15, \ldots, 200$ when $p = 0$ (solid line), $p = -1$ (dashed line) and $p = 1$ (dash-dot line).

2.- In the second example we will consider a rational modification of the Lebesgue measure

$$d\mu(\theta) = \frac{1-|r|^2}{|z-r|^2} \frac{d\theta}{2\pi}, \quad r \in \mathbb{D}, \quad z = e^{i\theta}$$

and we shall calculate the nodes and the weights for the n-point rational Szegő quadrature formulas for the measure μ.

Also in this case we know the Schur parameters for the ORF. They are all equal to zero except the first one, which is $a_1 = -r$. This can be derived from the formula (3.1) in [5] giving explicit expressings for the ORF, and knowing that $a_n = \phi_n(\alpha_{n-1})/\phi_n^*(\alpha_{n-1})$, we get this result. Note that this can also be obtained by considering this weight as a rational modification of the Lebesgue measure, [12, Thm. 7].

Testing the rational integrand (13) again, we get for all tested values for $r \in (-1, 1)$ and different values of v always the same result as with the matlab built-in function quad up to the accuracy that could be reached with quad. Even with only a 3-point rule, we get a relative error of $O(10^{-3})$ which runs up to machine precision for an n-point formula with $n \geq 7$.

Also the sinus integrand $z^p/(\sin(z^5) + 0.2)$ has been tested for this Poisson weight, and again, the method converges, but now for $p = -1, 0, 1$, the convergence behaviour is very similar to the case $p = 1$ for the Lebesgue measure (the dash-dot line in the right hand side of Figure 1). This was independent of the chosen value of $r \in \mathbb{D}$ and $v \in \mathbb{T}$.

3.- As a last example of a weight function we take the Chebyshev weights $w(\theta) = 1 \pm \cos\theta$. Explicit expressions for the ORF were obtained in [2]. We cite the following result.

Theorem 6.1. *Let $\nu \in \{1, -1\}$ and consider the measures $d\mu(\theta) = (1 - \nu\cos(\theta))\frac{d\theta}{2\pi}$. Define*

$$X_n(z) = c_n + z^2(z - b_n)\frac{B_{n-1}(z)}{1 - \overline{\alpha}_n z}$$

where

$$c_n = \frac{\nu B_{n-1}(\nu)}{(1-\nu\overline{\alpha}_n)Q_n + 1}, \quad b_n = \nu + \frac{1-\nu\overline{\alpha}_n}{B_{n-1}(\nu)}c_n, \quad \textit{and} \quad Q_n = 1 + \sum_{k=1}^{n-1} \frac{1-|\alpha_k|^2}{|\nu-\alpha_k|^2}.$$

Then the functions $\Phi_n \in \mathcal{L}_n$ *defined by*

$$\Phi_0 = 1, \quad \Phi_n(z) = \frac{X_n(z)}{(z-\nu)^2}, \quad n = 1, 2, \dots$$

form an orthogonal system for the measure $d\mu(\theta)$.

Remark 6.2. Note that $X_n(z)$ has a double zero in $z = \nu$ so that the Φ_n are indeed in the appropriate space.

The simple recursion (5) fixed uniquely the orthonormal basis, which implies that there must exist unimodular constants $\rho_n \in \mathbb{T}$ such that $\phi_n = \rho_n \Phi_n / \|\Phi_n\|$. Hence, the Schur parameters a_n that are used in this paper are given by

$$a_n = \frac{\phi_n(\alpha_{n-1})}{\phi_n^*(\alpha_{n-1})} = \rho_n^2 \frac{\Phi_n(\alpha_{n-1})}{\Phi_n^*(\alpha_{n-1})} = \rho_n^2 \hat{a}_n, \quad \hat{a}_n = \frac{\Phi_n(\alpha_{n-1})}{\Phi_n^*(\alpha_{n-1})}.$$

Hence we need to know the value of $\rho_n \in \mathbb{T}$, which is given in the following Lemma.

Lemma 6.3. *The* Φ_n *defined in Theorem 6.1 and the* ϕ_n *defined by (5) for one of the Chebyshev weights measures* $d\mu(\theta) = (1 - \nu\cos(\theta))\frac{d\theta}{2\pi}$ *are related by* $\phi_n = \rho_n \Phi_n / \|\Phi_n\|$ *with* ρ_n *recursively defined by* $\rho_0 = 1$ *and for* $n > 0$*:* $\rho_n = \rho_{n-1} e^{it}$ *where*

$$t = \arg\left(\frac{1 - b_{n-1}\overline{\alpha}_{n-1}}{1 - b_n\overline{\alpha}_{n-1}}\right),$$

b_k *as defined in the previous theorem and* $\arg(z) = z/|z|$ *for* $z \neq 0$.

Proof. Take the second relation of (5), set $\phi_n^* = \overline{\rho}_n \tilde{\Phi}_n^*$, with $\tilde{\Phi}_n = \Phi_n / \|\Phi_n\|$, and take $z = \alpha_{n-1}$, then this results in

$$\overline{\rho}_n \tilde{\Phi}_n^*(\alpha_{n-1}) = e_n \frac{\varpi_{n-1}(\alpha_{n-1})}{\varpi_n(\alpha_{n-1})} \overline{\rho}_{n-1} \tilde{\Phi}_{n-1}^*(\alpha_{n-1})$$

so that

$$\rho_n = \rho_{n-1} \frac{\tilde{\Phi}_n^*(\alpha_{n-1})\varpi_n(\alpha_{n-1})}{e_n \varpi_{n-1}(\alpha_{n-1})\tilde{\Phi}_{n-1}^*(\alpha_{n-1})}.$$

Since $e_n \varpi_{n-1}(\alpha_{n-1})$ is real, we only need to find $t = \arg\Big(\tilde{\Phi}_n^*(\alpha_{n-1})\varpi_n(\alpha_{n-1}) / \tilde{\Phi}_{n-1}^*(\alpha_{n-1})\Big)$. Using the expressions of the previous theorem, we easily get

$$t = \arg\left(\varpi_n(\alpha_{n-1}) \frac{\frac{1-\bar{b}_n\alpha_{n-1}}{\varpi_n(\alpha_{n-1})(1-\nu\alpha_{n-1})^2}}{\frac{1-\bar{b}_{n-1}\alpha_{n-1}}{\varpi_{n-1}(\alpha_{n-1})(1-\nu\alpha_{n-1})^2}}\right) = \arg\left(\frac{1 - b_{n-1}\overline{\alpha}_{n-1}}{1 - b_n\overline{\alpha}_{n-1}}\right).$$

□

Thus we may conclude with the following Corollary.

Corollary 6.4. *The Schur parameters for one of the Chebyshev weights* $w(\theta) = 1 - \nu\cos(\theta)$, $\nu \in \{-1, 1\}$, *are given by*

$$a_n = \rho_n^2 c_n \frac{1 - \overline{\alpha}_n \alpha_{n-1}}{1 - \overline{b}_n \alpha_{n-1}}$$

with c_n *and* b_n *as defined in Theorem 6.1 and* ρ_n *as defined in Lemma 6.3.*

Proof. Using the explicit expressions for Φ_n of Theorem 6.1, it is easily verified that

$$\hat{a}_n = \frac{\Phi_n(\alpha_{n-1})}{\Phi_n^*(\alpha_{n-1})} = \frac{\frac{c_n}{(\nu - \alpha_{n-1})^2}}{\frac{1 - \overline{b}_n \alpha_{n-1}}{(1 - \nu\alpha_{n-1})^2 (1 - \overline{\alpha}_n \alpha_{n-1})}} = c_n \frac{1 - \overline{\alpha}_n \alpha_{n-1}}{1 - \overline{b}_n \alpha_{n-1}}.$$

□

With these formulas, the Schur parameters for $\nu = 1$ and $\nu = -1$ are listed in Table 3. When computing the nodes and weights as in the previous case where we considered the integration of the function $z^p/(\sin(z^5) + 0.2)$, we get convergence as in Figure 2.

Table 3: The first Schur parameters for the Chebyshev weights.

$\mathrm{Re}(a_n)$, $\nu = 1$	$\mathrm{Im}(a_n)$, $\nu = 1$	$\mathrm{Re}(a_n)$, $\nu = -1$	$\mathrm{Im}(a_n)$, $\nu = -1$
6.484980244349786E − 1	−1.954313487379818E − 1	−3.764308327076129E − 1	−6.200419234988335E − 2
3.994179878071785E − 1	−2.347446214711291E − 2	1.051782778929641E − 1	+2.186201793920325E − 1
3.504916847850052E − 2	−1.073917665159091E − 1	−1.675443179555413E − 1	−5.900178117799699E − 1
1.322850313122597E − 2	−1.050230830923701E − 1	3.110623057484530E − 2	+5.677630782386957E − 1
1.524441654736276E − 1	−2.020098396942887E − 1	−1.614117089087093E − 2	−6.001993478955740E − 2
2.658482542075947E − 1	−3.905401355715618E − 1	2.760486516596777E − 2	+2.616086292739314E − 2
1.741035652409017E − 1	−7.355940937308936E − 2	−2.431920858452069E − 2	−5.293633515052289E − 2
1.051351423888892E − 2	−5.748546887279331E − 2	4.384436283821600E − 2	+2.580212169863806E − 1
4.576707730393982E − 3	−5.628360560630356E − 2	−1.816046710658346E − 2	−2.525265659969841E − 1
9.775024797623259E − 2	−1.106486424260548E − 1	1.063183069696326E − 2	+3.256707864057530E − 2

Since the nodes are easily available through these computations, we have plotted them in Figure 3 as a function of the parameter v, more precisely as a function of $\theta \in [\pi, \pi]$ where $v = e^{i\theta}$. The measures are the Chebyshev measures $d\mu(\theta) = (1 - \nu\cos(\theta))\frac{d\theta}{2\pi}$, and the poles are given by $r(-1)^{1/5}$ with as in the our previous examples $r \approx 0.72$. These poles do not have great influence on the position of the nodes, which vary smoothly with θ. However, when $r = 0.9$, then the influence is more pronounced resulting in more abrupt changes. The corresponding weights are plotted in Figure 4.

When the integral of $1/(\sin(z^5) + a^5)$ with respect to the Chebyshev weights is computed with for example $a = 0.9$, we can see the same kind of linear convergence that we have illustrated before, but the speed of convergence is much slower. For example with $a = 0.9$, a 200-point formula still has only 7 or 8 correct digits both for $\nu = 1$ and $\nu = -1$.

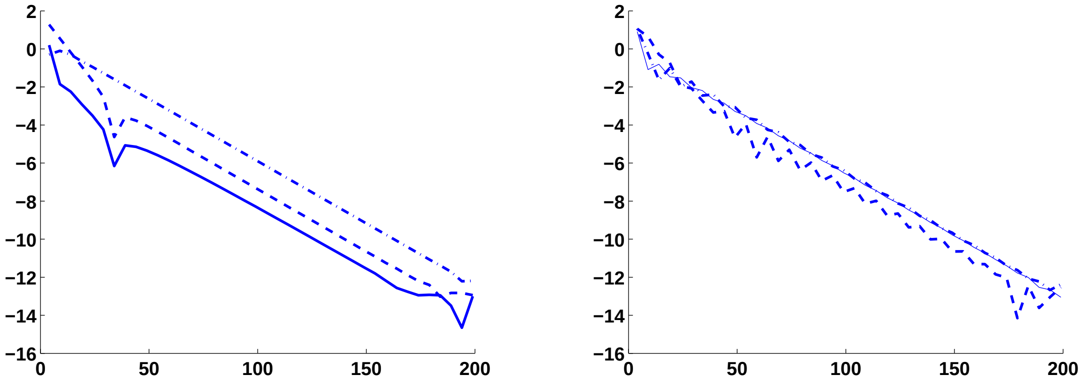

Figure 2: The relative error of the integral $\int \frac{z^p}{\sin(z^5)+0.2} d\theta$ when it is computed with an n-point formula for $n = 5, 10, 15, \ldots, 200$. The log of the relative error as a function of n is plotted. Here $p = 0$ (solid line), $p = -1$ (dashed line) and $p = 1$ (dash-dot line) are plotted on the left for $\nu = 1$ and on the right for $\nu = -1$.

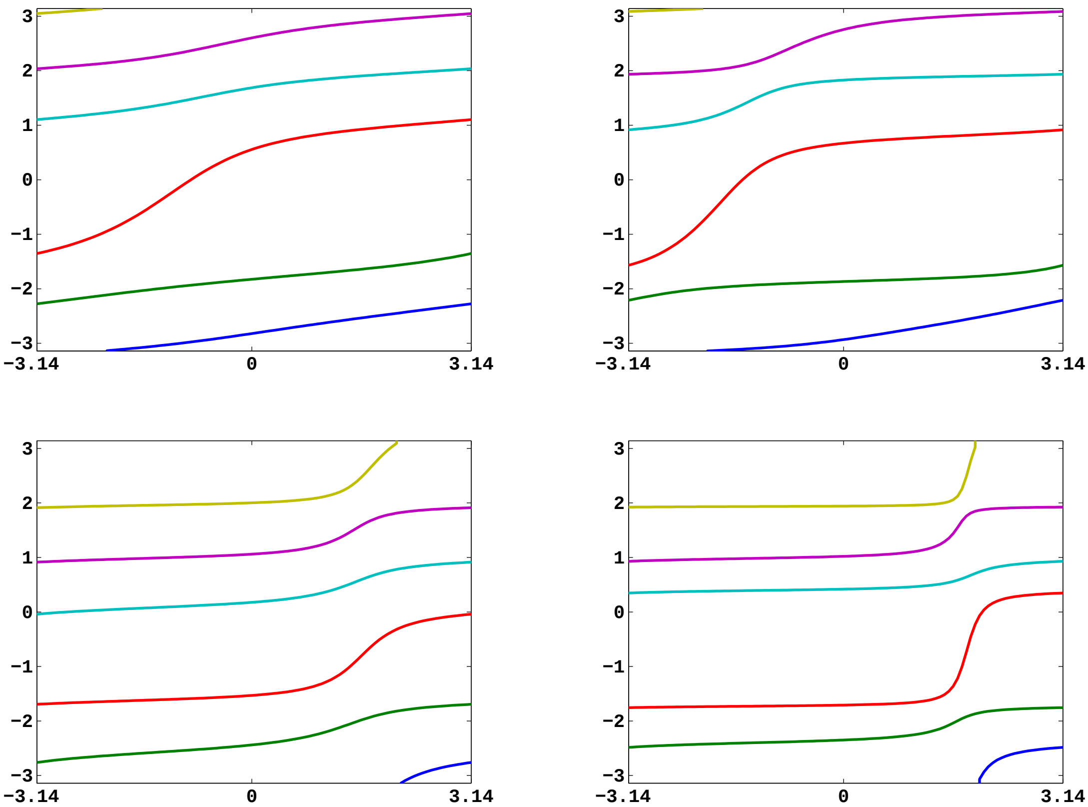

Figure 3: The nodes of the of the quadrature formula with $n = 5$ points as a function of $\theta \in [-\pi, \pi]$ where $v = e^{i\theta}$. The measure is $d\mu(\theta) = (1 - \nu\cos(\theta))d\theta/(2\pi)$. The poles are given by $r(-1)^{1/5}$. On the left for $r = 0.72$ and on the right for $r = 0.9$. On top for $\nu = 1$, at the bottom for $\nu = -1$.

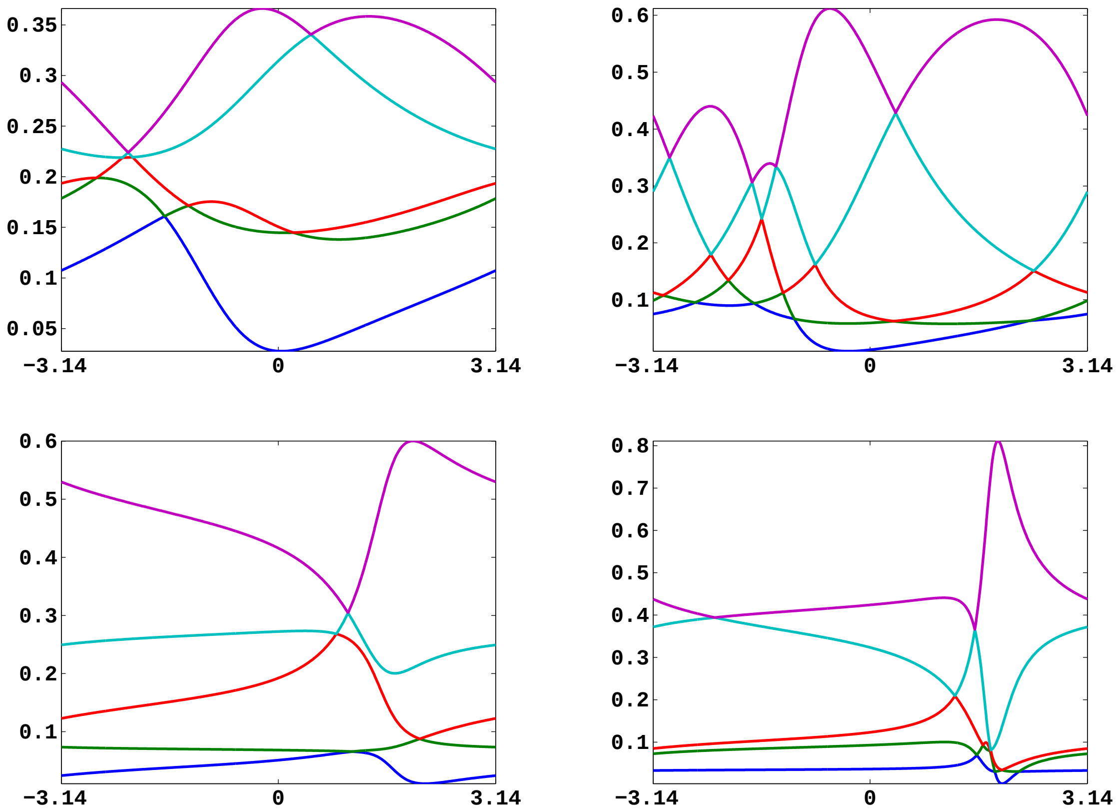

Figure 4: The weights of the of the quadrature formula with $n = 5$ points as a function of $\theta \in [-\pi, \pi]$ where $v = e^{i\theta}$. The measure is $d\mu(\theta) = (1 - \nu\cos(\theta))d\theta/(2\pi)$. The poles are given by $r(-1)^{1/5}$. On the left for $r = 0.72$ and on the right for $r = 0.9$. On top for $\nu = 1$, at the bottom for $\nu = -1$.

7 Concluding remarks

The previous examples show that these (generalized) eigenvalue methods for computing the nodes and weights of rational Szegő quadrature formulas are very efficient and stable as are the polynomial counterparts.

The main purpose of this paper was to show that also in the case of quadrature formulas on the unit circle based on orthogonal rational functions, the nodes and weights can be computed by solving an eigenvalue problem for a (structured) matrix. In this case, this matrix is a matrix Möbius transform of a unitary Hessenberg matrix $\mathcal{H}_n^{(n;u)}$ or a five-diagonal matrix $\mathcal{C}_n^{(n;u)}$. Equivalently one needs to solve a generalized eigenvalue problem for the numerator and denominator of the matrix Möbius transform. As long as finite dimensional matrices are involved, the Hessenberg and the corresponding five-diagonal matrix are basically the same because both these matrices can essentially be stored as a sequence of 2×2 Givens transforms. Only the ordering is different. So that they have the same structural complexity. Hence also their matrix Möbius transforms have similar complexity. This structure should be taken into account when solving larger problems. Many algorithms of QR type have been proposed for all kinds of structured matrices quite recently. Some of them might be usefull for the problem considered here, but the diagonal perturbations and the inversion involved in the matrix Möbius transform may lead to rather complex algorithms. When the problem is reformulated as a generalized eigenvalue problem for a matrix pencil, the structure of the matrices in the pencil is, at least conceptually, simpler. So that it is probably more convenient to design a QZ-type algorithm for the structured pencil. To the best of our knowledge, the literature on the QZ algorithm for structured matrices is quite limited. We know of [18, 26]. These solve the problem for (quasi-)separable matrices. Loosely speaking, we could say than an upper semiseparable matrix is the inverse of un upper Hessenberg matrix. So they are again in essence a sequence of 2×2 Givens transforms. In fact such an approach was taken in [8, 25] for the construction of orthogonal rational functions, which can be seen as an inverse eigenvalue problem for a matrix with semiseparable-plus-diagonal structure. This idea has been persued in [13] for the construction of a rational Gauss quadrature formula on the real line, which turns out to be a direct (instead of inverse) eigenvalue problem for a semiseparable-plus-diagonal matrix.

The case of the real line is in a sense simpler than the case of the unit circle because the recurrence can be represented by a (symmetric) tridiagonal (Jacobi) matrix. Hence its inverse is a symmetric semiseparable matrix. Moreover, the Gauss-nodes are the (real) zeros of the orthogonal functions (and not of the para-orthogonal ones). But these elements are only minor differences which do not influence the complexity of the computations. Neither do they explain the difference between the equations in [8, 25, 13] and in this paper. The difference comes from the fact that the recurrence in [8, 25, 13] is based on the fact that (we formulate it for the case of the unit circle)

$$\phi_n(z) \in \operatorname{span}\{(1-\overline{\alpha}_k z)\phi_k(z) : k = 0, \ldots, n+1\} = \left\{ \frac{p_{n+1}(z)}{\pi_n(z)} : p_{n+1} \in \mathbb{P}_{n+1} \right\},$$

i.e., $\phi_n \in \mathcal{L}'_{n+1} := (1-\overline{\alpha}_{n+1} z)\mathcal{L}_{n+1}$. This gives rise to a recurrence formulation of the form

$$\Phi_n(z) = \Phi_n(z)(\mathcal{I}_n - \mathcal{A}_n^\dagger z)\mathcal{H}'_n + E'_n$$

where $\Phi_n = (\phi_0, \ldots, \phi_n)$, $\mathcal{H}'_n$ is an upper Hessenberg matrix and E'_n is a vector taking care of the truncation effect at the end. However, it is clear that also $z\phi_n(z) \in \mathcal{L}'_{n+1}$, and thus also $(z - \alpha_n)\phi_n(z) \in \mathcal{L}'_{n+1}$. And this gives rise to a recurrence formulation with the form

$$\Phi_n(z)(z\mathcal{I}_n - \mathcal{A}_n) = \Phi_n(z)(\mathcal{I}_n - \mathcal{A}_n^{\dagger} z)\mathcal{H}_n + E_n$$

where again $\mathcal{H}_n$ is an upper Hessenberg matrix.

In the first case, this eventually leads to an eigenvalue problem (we skip all the details and assume for simplicity that all α_k are nonzero)

$$\Phi_n(\mathcal{I}_n - {\mathcal{H}'_n}^{-1})(\mathcal{A}_n^{\dagger})^{-1} = \Phi_n z$$

where the matrix involved, because of the inverse of the Hessenberg matrix, is essentially a semiseparable-plus-diagonal matrix.

In the second case, however, the generalized eigenvalue problem concerns

$$\Phi_n(\mathcal{I}_n + \mathcal{A}_n) = \Phi_n(\mathcal{I}_n + \mathcal{A}_n^{\dagger}\mathcal{H}_n)z$$

as has been considered in this paper. Similar formulations do exist for the case of the real line.

The details of how the different approaches relate exactly are far beyond the scope of this paper. We just conclude that, our approach using matrix Möbius transforms of Hessenberg or five-diagonal matrices is new in deriving the (generalized) eigenvalue problem giving nodes and weights of the quadrature formulas. It can be applied to the real line too, but it seems to lead an eigenvalue problem different from those obtained in the literature before. But whatever the formulation, be it Hessenberg, five-diagonal of semiseparable, in essence the matrices of the eigenvalue problem can all be stored as a sequence of 2×2 Givens rotations and hence the problems can all be solved with approximately the same computational effort. The even-odd decomposition of the generalized eigenvalue problem seems particularly attractive to us because it only involves two block diagonal matrices with 2×2 Givens blocks.

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