Convergence of general two-point Padé approximants to Stieltjes functions

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Stieltjes function

$$f(z) = \int_0^\infty \frac{d\alpha(x)}{1 - xz}, \quad z \in \mathbb{C} - \mathbb{R}_+$$

moments: $c_k = \int_0^\infty x^k d\alpha(x)$, $k \in \mathbb{Z}$

asymptotic expansions:

$$L_0 = \sum_{k=0}^{\infty} c_k z^k, \qquad z \to 0$$

$$L_\infty = -\sum_{k=1}^{\infty} c_{-k} z^{-k}, \quad z \to \infty$$

Padé approximant: matches $c_0, c_1, \ldots, c_{p+q}$

Two point Padé approximant: matches $c_{-p}, \ldots, c_0, \ldots, c_q$

More general pair

$L_0 = \sum_{k \geq \nu} c_k z^k$ and $L_\infty = \sum_{k \leq \mu} c^*_{-k} z^k$

is reduced to standard case

Standard case

$$L_\infty = \quad \cdots \quad c_3^* \quad c_2^* \quad c_1^* \qquad c_0 \quad c_1 \quad c_2 \quad c_3 \quad c_4 \quad c_5 \quad \cdots \quad = L_0$$

$$[k/n]_{(L_0,L_\infty)}(z)$$

General case

$$L^* = \quad \cdots \quad c_3^* \quad c_2^* \quad c_1^* \quad c_0^* \quad c_{-1}^* \quad c_{-2}^* \quad c_{-3}^* \qquad \begin{matrix} c_{-2} & c_{-1} & c_0 & c_1 & c_2 & c_3 & c_4 & c_5 & \cdots \end{matrix} = L$$

$$\Downarrow$$

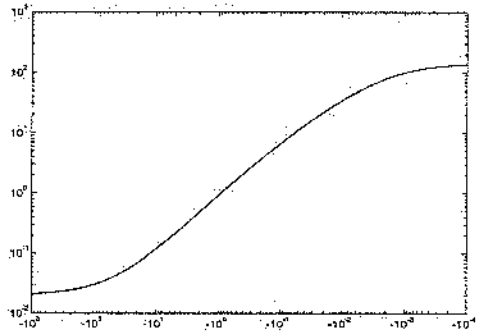
$L_0(z) = L(z) - P_+(z) - P_-(z)$	$L_\infty(z) = L^*(z) - P_+(z) - P_-(z)$
$P_+(z) = \sum_{k=\nu}^{-1} c_k z^k$	$P_-(z) = \sum_{k=0}^{\mu} c_k^* z^k$

$$\Downarrow$$

$$L_\infty \left\{ \begin{matrix} \cdots & c_3^* & c_2^* & c_1^* \\ & & - & - \\ & & c_{-2} & c_{-1} \end{matrix} \right. \qquad \left. \begin{matrix} c_0 & c_1 & c_2 & c_3 & c_4 & \cdots \\ - & - & - & - & & \\ c_0^* & c_{-1}^* & c_{-2}^* & c_{-3}^* & & \end{matrix} \right\} L_0$$

$$\Downarrow$$

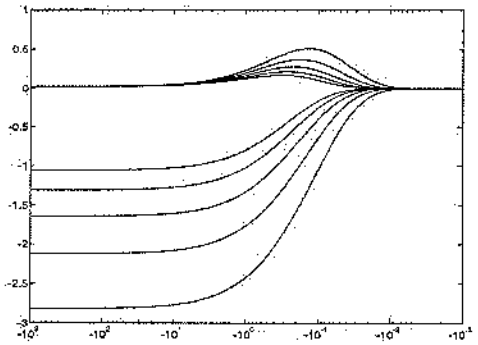
$$[k+\nu/n+\nu]_{(L,L^*)}(z) = [k/n]_{(L_0,L_\infty)}(z) + P_+(z) + P_-(z)$$



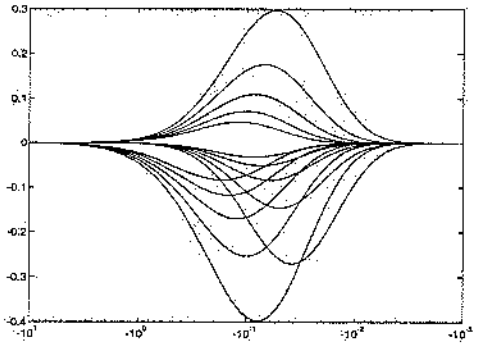
$$\alpha(x) = 0.02H(x) + \frac{1}{\log 1000} \begin{cases} 0, & 0 \le x \le 1 \\ x-1, & 1 < x < 1000 \\ 999, & x \ge 1000 \end{cases}$$

$$f(z) = b - \frac{1}{z \log a} \log\left(\frac{1-az}{1-z}\right), \quad -\infty < z < 0$$

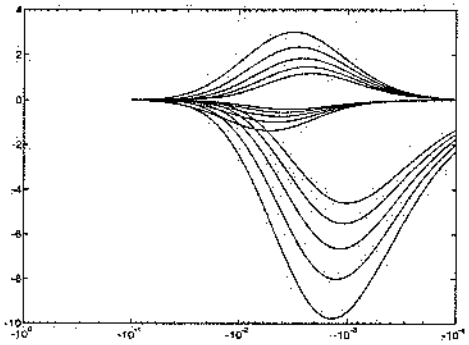
$$b = 0.02, \quad a = 1000$$



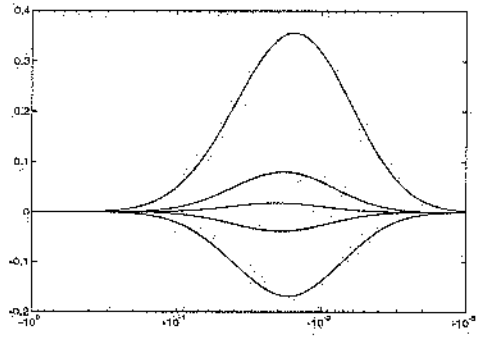
Padé approximants, type (n/n) and $(n-1/n)$, $n = 6, 7, 8, 9, 10$



Two-point Padé approximants,
type (n/n), $n = 6, 7, 8, 9, 10$
uses k coeficients from L_0, $k = 1, 2, 3$



Two-point Padé approximants,
type (n/n), $n = 6, 7, 8, 9, 10$
uses k coeficients from L_∞, $k = 1, 2, 3$



Two-point Padé approximants,
type (n/n), $n = 6, 7, 8, 9, 10$
uses n coeficients from L_0, $k = 1, 2, 3$
(symmetric case)

Quadrature

Stieltjes function is integral

$$f(z) = \int_0^\infty \frac{d\alpha(x)}{1 - xz}$$

approximant for $f(z)$ from quadrature formula

$$I_n\{g\} = \sum_{j=0}^{n} A_{j,n} g(x_{j,n})$$

for integral

$$I_\alpha\{g\} = \int_0^\infty g(x) d\alpha(x)$$

For $g(x) = 1/(1 - xz)$ some quadrature may result in

two-point Padé approximant for a Stieltjes function

Quadrature

L-polynomials $(p(n), q(n) \geq 0)$

$$\mathcal{R}_n = \Lambda_{-p(n),q(n)} = \text{span}\{z^{-p(n)}, \ldots, 1, \ldots, z^{q(n)}\}$$

$$\dim \mathcal{R}_n = n + 1 \Leftrightarrow p(n) + q(n) = n$$

$$\mathcal{R}_n \subset \mathcal{R}_{n+1} \Leftrightarrow 0 \leq p(n) - p(n-1) \leq 1$$

sequence $\{p(n)\}$ uniquely defines nested sequence $\{\mathcal{R}_n\}$

Find $A_{j,n}$ and $x_{j,n}$ such that

$$I_\alpha\{R\} = I_n\{R\}, \quad \forall R \in \mathcal{R}_N$$

with N as large as possible.

Gaussian approach: exact in $\mathcal{R}_{2n-1} = \Lambda_{-p(2n-1),q(2n-1)}$

Orthogonal approach: exact in $\Lambda_{-p(n)-p(n-1),q(n)+q(n-1)}$

they are the same iff $p(n) + p(n-1) = p(2n-1)$

$x_{j,n}$ zeros of n-th OP w.r.t.

$$d\tilde{\alpha}(x) = x^{-p(2n-1)} d\alpha(x) \quad \text{or} \quad d\hat{\alpha}(x) = x^{-p(n)-p(n-1)} d\alpha(x)$$

$$\langle f, g \rangle_\mu = \int_0^\infty f(x) g(x) d\mu(x)$$

Quadrature

$$\boxed{p(n)+p(n-1)=p(2n-1)}$$

set $\lambda(n)=p(n)-p(n-1)\in\{0,1\}$ then

$x_j=x_{j,n}$ are zeros of n-th OLP Q_n: $Q_n\perp_\alpha \mathcal{R}_{n-1}$

$$A_j=A_{j,n}=\frac{1}{x_j^{\lambda(n)}}\int_0^\infty x^{\lambda(n)}\left[\frac{Q_n(x)}{(x-x_j)Q_n'(x_j)}\right]^2 d\alpha(x)$$

for example

$$\begin{array}{lll} p(n)=0 & \rightarrow & \{1,x,x^2,\ldots\}\\ p(n)=n & \rightarrow & \{1,x^{-1},x^{-2},\ldots\}\\ p(n)=E\left[\frac{n}{2}\right] & \rightarrow & \{1,x,x^{-1},x^2,x^{-2},\ldots\}\\ p(n)=E\left[\frac{n+1}{2}\right] & \rightarrow & \{1,x^{-1},x,x^{-2},x^2,\ldots\}\end{array}$$

$p(2n-1)=n-M \rightarrow \tilde{p}(n)=E[\frac{n+1}{2}],\ d\tilde{\alpha}(x)=x^M d\alpha(x)$

$x_j=x_{j,n}$ are zeros of n-th OLP $\tilde{Q}_n$: $\tilde{Q}_n\perp_{\tilde{\alpha}}\tilde{\mathcal{R}}_{n-1}$

$$A_j=\frac{1}{x_j^{M+\lambda(n)}}\int_0^\infty x^{M+\lambda(n)}\left[\frac{\tilde{Q}_n(x)}{(x-x_j)\tilde{Q}_n'(x_j)}\right]^2 d\alpha(x)$$

$\lambda(n)=0$ for n even; $\lambda(n)=1$ for n odd

Two-point Padé Approx.

Lemma.

$$f_n(z)=I_n\{\frac{1}{1-xz}\}=\sum_{j=1}^n\frac{A_j}{1-x_jz}$$

is rational of type $(n-1/n)$ and if $I_n\{f\}$ exact in $\Lambda_{-p,q}$, then for

$$f(z)=I_\alpha\{\frac{1}{1-xz}\}\sim\begin{cases}L_0, & z\rightarrow 0\\ L_\infty, & z\rightarrow\infty\end{cases}$$

$$L_0(z)-f_n(z)=O(z^{q+1});\qquad L_\infty(z)-f_n(z)=O\Big((z^{-1})^{p+1}\Big).$$

Thus $p+q=2n-1\Rightarrow$ 2PA $f_n(z)=[q+1/n]_f(z)$

Approximation error

Theorem.
$p(2n-1)+q(2n-1)=2n-1$
$\lambda(n)=p(n)-p(n-1)\in\{0,1\}$

$$\begin{aligned} E_n(z) &= f(z)-f_n(z)=f(z)-[q(2n-1)+1/n]_f(z) \\ &= \frac{z^{\lambda(n)}}{Q_n(1/z)^2}\int_0^\infty \frac{x^{\lambda(n)}Q_n(x)^2 d\alpha(x)}{1-xz} \end{aligned}$$

$$\mathcal{R}_{n-1} \perp_\alpha Q_n \in \mathcal{R}_n$$

$\boxed{\lambda(n) \text{ fixes the sign of } E_n(z) \text{ for } z\in\mathbb{R}_-}$

examples ($z\in\mathbb{R}_-$)

$$p(n)=E[\tfrac{n}{2}]\rightarrow \quad f(z)-[n/n]_f(z) \quad \begin{cases} <0 & n \text{ even} \\ >0 & n \text{ odd} \end{cases}$$

$$p(n)=E[\tfrac{n+1}{2}]\rightarrow \quad f(z)-[n-1/n]_f(z) \quad \begin{cases} >0 & n \text{ even} \\ <0 & n \text{ odd} \end{cases}$$

main horizontals in 2PA table

			[0/3]	⋯
		[0/2]	[1/3]	⋯
	[0/1]	[1/2]	[2/3]	⋯
[0/0]	[1/1]	[2/2]	[3/3]	⋯
	[2/1]	[3/2]	[4/3]	⋯
		[4/2]	[5/3]	⋯
			[6/3]	⋯

Diagonals

Theorem. $k\le n$
(by considering the cases $p(2n-1)=2n-k$ and $p(2n-1)=k$)

$$E_n(z) = f(z)-[k/n]_f(z)=\frac{z^{2n-k}}{Q_n(1/z)^2}\int_0^\infty \frac{Q_n(x)^2 d\alpha(x)}{x^{2n-k}(1-xz)} \quad (1)$$

$$E_n(z) = f(z)-[2n-k/n]_f(z)=\frac{z^{k}}{Q_n(1/z)^2}\int_0^\infty \frac{Q_n(x)^2 d\alpha(x)}{x^{k}(1-xz)} \quad (2)$$

zeros of Q_n are the nodes of the quadrature

$\boxed{k \text{ fixes the sign of } E_n(z) \text{ for } z\in\mathbb{R}_-}$

examples ($z\in\mathbb{R}_-$)

$k=0$ in (1) $\rightarrow$ Padé of type $\frac{n-1}{n}$ at ∞ $\rightarrow E_n(z)>0$
$k=0$ in (2) $\rightarrow$ Padé of type $\frac{n-1}{n}$ at 0 $\rightarrow E_n(z)>0$

diagonals in 2PA table

			[0/3]	⋯
		[0/2]	[1/3]	⋯
	[0/1]	[1/2]	[2/3]	⋯
[0/0]	[1/1]	[2/2]	[3/3]	⋯
	[2/1]	[3/2]	[4/3]	⋯
		[4/2]	[5/3]	⋯
			[6/3]	⋯

$(\frac{n}{n})PA_f = c_0 + z(\frac{n-1}{n})PA_F, \qquad F(z)=[f(z)-c_0]/z$
z causes sign change of error

Horizontals

Theorem. $n > |M|$
(by considering the cases $p(2n-1) = n - M - 1$)

$$E_n(z) = f(z) - [n + M/n]_f(z) = \frac{z^{M+\lambda(n)}}{Q_n(1/z)^2} \int_0^\infty \frac{Q_n(x)^2 d\alpha(x)}{x^{M+\lambda(n)}(1-xz)}$$

zeros of Q_n are the nodes of the quadrature

$\lambda(n) = 0$ for n even, $\lambda(n) = 1$ for n odd

$\boxed{n + M \text{ fixes the sign of } E_n(z) \text{ for } z \in \mathbb{R}_-}$

namely $(z \in \mathbb{R}_-)$

$n + M$ even $\rightarrow E_n(z) > 0$
$n + M$ odd $\rightarrow E_n(z) < 0$

horizontals in 2PA table

			[0/3]	⋯
		[0/2]	[1/3]	⋯
	[0/1]	[1/2]	[2/3]	⋯
[0/0]	[1/1]	[2/2]	[3/3]	⋯
	[2/1]	[3/2]	[4/3]	⋯
		[4/2]	[5/3]	⋯
			[6/3]	⋯

sign of $E_n(z)$, $z \in \mathbb{R}_-$ { fixed on diagonals
alternates between diagonals
alternates on horizontals and verticals

Monotone decrease

$$f_n(z) = I_n\{\frac{1}{1-xz}\} = [q + 1/n]_f(z) = \int_0^\infty \frac{d\alpha_n(x)}{1-xz}$$

$$\alpha_n(x) = \sum_{j=1}^{n} A_j \delta_{x_j}(x), \quad A_j > 0.$$

Hence $f_n(z)$ is a Stieltjes function.

$$\mathcal{R}_n \subset \mathcal{R}_{n+1} \Rightarrow f_n(z) = [q + 1/n]_f(z) = [q + 1/n]_{f_{n+1}}(z)$$

because the moments defining f_n are the same for $f(z)$ and $f_{n+1}(z)$

$$E_n(z) - E_{n+1}(z) = f_{n+1}(z) - f_n(z) = f_{n+1}(z) - [q + 1/n]_{f_{n+1}}(z)$$

$\Rightarrow \boxed{E_n(z) - E_{n+1}(z) \text{ has same sign as } E_n(z) \text{ on } \mathbb{R}_-}$

$\Rightarrow \boxed{|E_n(z)| \text{ monotonically decreasing on } \mathbb{R}_-}$