ON THE BLOCK STRUCTURE OF THE LAURENT-PADE TABLE

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Abstract. Some ideas on the block structure of a formal Laurent-Padé table will be given. The structure is derived from the block structure of the table of Toeplitz determinants which also defines the blocks of a classical Padé table.

1. Formal Laurent series

With a bi-infinite sequence of complex numbers \( \{f_k\}_{-\infty}^{\infty} \) we associate a formal Laurent series (fls)

\[
F(z) = \sum_{k=-\infty}^{\infty} f_k z^k.
\]  
(1.1)

\( F(z) \) will be fixed throughout this paper. With \( F \) we associate the series

\[
Z(z) = \frac{1}{2} f_0 + \sum_{k=1}^{\infty} f_k z^k \quad \text{and} \quad \bar{Z}(z) = \frac{1}{2} f_0 + \sum_{k=1}^{\infty} f_{-k} z^{-k}.
\]  
(1.2)

A projection operator \( \Pi_{m:n} \) is defined by

\[
\Pi_{m:n} F(z) = \sum_{k=m}^{n} f_k z^k.
\]  
(1.3)

A Laurent polynomial is an expression like in the right hand side of (1.3) with \( m \) and \( n \) finite. Its degree is \( \max \{ |c| : f_c \neq 0 \} \). The notations \( O_+ \) and \( O_- \) are introduced via

\[
F(z) = O_+ (z^m) \iff \Pi_{-\infty:m-1} F(z) = 0,
\]  
(1.4)

\[
F(z) = O_- (z^n) \iff \Pi_{n+1:\infty} F(z) = 0.
\]
Sup \( \{ m \mid F(z) = O_+ (z^m) \} \) is called the \(+\) order of \( F(z) \) and we denote it as \( \text{ord}_+ (F) \).
Similarly \( \text{Inf} \{ n \mid F(z) = O_- (z^n) \} \) is called the \(-\) order of \( F(z) \) and we denote it as \( \text{ord}_- (F) \).

The product of two fls is defined in the usual way as a Cauchy product, for those cases where both have finite \(+\) order, finite \(-\) order or if one of them is a Laurent polynomial. This is all that we shall need. The reciprocal of a fls with finite \(+\) order \((-\) order) is also defined in the usual way. It is again a fls with finite \(+\) order \((-\) order). If it is a Laurent polynomial, it is not clear which reciprocal we want: the one with finite \(+\) order or the one with finite \(-\) order. To distinguish between both we use the notation \( L_+ (1/F(z)) \) to indicate the reciprocal with finite \(+\) order and \( L_- (1/F(z)) \) for the other one. Also if \( F(z) \) and \( G(z) \) are Laurent polynomials, then

\[
L_+(F(z)/G(z)) \text{ means } F(z) L_+(1/G(z)) \text{ etc...}
\]

By a ratio of two Laurent polynomials we mean a complete equivalence class. I.e., we do not distinguish between \( P(z)/Q(z) \) and \( [A(z)P(z)]/[A(z)Q(z)] \) for \( A(z) \neq 0 \) and \( P_1(z)/Q_1(z) = P_2(z)/Q_2(z) \) means that they are in the same equivalence class, i.e.,

\[
P_1(z)Q_2(z) = P_2(z)Q_1(z)
\]

We call the fls (1.1) normal if all the Toeplitz determinants \( T_{mn} = \det (f_{m+i-j})_{i,j=1}^n \) for \( m = 0, \pm 1, \pm 2, \ldots ; n = 1,2,\ldots \) are nonzero. It will be convenient to define \( T_{m0} = 1 \).

2. Laurent-Padé Approximants and Laurent-Padé Forms

The notion of Padé approximant for a formal power series can be extended to what is called a Laurent-Padé approximant (LPA) for a fls. Let \( m \) and \( n \) be nonnegative integers and suppose \( R(z) \) and \( \hat{R}(z) \) are both ratios of two Laurent polynomials such that:

\[
\begin{align*}
(1) \quad & Z(z) - L_+ (R(z)) = O_+ (z^{m+n+1}), \\
(2) \quad & \hat{Z}(z) - L_- (\hat{R}(z)) = O_- (z^{-(m+n+1)}), \\
(3) \quad & R(z) + \hat{R}(z) = A_m (z)/B_n (z) \text{ where } A_m (z) \text{ and } B_n (z) \text{ are Laurent polynomials of degree at most } m, \text{respectively } n.
\end{align*}
\]

Then we call the pair \( (R(z), \hat{R}(z)) \) an \((m,n)\) LPA for \( F(z) \).

Remark 1: This is clearly an extension of the classical notion of Padé approximant since \( L_+ (R(z)) + L_- (\hat{R}(z)) \) is a fls which matches the coefficients \( a_k \) for \( |k| \leq m+n \) in the given fls while \( R(z) + \hat{R}(z) \) is the ratio of two Laurent polynomials with numerator degree \( m \) and denominator degree \( n \).
Remark 2: We preferred to define a LPA as a couple of ratios rather than as the sum of these because in our formal setting there is not an unambiguous way to return from $A_m(z)/B_n(z)$ to $R(z)$ and $\hat{R}(z)$, thus to the fls that we associate with it.

A LPA may not exist. What always exists is a Laurent-Padé form (LPF) which we now define by relaxing the conditions (1) and (2).

We call $(R(z), \hat{R}(z))$ an $(m,n)$ LPF if $R(z)$ and $\hat{R}(z)$ have representants $R(z) = \frac{P_m(z)}{Q_m(z)}$ and $\hat{R}(z) = \frac{\hat{P}_{m,n}(z)}{\hat{Q}_{m,n}(z)}$, such that

1. $Z(z) Q_{m,n}(z) - P_{m,n}(z) = O(z^{-m+n+1}),$

2. $\hat{Z}(z) \hat{Q}_{m,n}(z) - \hat{P}_{m,n}(z) = O(z^{-(m+n+1)}),$

(3) is as above.

A constructive way to find the $(m,n)$ LPF goes as follows:

Theorem 2.1 Let $\mu = 0, \pm 1, \pm 2, \ldots$ and $\nu = 0, 1, 2, \ldots$.

Let $Q_{\mu \nu}(z)$ and $\hat{Q}_{\mu \nu}(z)$ be nontrivial polynomials of degree $\nu$ which are (not necessarily unique) solutions of

$$\prod_{\mu + 1: \mu + \nu} (F(z)Q_{\mu \nu}(z)) = 0 \quad \text{and} \quad \prod_{\mu: \mu + \nu - 1} (F(z)\hat{Q}_{\mu \nu}(z)) = 0. \tag{2.2}$$

Define Laurent polynomials $P_{\mu \nu}(z)$ and $\hat{P}_{\mu \nu}(z)$ by

$$P_{\mu \nu}(z) = \prod_{k(\mu, \nu): \nu(\mu, \nu)} (z) Q_{\mu \nu}(z) \tag{2.3a}$$

and

$$\hat{P}_{\mu + 1, \nu}(z) = \prod_{k(\mu, \nu): \nu(\mu, \nu)} (\hat{Z}(z)\hat{Q}_{\mu + 1, \nu}(z)), \tag{2.3b}$$

where $k(\mu, \nu) = \min(0, \mu + \nu + 1)$ and $\nu(\mu, \nu) = \max(\mu, \nu)$.

Then $Q_{mn}(z) = (P_{mn}(z)/Q_{mn}(z), \hat{P}_{m,n}(z)/\hat{Q}_{m,n}(z))$ and

$\hat{Q}_{m+1,n}(z) = (\hat{P}_{m+1,n}(z)/\hat{Q}_{m+1,n}(z), \hat{P}_{m,n}(z)/\hat{Q}_{m,n}(z))$ are $(m,n)$ LPF’s for $m > 0$.

Proof

We remark that equations (2.1) always have nontrivial solutions. We define auxiliary formal series by

$$V_{\mu \nu}(z) = \prod_{\omega: \mu} (F(z) Q_{\mu \nu}(z)), \quad \hat{V}_{\mu \nu}(z) = \prod_{\omega: \mu - 1} (F(z) \hat{Q}_{\mu \nu}(z)), \tag{2.4}$$

$$W_{\mu \nu}(z) = \prod_{\omega: \nu + 1: \mu} (F(z) \hat{Q}_{\mu \nu}(z)), \quad \hat{W}_{\mu \nu}(z) = \prod_{\omega: \nu + \omega} (F(z) \hat{Q}_{\mu \nu}(z)). \tag{2.5}$$
Clearly $F(z)Q_{\mu\nu}(z) = V_{\mu\nu}(z) + W_{\mu\nu}(z)$ and $F(z)\tilde{Q}_{\mu\nu}(z) = \tilde{V}_{\mu\nu}(z) + \tilde{W}_{\mu\nu}(z)$.

From this it follows that (remember $F(z) = Z(z) + \tilde{Z}(z)$)

\begin{align}
Z(z)Q_{\mu\nu}(z) - W_{\mu\nu}(z) &= -\tilde{Z}(z)Q_{\mu\nu}(z) + V_{\mu\nu}(z) = P_{\mu\nu}(z)
\tag{2.6}
\end{align}

\begin{align}
Z(z)\tilde{Q}_{\mu\nu}(z) - \tilde{W}_{\mu\nu}(z) &= -\tilde{Z}(z)\tilde{Q}_{\mu\nu}(z) + \tilde{V}_{\mu\nu}(z) = -\tilde{P}_{\mu\nu}(z)
\tag{2.7}
\end{align}

These could be seen as alternatives for the definitions (2.3).

From this we find that

\begin{align}
Z(z)Q_{mn}(z) - P_{mn}(z) = W_{mn}(z) = O_+(z^{m+n+1})
\end{align}

and

\begin{align}
\tilde{Z}(z)(z^{-n}\tilde{Q}_{-m,n}(z)) - (z^{-n}\tilde{P}_{-m,n}(z)) = z^{-n}\tilde{V}_{-m,n}(z) = O_-(z^{-(m+n+1)}),
\end{align}

so that the order of approximation is as required. Consider now

\begin{align}
\frac{P_{mn}(z)}{Q_{mn}(z)} + \frac{z^{-n}\tilde{P}_{-m,n}(z)}{z^{-n}\tilde{Q}_{-m,n}(z)} = \frac{z^{-n}(P_{mn}(z)\tilde{Q}_{-m,n}(z) + \tilde{P}_{-m,n}(z)Q_{mn}(z))}{z^{-n}Q_{mn}(z)\tilde{Q}_{-m,n}(z)}.
\end{align}

The degree of the Laurent polynomial in the denominator is as it should be. Let $L_{mn}$ be the numerator. Then we only have to prove that it is a Laurent polynomial of degree at most $m$. We drop the argument in the notation for simplicity. For $m,n \geq 0$ we have

\begin{align}
L_{mn} &= [zQ_{mn} - W_{mn}]\tilde{Q}_{-m,n} + [\tilde{Z}\tilde{Q}_{-m,n} - \tilde{V}_{-m,n}]Q_{mn}
\end{align}

\begin{align}
&= [V_{mn} + W_{mn}]\tilde{Q}_{-m,n} - W_{mn}\tilde{Q}_{-m,n} - \tilde{V}_{-m,n}Q_{mn}
\end{align}

\begin{align}
&= V_{mn}\tilde{Q}_{-m,n} - \tilde{V}_{-m,n}Q_{mn} = O_-(z^{m+n})
\end{align}

\begin{align}
&= [\tilde{V}_{-m,n} + \tilde{W}_{-m,n}]Q_{mn} - W_{mn}\tilde{Q}_{-m,n} - \tilde{V}_{-m,n}Q_{mn}
\end{align}

\begin{align}
&= \tilde{W}_{-m,n}Q_{mn} - W_{mn}\tilde{Q}_{-m,n} = O_+(z^{n-m}).
\end{align}

This completes the proof for the first part of the theorem. The second part has a similar proof. \qed
We shall call Laurent-Padé table, the table with entries $K_{\mu\nu}^\prime; \nu=0,1,2,\ldots; \nu=0,1,2,\ldots$ before we investigate its structure, we consider:

3. The T-table.

We define the T-table as the infinite matrix with $(m,n)$-th entry

$T_{mn}$, $m=0,1,2,\ldots; n=0,1,2,\ldots$, where $T_{mn}$ is the Toeplitz determinant defined in section 1. It is known that the corresponding table in the classical Padé case has a characteristic block structure. (see theorem 3.2 of [1]). Let

$\text{ord}_+ F(z) = m^+ < \infty$. Then, except for a shift over $m^+$ rows, the block structure of the T-table is as described and proved in Gragg’s theorem 3.2. Similarly with the transformation $z \rightarrow 1/z$, the case $\text{ord}_- F(z) = m^- = -\infty$ is covered by this theorem.

In the regions $m > m^+$ and $m < m^-$, the $T_{mn}$ are trivially zero. Therefore we concentrate on $T_{\mu\nu}$ with $m^+ \leq \mu < m^- \quad \text{and} \quad \nu \geq 0$.

For such $(\mu, \nu)$ we suppose that $Q_{\mu\nu}^\prime, P_{\mu\nu}^\prime, V_{\mu\nu}, W_{\mu\nu}$ are solutions of the equations (2.2-5). We shall associate with it a reduced solution as follows. Let

$P_{\mu\nu}(z) = z^{k(\mu, \nu)} A_{\mu\nu}$ so that $A_{\mu\nu}$ is a genuine polynomial of degree at most $k(\mu, \nu) - k(\mu, \nu)$. Let $A/Q$ be the unique reduced form of $A_{\mu\nu}/Q_{\mu\nu}$ such that $Q(0) = 1$.

Set further

$$P(z) = z^{k(\mu, \nu)} A(z), \quad V(z) = \tilde{Z}(z) Q(z) + P(z), \quad W(z) = Z(z) Q(z) - P(z). \quad (3.1)$$

We call $S = (P, Q, V, W)$ the reduced solution for equations (2.2-5). Let the greatest common divisor of $A_{\mu\nu}$ and $Q_{\mu\nu}$ producing the reduced form be $z^\lambda D(z)$ with

$D(z) = d_0 + d_1 z + \ldots + d_k z^k, \quad d_0 d_k \neq 0$ then we clearly have for

$S_{\mu\nu} = (P_{\mu\nu}^\prime, Q_{\mu\nu}^\prime, V_{\mu\nu}, W_{\mu\nu})$ that

$$S_{\mu\nu}(z) = z^{\lambda} D(z) S(z). \quad (3.2)$$

Now $\text{ord}_- V$ and $\text{ord}_+ W$ will be used to define the top row, respectively the bottom row of the blocks in the T-table. In a classical Padé situation $\text{ord}_- V$ will always be nonnegative, but now $\text{ord}_- V$ can be $-\infty$. In this case, it is impossible to define the bottom row as $\text{ord}_- V + k$ because $k$ must then be $-\infty$. Finally we note that $\text{ord}_- V = -\infty$ and $\text{ord}_+ W = \infty$ can not occur simultaneously if $F \neq 0$ because $F Q = V + W$.

Theorem 3.1

Let $S = (P, Q, V, W)$ be the reduced solution of equations (2.2-5). Let

$$\text{ord}_- Q = n, \quad \text{ord}_- V = m, \quad \text{ord}_+ W = \tilde{m} + n + 1 \quad (3.3)$$

and define $k = \tilde{m} - m$. 


Then the following statements are true:

(a) \( k > 0 \),

(b) \( S = (P, Q, V, W) \) is a reduced solution of equations (2.2-5) if and only if
\[
m \leq \nu \leq \tilde{m} \quad \text{and} \quad n \leq \nu \leq n + k.
\]
(3.4)

For \((\nu, \nu)\) satisfying (3.4):

(c) \( S_{\nu \nu} = (P_{\nu \nu}, Q_{\nu \nu}, V_{\nu \nu}, W_{\nu \nu}) \) is a solution of (2.2-5) if and only if
\[
S_{\nu \nu} = z^{\nu \nu} D(z) S(z)
\]
with
\[
\lambda_{\nu \nu} = \max \{0, (\nu - m) + (\nu - n) - k\} \quad \text{if} \quad m > -\infty
\]
\[
= \max \{0, (\nu - \tilde{m}) + (\nu - n)\} \quad \text{if} \quad \tilde{m} < \infty
\]
(3.5)

and \( D \) a nonzero polynomial of degree at most
\[
\kappa_{\nu \nu} = \nu - \text{rank} \left( f^{\nu + i - j}_{i=1, \ldots, \nu; j=0,1, \ldots, \nu} \right)
\]
(3.6)

(d) \( T_{\mu n} \neq 0 \quad m \leq \nu \leq \tilde{m} \),

\( T_{mv} \neq 0 \quad n \leq \nu \leq n + k \), \quad \text{if} \quad m > -\infty,

\( T_{m+1, \nu} \neq 0 \quad n \leq \nu \leq n + k \), \quad \text{if} \quad \tilde{m} < \infty,

\( T_{\mu, n+k+1} \neq 0 \quad m \leq \nu \leq \tilde{m} \), \quad \text{if} \quad k < \infty,

\( T_{\nu \nu} = 0 \quad m \leq \nu \leq \tilde{m} \quad \text{and} \quad n \leq \nu \leq n + k \).

Proof

We give the proof only for \( m > -\infty \), in which case \( \tilde{m} = m + k \). If \( m = -\infty \), \( \tilde{m} \) will be finite and we can rewrite the following proof with \( m \) replaced by \( \tilde{m} - k \).

Like in [1, p14] we easily derive from (3.2) that we must have the relations

\[
\kappa > 0,
\]

\[
\lambda \geq \max \{0, (\nu - m) + (\nu - n) - k\},
\]

(3.7)

\[
\kappa + \lambda \leq \min \{\nu - m, \nu - n\}.
\]
Conversely, let $S = (P, Q, V, W)$ satisfy (2.2-5) with $(\mu, \nu)$ replaced by $(m, n)$ and suppose $Q(0) = 0$, $\text{ord}_Q = n$, $\text{ord}_V = m$ and $\text{ord}_W = m + n + k + 1$.

Then for $\lambda$ and $\kappa$ solutions of (3.7) and for an arbitrary polynomial $D(z) = d_0 + d_1 z + \ldots + d_k z^\kappa$, $d_0, d_k \neq 0$, $S_{\mu \nu}$, given by (3.2) will be a solution of (2.2-5).

Since $S$ is a reduced solution there must exist integers $\kappa, \lambda, u, v$ satisfying (3.7).

Thus

$$k \geq (\mu - m) + (\nu - n) - \lambda \geq 2\kappa + \lambda \geq 0.$$ 

This proves (a).

Furthermore (3.7) can only have a solution if

$$\max \{0, (\mu - m) + (\nu - n) - k\} \leq \min \{\mu - m, \nu - n\}, \quad (3.8)$$

which is equivalent with (3.4). Thus also (b) is proved.

To prove (c) we remark that the most general solution of (2.2-5) is obtained by choosing $\lambda$ as small as possible and $\kappa$ as large as possible. The minimal $\lambda$ satisfying (3.7) is given by $\lambda_{\mu \nu}$ in (3.5) and the maximal $\kappa$ is $\kappa_{\mu \nu}$ as in (3.6) because $\kappa_{\mu \nu} + 1$ is then the number of degrees of freedom we have in the choice of a solution $Q_{\mu \nu}$ for (2.2).

We know that $T_{\mu \nu} = 0$ if and only if (2.2) has a nontrivial solution for $Q_{\mu \nu}$ with $Q_{\mu \nu}(0) = 0$, i.e. with $\lambda > 0$. Because of (3.7) this occurs if and only if $\min \{\mu - m, \nu - n\} > 0$ and (3.8) is satisfied. This is equivalent with statement (d).

\[\square\]

The numbers $m, n, k$ and $m$ are related to the unique reduced solution of (2.2-5) and depend for a given $F$ only on $\mu$ and $\nu$. Therefore we define with the notation of previous theorem:

$$N_{\mu \nu} = 0 \quad \text{if} \quad \mu < m^+ \text{ or } \mu > m^-$$

$$= n \quad \text{otherwise},$$

$$K_{\mu \nu} = \infty \quad \text{if} \quad \mu < m^+ \text{ or } \mu > m^-$$

$$= k \quad \text{otherwise},$$

$$M_{\mu \nu} = -\infty \quad \text{if} \quad \mu < m^+$$

$$= m^- \quad \text{if} \quad \mu > m^-$$

$$= m \quad \text{otherwise},$$
\[ \hat{M}_{\mu\nu} = m^{+1} \quad \text{if} \quad \mu < m^+ \]
\[ = \quad \text{if} \quad \mu > m^- \]
\[ = \quad \text{otherwise}. \]

With these we define square blocks of indices:
\[ B_{\mu\nu} = \{(m,n) | M_{\mu\nu} \leq m \leq \hat{M}_{\mu\nu}, N_{\mu\nu} \leq n \leq N_{\mu\nu} + K_{\mu\nu} \} \]
and by deleting its first row and first column we obtain
\[ \tilde{B}_{\mu\nu} = \{(m,n) | M_{\mu\nu} < m \leq \hat{M}_{\mu\nu}, N_{\mu\nu} < n \leq N_{\mu\nu} + K_{\mu\nu} \}. \]

Part (d) of previous theorem says that \( T_{mn} = 0 \) for \((m,n) \in \tilde{B}_{\mu\nu}\) and that for all the finite couples \((m,n)\) bordering \( \tilde{B}_{\mu\nu} \), \( T_{mn} \) will be nonzero.

4. The Laurent-Padé table

We consider now the table with entries \( K_{\mu\nu} \) as defined in theorem 2.1. It is a simple consequence of theorem 3.1 (c) that \( P_{mn}/Q_{mn} = P_{\mu\nu}/Q_{\mu\nu} \) for all \((m,n) \in B_{\mu\nu}\). Also it follows from defining equations (2.2-5) that \( \hat{P}_{\mu+1,\nu}/\hat{Q}_{\mu+1,\nu} = P_{\mu\nu}/Q_{\mu\nu} \). Thus we must also have \( \hat{P}_{mn}/\hat{Q}_{mn} = \hat{P}_{\mu\nu}/\hat{Q}_{\mu\nu} \) for all \((m,n)\) in the shifted grid \((1,0) + B_{-\mu-1,\nu} = \{(p+1,q) | (p,q) \in B_{\mu-1,\nu}\} \). The (rectangular) blocks of equal entries in the Laurent-Padé table are found as follows. Produce a plot of the square blocks \( B_{\mu\nu} \). Reflect this plot in the line \( \mu = -1/2 \) and take the intersections of the original and the reflected squares. These are rectangles with equal \( K_{mn} \) entries. More precisely we have

Theorem 4.1.

Define \( D_{\mu\nu} = B_{\mu\nu} \cap \tilde{B}_{\mu\nu} \) with \( \tilde{B}_{\mu\nu} = \{(m,n) | (-m,n) \in (1,0) + B_{-\mu-1,\nu}\} \). Then \( K_{mn} = K_{\mu\nu} \) for all \((m,n) \in D_{\mu\nu}\).

Proof

This is very simple to prove. By the remark in the beginning of this section we know that \( P_{mn}/Q_{mn} = P_{\mu\nu}/Q_{\mu\nu} \) for \((m,n) \in B_{\mu\nu}\) and \( \hat{P}_{-m,n}/\hat{Q}_{-m,n} = \hat{P}_{-\mu,\nu}/\hat{Q}_{-\mu,\nu} \) for \((m,n) \in \tilde{B}_{\mu\nu}\).

This proves the theorem. \( \square \)

We remark that not all of these LPP's are LPA's. As in a classical Padé table, only the elements with indices in the left upper triangular part of a singular block, diagonal included, are Padé approximants. Here we have a similar situation: Let \( \hat{L}_{\mu\nu} \) be the left upper part of \( \hat{B}_{\mu\nu} \) and \( \hat{L}_{\mu\nu} \) the left upper part of \( \hat{B}_{\mu\nu} \). Then we have
Theorem 4.2

For \( u \geq 0 \)
\[
(\mu, \nu) \in L_{\mu \nu} \iff \text{ord}_+ (2 - \mathcal{L}_+ (P_{\mu \nu} / Q_{\mu \nu})) \geq u + v + 1 \]
\[
(\mu, \nu) \in L_{\mu \nu} \iff \text{ord}_- (\bar{2} - \mathcal{L}_- (\bar{P}_{\mu \nu} / \bar{Q}_{\mu \nu})) \leq -(u + v + 1)
\]
and

\( k_{\mu \nu} \) is a LPA iff \( (\mu, \nu) \in L_{\mu \nu} \cap \bar{L}_{\mu \nu} \).

Proof

Let \( S = (P, Q, V, W) \) be the reduced solution for (2.2-5) with \( \text{ord}_- V = m; \text{ord}_- Q = n \) and \( \text{ord}_+ W = \bar{m} + \bar{n} + 1 \).

Then \( \text{ord}_+ (2 - \mathcal{L}_+ (P_{\mu \nu} / Q_{\mu \nu})) = \text{ord}_+ (2 - \mathcal{L}_+ (P/Q)) = \text{ord}_+ (2 - \mathcal{L}_+ (P/Q)) = \text{ord}_+ W = \bar{m} + \bar{n} + 1 \). Thus \( \text{ord}_+ (2 - \mathcal{L}_+ (P_{\mu \nu} / Q_{\mu \nu})) \geq u + v + 1 \) iff \( \bar{m} + \bar{n} \geq u + v \). I.e.

\( (\mu, \nu) \in L_{\mu \nu} \). The second relation can be proved similarly. Both relations are true iff \( k_{\mu \nu} \) is a LPA. \( \square \)

5. The Chebyshev-Padé table

For the special case that \( f_{-k} = f_{k} \) for all \( k \), we may replace "Laurent-Padé" by "Chebyshev-Padé" in the previous text. It can directly be verified from their definitions that the symmetry of \( F \) implies the following properties:

Theorem 5.1

If \( F(z) = F(1/z) \), then for all \( \nu = 0, \pm 1, \pm 2, \ldots \) :

\[
Q_{-\mu, \nu}(z) = z^\nu Q_{\mu \nu}(1/z); \quad V_{-\mu, \nu}(z) = z^\nu \bar{W}_{\mu \nu}(1/z); \quad W_{-\mu, \nu}(z) = z^\nu \bar{W}_{\mu \nu}(1/z) \text{ and}
\]

\[
P_{-\mu, \nu}(z) = z^\nu \bar{P}_{\mu \nu}(1/z). \quad \text{Also } T_{-\mu, \nu} = T_{\mu \nu} \text{ and therefore } B_{\mu \nu} = \bar{B}_{\mu \nu}, \text{ so that}
\]

\[
D_{-\mu \nu} = D_{\mu \nu}.
\]

This means that the Chebyshev-Padé table is symmetric with respect to \( \mu = -1/2 \).

The square blocks \( D_{\mu \nu} \) define equal entries and in \( L_{\mu \nu} \) these entries are Chebyshev-Padé approximants in the strict sense.

6. Conclusion

We derived the block structure for a Laurent-Padé table. It contains rectangular blocks but these become square again in the symmetric case of Chebyshev-Padé. We derived the results in a formal setting. If the fls is a Laurent series of a function convergent in an annular region of the complex plane one should be more careful in the definition of a LPA as was done in [3].
7. References

