Fourier analysis and the Takenaka-Malmquist basis

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***Abstract*— We consider problems related to the generalization of the classical Fourier basis to a basis of rational functions with prescribed poles outside the unit disk. We give some generalizations about the convergence and estimation of the Fourier coefficients with respect to this generalized basis. We also consider a rational generalization of the classical Toeplitz matrices and consider the asymptotic distribution of their spectrum. These bases and general Toeplitz matrices were considered by Ninness *et al.* in the context of least squares system estimation where the prescribed poles allow to incorporate a priori knowledge into the system dynamics of the model.**

I. INTRODUCTION

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk in the complex plane. We use the following function spaces: $\mathcal{L}_2(\mathbb{T})$ the Hilbert space of square integrable functions on the unit circle, $\mathcal{H}_2(\mathbb{D}) \subset \mathcal{L}_2(\mathbb{T})$ the Hardy space of complex functions analytic in $\mathbb{D}$, $\mathcal{C}_{2\pi}$ the space of 2π-periodic functions, and $\mathcal{C}^q_{2\pi} \subset \mathcal{C}_{2\pi}$ the $\mathcal{C}_{2\pi}$ with continuous qth derivative. $\mathcal{H}_2(\mathbb{D})$ is considered to be a subset of $\mathcal{L}_2(\mathbb{T})$ in the sense that the boundary functions are in $\mathcal{L}_2(\mathbb{T})$. Via $f(\omega) = F(\mathrm{e}^{\mathrm{j}\omega})$ we may identify the 2π-periodic function f and the function F of a complex variable, evaluated on $\mathbb{T}$ when F is continuous in $\mathbb{D} \cup \mathbb{T}$. The inner product is defined as $\langle F, G\rangle = \frac{1}{2\pi}\int_0^{2\pi} F(\mathrm{e}^{\mathrm{j}\omega})\overline{G(\mathrm{e}^{\mathrm{j}\omega})}d\omega = \langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\omega)\overline{g(\omega)}d\omega$ and the induced norm by $\|\cdot\|_2$.

It is well known that the orthonormal Fourier basis $\{\mathrm{e}^{\mathrm{j}n\omega}\}_{n\in\mathbb{Z}}$ is complete in $\mathcal{L}_2(\mathbb{T})$ as well as in $\mathcal{C}_{2\pi}$. However, this is not the only basis with these properties. For example the Takenaka-Malmquist (TM) basis [15], [10], [16, p. 227], which is given by

$$\mathcal{B}_n(z) = \frac{\sqrt{1-|\xi_n|^2}}{1-\overline{\xi}_n z}\varphi_n(z) \quad \mathcal{B}_{-n}(z) = \overline{\mathcal{B}_n\,(1/\overline{z})}, \qquad (1)$$

for $n = 0, 1, 2, \ldots$ with finite Blaschke product

$$\varphi_p(z) = \prod_{k=0}^{p-1} \frac{z-\xi_k}{1-\overline{\xi}_k z} \qquad (2)$$

where $\xi_k \in \mathbb{D}$ for all $k \geq 0$ and $\xi_0 = 0$. The basis $\{\mathcal{B}_n : n \geq 0\}$ is known to be complete in $\mathcal{H}_2(\mathbb{D})$ and the basis $\{\mathcal{B}_n : n \in \mathbb{Z}\}$ is complete in $\mathcal{L}_2(\mathbb{T})$ iff $\sum_{n\geq 0}(1-|\xi_n|) = \infty$ [1, p. 244], [5, Chap. 7], [12, App. C].

The condition $\sum_{n\geq 0}(1-|\xi_n|) = \infty$ implies that the Blaschke product (2) diverges to 0. The simplest way to satisfy this condition is to assume as we shall do in the rest of this paper

Assumption 1: The ξ_k satisfy $|\xi_k| \leq c < 1$ for $k \geq 0$ and $\xi_0 = 0$.

Note that by choosing all $\xi_k = 0$, we obtain the classical Fourier basis as a special case. We shall denote this basis by $\hat{\mathcal{B}}_n(e^{\mathrm{j}\omega}) = \mathrm{e}^{\mathrm{j}\omega}$. Similarly we shall put a hat on all objects referring explicitly to the classical Fourier case. It is the purpose of this note to generalize some of the classical Fourier results to this more general TM basis. The motivation for this generalized basis and for the interest in generalizing these properties can be found in papers by Ninness and coworkers [11], [12], [13], [14]. See also [5].

The specific results that are presented in this paper are important for rational approximation in general, but our main interest is motivated by the paper [13] and the arguments given there. In particular in the context of system identification and models derived from orthonormal rational bases as they were introduced by several authors with different degrees of generality (see e.g. [13] for some history). The result in Theorem 3 below is a far reaching generalization of the convergence of simple Cesàro means in the classical Fourier case. The spectral properties of the general Toeplitz matrix $M_p(f)$ (5) are important from a numerical point of view because the condition number of this matrix will dictate the numerical robustness of the least-squares estimation problem in the underlying model identification.

II. RATE OF CONVERGENCE

For a function $f \in \mathcal{C}^q_{2\pi}$ it is known that the Fourier coefficients $\hat{c}_n = \left\langle f, \hat{\mathcal{B}}_n \right\rangle$ satisfy $|\hat{c}_0| \leq \|f\|_1$ and $|\hat{c}_{\pm n}| \leq \|f^{(q)}\|_1/n^q$ [7], [8]. This result can be generalized as follows for the TM basis.

Theorem 1: [2] Let Assumption 1 be satisfied. If $f \in \mathcal{C}^q_{2\pi}$ with $q > 2$, then the generalized Fourier coefficients $c_n = \langle f, \mathcal{B}_n\rangle$ for the TM basis $\{\mathcal{B}_n\}$ satisfy

$$|c_0| \leq \|f\|_1 \quad \text{and} \quad |c_{\pm n}| \leq C(q,c,n)\|f^{(q)}\|_1/(\epsilon n)^q, \quad n \geq 1$$

where $C(q,c,n)$ is a constant only depending on q, c and n.

$$C(q,c,n) = [K_1/\ln\tilde{\rho} + K_2(c)\epsilon n/(q-1)],$$

$K_2(0^+) = 0$, $\epsilon = \ln[(\rho+c)/(1+\rho c)]/\ln\rho$, $\rho = \rho(c,q,n)$ is given by

$$\rho = \min(\rho_1 + \rho_2, \mathrm{e}^{q/2}),$$

$$\rho_1 = \left(\frac{q}{n-1}\,\frac{-4}{\ln c}\right)^{q/4} + \left(\frac{n}{2}\right)^{q/(n-2)}$$

$$\rho_2 = \frac{1+c}{\sqrt{cq}} + \frac{1}{c(q-1)},$$

and $\tilde{\rho} = \lim_{n\to\infty} \rho(c,q,n)$.

The precise behavior of $C(q,c,n)$ is important to prove the (speed of) convergence of the partial sum

$$f_p(\omega) = \sum_{|n|<p} c_n \mathcal{B}_n(\mathrm{e}^{\mathrm{j}\omega}) \tag{3}$$

to $f(\omega)$. We have

Corollary 2: With the same assumption as in Theorem 1, the partial sum $f_p(\omega)$ converges to $f(\omega)$ i.e.

$$\lim_{p\to\infty} f_p(\omega) = f(\omega)$$

uniformly in ω with a convergence rate at least as fast as $1/p^{q-2}$.

Like in the Fourier case, this result is based on the fact that the expansion

$$f(\omega) = \sum_{n\in\mathbb{Z}} c_n \mathcal{B}_n(\mathrm{e}^{\mathrm{j}\omega}), \quad c_n = \langle f, \mathcal{B}_n \rangle \tag{4}$$

is absolutely convergent and that the limiting set $\{\mathcal{B}_k : k \in \mathbb{Z}\}$ constitutes an orthonormal basis for $\mathcal{L}_2(\mathbb{T})$. This complements results given in [3].

III. GENERALIZED TOEPLITZ MATRICES

For $f \in \mathcal{C}_{2\pi}$, we can define the matrices

$$M_p(f) = \frac{1}{2\pi}\int_0^{2\pi} \Gamma_p(\omega) f(\omega) \Gamma_p^*(\omega) \mathrm{d}\omega, \quad p \geq 1, \tag{5}$$

where Γ_p is the column vector containing the first p basis functions. (The superscript * denotes complex conjugate transpose.)

In the classical Fourier case $\hat{\Gamma}_p(\omega) = [1,\ \mathrm{e}^{-\mathrm{j}\omega}, \ldots, \mathrm{e}^{-\mathrm{j}(p-1)\omega}]^*$, resulting in a Toeplitz matrix $\hat{M}_p(f)$.

Its TM generalization is still given by (5) but with TM basis

$$\Gamma_p(\omega) = \left[\overline{\mathcal{B}_0(\mathrm{e}^{\mathrm{j}\omega})}, \ldots, \overline{\mathcal{B}_{p-1}(\mathrm{e}^{\mathrm{j}\omega})}\right]^*. \tag{6}$$

In this case we call $M_p(f)$ a generalized Toeplitz matrix with symbol f. Clearly, if $f(\omega)$ is real-valued, then $M_p(f)$ is Hermitian, i.e. $M_p(f) = M_p^*(f)$.

The reproducing kernel for the subspace $\mathcal{S}_p = \mathrm{span}\{\mathcal{B}_n\}_{n=0}^{p-1}$ is given by

$$K_p(\mu,\sigma) = \sum_{n=0}^{p-1} \mathcal{B}_n(\mathrm{e}^{\mathrm{j}\mu})\overline{\mathcal{B}_n(\mathrm{e}^{\mathrm{j}\sigma})} = \Gamma_p^*(\sigma)\Gamma_p(\mu).$$

Note that as a vector from $\mathbb{C}^p$, the squared 2-norm of $\Gamma_p(\omega)$ is $\gamma_p(\omega) = \Gamma_p^*(\omega)\Gamma_p(\omega) = K_p(\omega,\omega)$. For the classical Fourier case, $\hat{\gamma}_p(\omega) = p$, but in general $\gamma_p(\omega)$ will depend on ω. Suppose we normalize the vectors Γ_p as

$$G_p(\omega) = \frac{\Gamma_p(\omega)}{\sqrt{\gamma_p(\omega)}} = \frac{1}{\sqrt{\gamma_p(\omega)}}\left[\overline{\mathcal{B}_0(\mathrm{e}^{\mathrm{j}\omega})}, \ldots, \overline{\mathcal{B}_{p-1}(\mathrm{e}^{\mathrm{j}\omega})}\right]^*. \tag{7}$$

The Toeplitz matrix $\hat{M}_p(f)$ and the normalized vectors $\hat{G}_p$ play an important role in classical Fourier case.

For example for functions $f \in \mathcal{C}_{2\pi}$, the convergence of the classical Fourier series may not be uniform, and then other summation techniques like Cesàro means are used. It takes only a little thought to see that such a sum can be written as (see for example [13])

$$\hat{G}_p^*(\omega)\hat{M}_p(f)\hat{G}_p(\omega) = \sum_{k=-(p-1)}^{p-1} \left(1 - \frac{|k|}{p}\right)\hat{c}_k \mathrm{e}^{\mathrm{j}\omega k}$$

where the $\hat{c}_k$ are the classical Fourier coefficients of f. So

$$\lim_{p\to\infty} \hat{G}_p^*(\omega)\hat{M}_p(f)\hat{G}_p(\omega) = f(\omega). \tag{8}$$

Replacing $\hat{M}_p(f)$ by $\hat{M}_p(f)\hat{M}_p(g)$ results in a convergence to $f(\omega)g(\omega)$. This and other more general situations leads us to the following far reaching generalization to the TM case [2].

Theorem 3: Let Assumption 1 be satisfied. Assume $f^{[k]}(\omega) \in \mathcal{C}_{2\pi}^q$ with $q > 4$ for $k = 1, \ldots, n$. Let $T(\cdot)$ be an n-variable analytic function (i.e. having convergent Taylor expansion) on the range of the $f^{[k]}(\omega)$'s. Then,

$$\lim_{p\to\infty} G_p^*(\sigma) T\Big(M_p(f^{[1]}), \ldots, M_p(f^{[n]})\Big) G_p(\mu) = \begin{cases} \left[T(f^{[1]}, \ldots, f^{[n]})(\mu)\right], & \text{if } \sigma = \mu \\ 0, & \text{otherwise.} \end{cases}$$

The rate of convergence is at least as fast as $1/p$ if $\sigma = \mu$ and as $\ln p/p$ otherwise.

IV. SPECTRAL ASYMPTOTICS

The previous theorem is one of the main results in this paper. Another one is related to the spectrum of the generalized Toeplitz matrices. For example, a result by Grenander and Szegő [9, p. 65] says that for certain f, the average of the spectrum of $\hat{M}_p(f)$ converges to the average value of f as $p \to \infty$. In many applications (see e.g. [13] and the references mentioned there), it is important to know the asymptotic distribution of the spectrum of the (generalized) Toeplitz matrices. Let $\lambda_i(X)$ denote the ith (in any order) eigenvalue of the Hermitian matrix X. A general formulation of asymptotics for generalized Toeplitz matrices for the TM basis is the following [2]

Theorem 4: Let Assumption 1 be satisfied. Suppose the real function $f \in \mathcal{C}_{2\pi}^q$ with $q > 4$. Let $T(\cdot)$ stand for a continuous function on the range of $f(\omega)$. Furthermore, let $\chi_p(\omega)$ be the phase of $\varphi_p(\mathrm{e}^{\mathrm{j}\omega})$ for $\omega \in [0, 2\pi)$ where $\varphi_p(z)$ is the Blaschke product (2). Then, $\tilde{\chi}(\omega) = \lim_{p\to\infty} \chi_p(\omega)/p$ exists and

$$\lim_{p\to\infty} \frac{1}{p}\sum_{i=0}^{p-1} T\Big(\lambda_i\big(M_p(f)\big)\Big) = \frac{1}{2\pi}\int_0^{2\pi} T\Big(f\big(\tilde{\chi}^{-1}(\omega)\big)\Big)\mathrm{d}\omega$$

The convergence rate to the limiting value is at least as fast as $1/p$.

To prove the convergence displayed in Theorems 3 and 4, we shall use an approximation of the general Toeplitz matrices $M_p(f)$ which is obtained by replacing the integral in (5) by a Szegő quadrature formula. The bound for the approximation error play an important role in deriving the previous convergence results. This is briefly explained in the next two sections.

V. QUADRATURE FORMULAS

We can approximate the integral $I(f) = \frac{1}{2\pi}\int_0^{2\pi} f(\omega)d\omega$ for a 2π-periodic function $f(\omega) = F(\mathrm{e}^{\mathrm{j}\omega})$ by a quadrature formula of the form $I_p(f) = \sum_{k=0}^{p-1} H_k f(\omega_k)$. Applying this in the formula (5) gives an approximation of the form $W_p(f) = \Upsilon_p F_p \Upsilon_p^*$ where $F_p = \mathrm{diag}(f(\omega_0),\ldots,f(\omega_{p-1}))$, $\Upsilon_p = [G_p(\omega_0),\ldots,G_p(\omega_{p-1})]$, with $G_p(\omega_k) = \Gamma_p(\omega_k)\sqrt{H_k}$, $k = 0,\ldots,p-1$. Exactness is obtained for all F in $\mathcal{R}_p = \mathcal{S}_p\cdot\mathcal{S}_{-p} = \mathrm{span}\{\mathcal{B}_k(\mathrm{e}^{\mathrm{j}\omega})\overline{\mathcal{B}_l(\mathrm{e}^{\mathrm{j}\omega})} : k,l = 0,\ldots,p-1\}$ for (rational) Szegő quadrature formulas [3], [4], [5], [6]. Such formulas have nodes ω_i such that $\mathrm{e}^{\mathrm{j}\omega_i}$ are the p simple zeros on $\mathbb{T}$ of the para-orthogonal function $\mathcal{B}_p(\mathrm{e}^{\mathrm{j}\omega}) - \eta\mathcal{B}_p(\mathrm{e}^{\mathrm{j}\omega})\overline{\mathcal{B}_p(\mathrm{e}^{\mathrm{j}\omega})}$ where $\eta = \mathrm{e}^{\mathrm{j}\theta} \in \mathbb{T}$ can be chosen arbitrarily. The weights of these quadrature formulas are $H_k = 1/\gamma_p(\omega_k)$. Thus for Szegő formulas, $\Upsilon_p = [G_p(\omega_0),\ldots,G_p(\omega_{p-1})]$ is a matrix with normalized columns in general.

In the classical Fourier case Υ_p is a unitary matrix. It is in fact the FFT matrix with entries $\mathrm{e}^{\mathrm{j}k\omega_l}/\sqrt{p}$ for $k,l = 0,\ldots,p-1$ since the nodes $\omega_l = 2\pi l/p$ correspond to the roots of unity (or a rotation thereof).

In the more general TM situation, the nodes correspond to zeros of the para-orthogonal functions, which means that ω_i are such that they solve $\varphi_p(\mathrm{e}^{\mathrm{j}\omega}) = \eta$ with $\eta \in \mathbb{T}$, i.e., the phase $\chi_p(\omega)$ of $\varphi_p(\mathrm{e}^{\mathrm{j}\omega})$ has a constant value θ, modulo 2π. Thus $\Omega_p(\theta) = \{\omega_0,\ldots,\omega_{p-1}\}$ is given by

$$\Omega_p(\theta) = \{\omega \in [0,2\pi) : \chi_p(\omega) = \theta \bmod 2\pi\}. \qquad (9)$$

Note that just as in the classical case this implies that the matrices Υ_p are unitary because $G_p^*(\mu)G_p(\sigma) = 0$ iff $\Gamma_p^*(\mu)\Gamma_p(\sigma) = 0$. The Christoffel-Darboux relation (see e.g., [13], [5])

$$K_p(\mu,\sigma) = \frac{1 - \varphi_p(\mathrm{e}^{\mathrm{j}\mu})\overline{\varphi_p(\mathrm{e}^{\mathrm{j}\sigma})}}{1 - \mathrm{e}^{\mathrm{j}(\mu-\sigma)}}. \qquad (10)$$

implies that $G_p^*(\mu)G_p(\sigma) = 0$ iff $\overline{\varphi_p(\mathrm{e}^{\mathrm{j}\sigma})}\varphi_p(\mathrm{e}^{\mathrm{j}\mu}) = 1$, which gives the set of points described by $\Omega_p(\theta)$.

The orthogonality of the rows of the matrix Υ_p implies that the $\mathcal{B}_n$ are also orthogonal with respect to summation:

$$\sum_{i=0}^{p-1} \frac{\mathcal{B}_n(\mathrm{e}^{\mathrm{j}\omega_i})\overline{\mathcal{B}_m(\mathrm{e}^{\mathrm{j}\omega_i})}}{\gamma_n(\omega_i)} = \delta_{n-m}, \quad n,m = 0,\ldots,p-1.$$

VI. APPROXIMATION OF THE GENERALIZED TOEPLITZ MATRIX

To analyse the asymptotics of the spectrum of $M_p(f)$, we need some explicit expression for its eigenvalues or at least for an approximate value. This can be obtained through the previous quadrature formulas.

Indeed, the quadrature formulas of the previous section give rise to an approximation of the general Toeplitz matrix $M_p(f)$ of (5). This approximation had the form

$$\begin{aligned} W_p(f) &= \Upsilon_p F_p(f) \Upsilon_p^*, \\ F_p(f) &= \mathrm{diag}(f(\omega_0),\ldots,f(\omega_{p-1})), \\ \Upsilon_p &= [G_p(\omega_0),\ldots,G_p(\omega_{p-1})], \end{aligned} \qquad (11)$$

where the $\omega_k \in \Omega_p(\theta)$. Note that this corresponds to the eigenvalue decomposition of $W_p(f)$, since the columns of the unitary matrix Υ_p are the eigenvectors and the eigenvalues are the diagonal elements of F_p. Thus $f(\omega_k) \approx f_p(\omega_k)$, $\omega_k \in \Omega_p(\theta)$ are approximations for the eigenvalues of $M_p(f)$. Although the quadrature formula is exact in $\mathcal{R}_p$, in this case we approximate the integral of $\mathcal{B}_k(\mathrm{e}^{\mathrm{j}\omega})f(\omega)\overline{\mathcal{B}_l(\mathrm{e}^{\mathrm{j}\omega})}$, which is not in $\mathcal{R}_p$ in general. Using an expansion like (4) for f, we need to integrate products of three basis functions $\mathcal{B}_k$ where one or two of them may be complex conjugated. In order to simplify the analysis, we shall also replace the sample values $f(\omega_k)$ in the diagonal matrix F_p by the approximate values $f_p(\omega_k)$ with f_p the partial sum (3). With this modification of F_p all the indices in the triple products of the $\mathcal{B}_k$ can be restricted to values $0,\ldots,p-1$. This allows us to obtain upper bounds for the elements in $\Delta_p(f) = M_p(f) - W_p(f)$, from which the convergence results can be obtained. Indeed, we have

Theorem 5: Let $\Delta_p(f) = M_p(f) - W_p(f)$, with $W_p(f)$ as in (11) with f replaced by f_p, then

$$|G_p^*(\sigma)\Delta_p(f)G_p(\mu)| \le K_c^2 K_{f,p}/p, \quad \forall\sigma,\mu \in [0,2\pi),$$

$\|\Delta_p(f)\|_F \le K_{f,p}$, and

$$\|\Delta_p(f)G_p(\omega)\|_2 \le K_c K_{f,p}/p^{1/2}, \quad \forall\omega \in [0,2\pi)$$

where $\|\cdot\|_F$ is the Frobenius norm, $K_c = (1+c)/(1-c)$, while

$$K_{f,p} = 2K_c \sum_{k=0}^{p-1} K_0(k)\max(|a_k|,|a_{-k}|)$$

with $K_0(k) = [k^2(\epsilon_0-1)+(2k+1)/(1-\eta_0)]$, where for some $\rho_0 \in (\rho^*,1)$ with $\rho^* \in (c,1)$, $\eta_0 = (\rho_0+c)/(1+\rho_0 c) < 1$, and

$$\epsilon_0 = \left\lceil \frac{\ln(1-\rho_0 c) - \ln(\rho_0 - c)}{\ln(1+\rho_0 c) - \ln(\rho_0 + c)} \right\rceil,$$

where $\lceil x \rceil$ is the smallest integer not less than x.

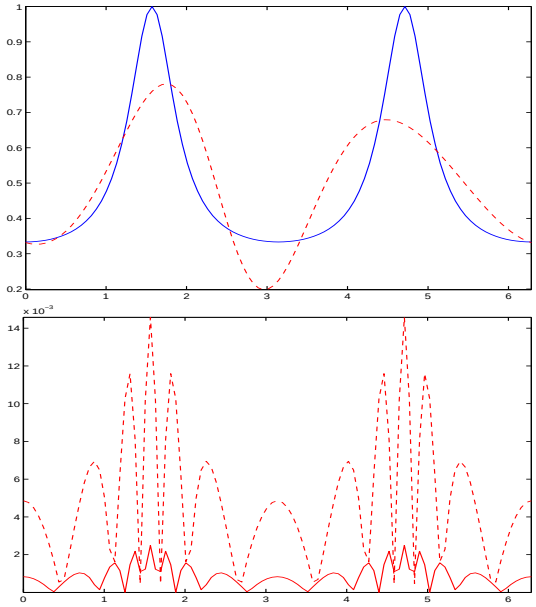


Fig. 1. Top: The function $f(\omega)$ (solid) and its approximation $f_2(\omega)$ (dashed) when choosing random poles. Bottom: Error of the approximation by $f_p(\omega)$ for $p = 4, 6$, (dashed,solid) when choosing $\xi_{2k} = \mathrm{j}0.6$, $\xi_{2k+1} = -\mathrm{j}0.6$.

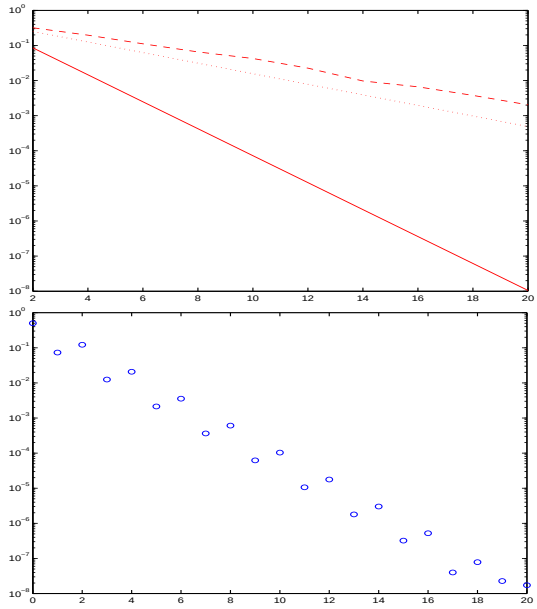


Fig. 2. Top: The error $\|f - f_p\|_\infty$ as a function of p for different choices of the poles. Bottom: The absolute value of the generalized Fourier coefficients with $\xi_k = \pm\mathrm{j}0.6$.

VII. SIMULATION RESULTS

In this section we shall check and illustrate some of the results that were summarized in the previous sections.

First we consider the function $f(\omega) = (\cos 2\omega + 2)/(4\cos(2\omega)+5) = \mathrm{Re}F(\mathrm{e}^{\mathrm{j}\omega})$ with $F(z) = 1/(z^2+2)$. Its poles are $\pm\mathrm{j}\sqrt{2}$, which corresponds to placing ξ_i at $\pm\mathrm{j}/\sqrt{2}$ which is approximately $\pm\mathrm{j}0.7$.

We shall experiment with either ξ_k that are randomly distributed in a circle around the origin of radius 0.4, or we assume that we have an accurate estimate of the poles and choose $\xi_k = \pm\mathrm{j}0.6$. Or we take another extreme case where all the $\xi_k = 0$.

Figure 1 shows the function $f(\omega)$ and an approximant $f_2(\omega)$ when choosing the random ξ_k. At the bottom the errors $f(\omega) - f_p(\omega)$ are plotted for $p = 2, 4$ where the poles are chosen in $\pm\mathrm{j}0.6$. In Figure 2 we see the error $\|f - f_p\|_\infty$ as a function of p for random ξ_k (largest), for all $\xi_k = 0$ (middle) or $\xi_k = \pm\mathrm{j}0.6$ (smallest). Only the error for the f_{2k} are plotted, because of the symmetry in f. It illustrates clearly that an appropriate choice of the ξ_k may improve the approximation quality considerably. The introduction of randomly chosen poles may be worse than choosing them all at the origin. At the bottom of Figure 2 you see the absolute value of the 21 first generalized Fourier coefficients (logarithmic scale) for the case of $\xi_k = \pm\mathrm{j}0.6$. To illustrate where the nodes of the quadrature formulas are located, we have plotted in Figure 3 the argument of the Blaschke product φ_{10} for some choices of the ξ_k. At the top you see the

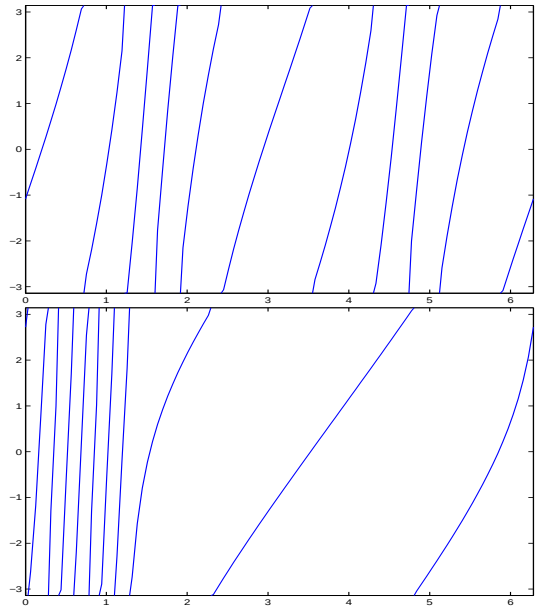


Fig. 3. The argument of φ_{10}. Top: when $\xi_k = \pm\mathrm{j}0.6$. Bottom: when $\xi_k = 0.9\mathrm{e}^{\mathrm{j}k\pi/10}$, $k = 0, \ldots, 10$.

argument when $\xi_k = \pm$j0.6. At the bottom, the choice was to take 10 ξ_k's of modulus 0.9 that were uniformly distributed on the upper half circle with radius 0.9. You can see how this distribution results in a concentration of 8 nodes in $(0, \pi)$, while there are only 2 in $(\pi, 2\pi)$. For all $\xi_k = 0$ at the origin, the distribution of the nodes is uniform over $[0, 2\pi)$ and for randomly distributed ξ_k's in a circle of radius 0.4, the distribution of the nodes is almost uniform over $[0, 2\pi)$. These pictures illustrate that the nodes of the quadrature formula which are the elements from the set $\Omega_p(\theta)$, tend to flock together on the circle in the neighborhood of dominant poles.

In the next figures we illustrate the approximation of the Toeplitz matrix $M_p(f)$ as described in Theorem 5. From now on we only consider the case where $\xi_k = \pm$j0.6. In Figure 4 on the top we show as a function of p the Frobenius norms $\|\Delta_p(f)\|_F$ for our example when $\xi_k = \pm$j0.6. At the top, we show $\|\Delta_p(f)G_p(\omega)\|_2$ as a function of p for the same example.

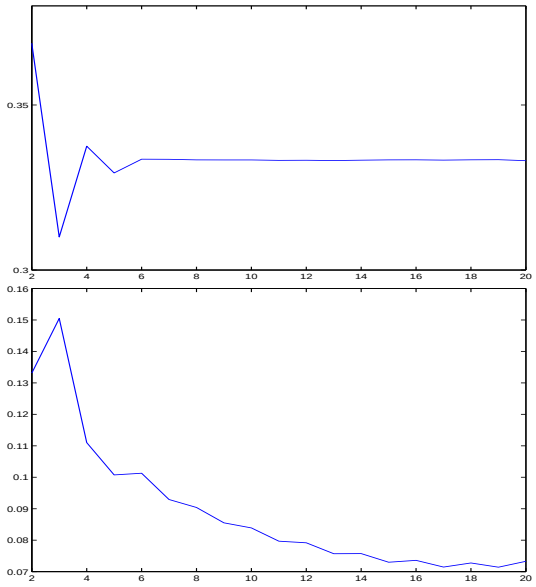


Fig. 4. Top: The error $\|\Delta_p(f)\|_\infty$ as a function of p. Bottom: The error $\|\Delta_p(f)G_p(\omega)\|_2$ as a function of p.

The approximation of the eigenvalues, described in Theorem 3 is illustrated in Figure 5. At the top you can observe $\|\vec{\lambda}_p - \vec{\rho}_p\|_2$ where $\vec{\lambda}_p$ is the vector of sorted eigenvalues of $M_p(f)$ and $\vec{\rho}_p$ is the vector of function values at the nodes of the quadrature formula: $\vec{\rho}_p = [f(\omega_0), \ldots, f(\omega_{p-1})]$, the ω_k being the elements from $\Omega_p(\theta)$ (we took here $\theta = 1$, but the influence of θ is minimal). At the top of Figure 5 we have plotted for $p = 20$ the diagonal elements of $\Upsilon_p^* M_p(f) \Upsilon_p$ (circles) against $f(\omega)$. These diagonal elements should, according to Theorem 3, be approximately equal to the values $f(\omega_k)$ for $\omega_k \in \Omega_p(\theta)$. It is clear that when evaluated at the intermediate values of ω, the expression $G_p^*(\omega)M_p(f)G_p(\omega)$ shall for $p = 20$ be a pretty good approximation of $f(\omega)$.

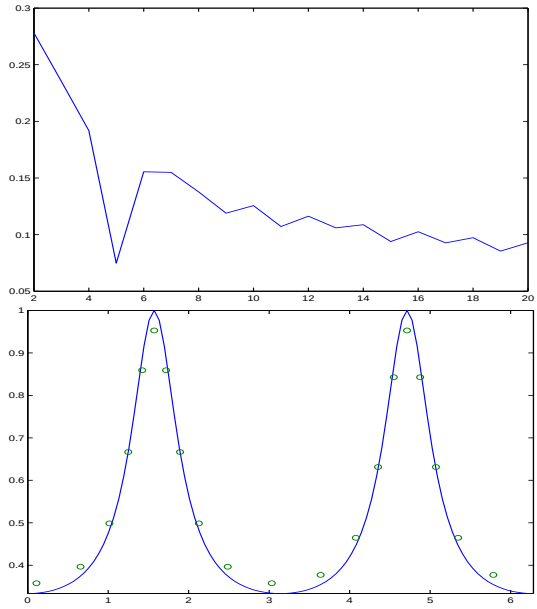


Fig. 5. Top: The 2-norm of the vector of differences $\lambda_i(M_p(f)) - f(\omega_i)$ for $\omega_i \in \Omega_p(\theta)$. Bottom: Plot diag$(\Upsilon_p^* M_p(f) \Upsilon_p)$ (circles) and $f(\omega)$.

In Figure 6, we have illustrated Theorem 4 by plotting on the top the value of $(1/p)\sum_{k=0}^{p-1} \lambda_k(M_p(f))$ for $p = 2, \ldots, 20$. As $p \to \infty$, this should approximate the integral as indicated in the theorem. In this case the integral can be evaluated to be ≈ 0.6098. At $p = 20$, the value of the left hand sum is 0.6045. To compute the value of the integral, we may observe that $\int_0^{2\pi} f(\tilde{\chi}^{-1}(\omega))\mathrm{d}\omega = \int_0^{2\pi} f(\mu)\tilde{\chi}'(\mu)\mathrm{d}\mu$. Where the $\tilde{\chi}$ andt $\tilde{\chi}'$ are plotted in Figure 7

At the bottom of Figure 6 we have plotted with dashed line the cumulative distribution: $(1/p)\sum_{k=0}^{p-1} f(\omega_k)$ where $\Omega_p(\theta) = \{\omega_0, \ldots, \omega_{p-1}\}$ (the $f(\omega_k)$ are sorted). The solid line is the cumulative distribution of the eigenvalues: $(1/p)\sum_{k=0}^{p-1} \lambda_k(M_p(f))$, where the (real) eigenvalues are also assumed to be sorted. The plot corresponds to $p = 20$ and $\theta = 1$. Both of these curves reach the values 0.6045 in their rightmost point which corresponds to the rightmost point of the figure on the top.

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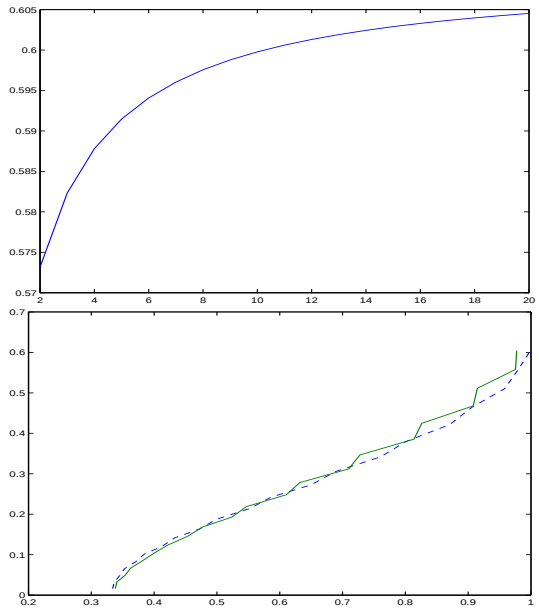


Fig. 6. Top: $(1/p)\sum_{k=0}^{p-1}\lambda_k(M_p(f))$ as a function of p. Bottom: The cumulative distribution of $f(\omega_k)$, $\omega_k \in \Omega_{20}(1)$(dashed) and of $\lambda_k(M_{20}(f))$.

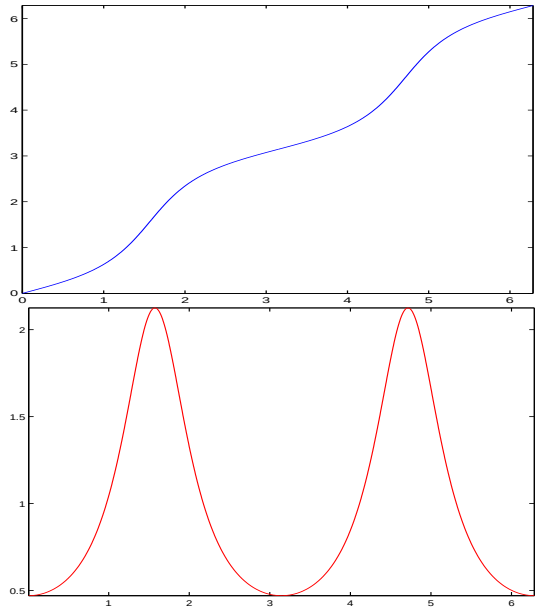


Fig. 7. Top: $\tilde{\chi}$ for our example: $\xi_k = \pm 0.6\mathrm{j}$. Bottom: The derivative $\tilde{\chi}'$ of the top figure.

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