CONTINUED FRACTIONS AND ORTHOGONAL RATIONAL FUNCTIONS

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Abstract

To a given modified PC-fraction with approximants A_n/B_n there exists a unique pair (L_0, L_∞) of formal power series at 0 and ∞ such that A_{2m}/B_{2m} is the weak (m,m) two-point Padé approximant to L_0 of order $m+1$ and to L_∞ of order m, while A_{2m+1}/B_{2m+1} is the weak (m,m) two-point Padé approximant to L_0 of order m and to L_∞ of order $m+1$. The canonical denominators of the even contraction of the modified PC-fraction (which is a modified T-fraction) are orthogonal Laurent polynomials obtained from the basis $\{1, z^{-1}, z, z^{-2}, z^2, \ldots\}$, while the canonical denominators of the odd contractions of the modified PC-fraction (which is a modified M-fraction) are orthogonal Laurent polynomials obtained from the basis $\{1, z, z^{-1}, z^2, z^{-2}, \ldots\}$.

An analogous situation arises when the pair of power series (L_0, L_∞) is replaced by Newton series determined by a general interpolation table of points on the real line, the modified PC-fraction and its contractions are replaced by appropriate analogous continued fractions, and orthogonal Laurent polynomials are replaced by orthogonal rational functions with poles in the set of interpolation points. In this paper an investigation of these relationships is carried out.

1 Introduction

By a *modified PC-fraction (modified Perron-Carathéodory fraction)* we mean a continued fraction

$$\kappa_0 + K_{n=1}^{\infty} \frac{a_n}{b_n} \tag{1.1}$$

where

$$a_{4m} = 1, \quad b_{4m} = \mu_{4m}, \quad m = 1, 2, \ldots \tag{1.2}$$

$$a_{4m+1} = \lambda_{4m+1}, \quad b_{4m+1} = \mu_{4m+1}, \quad m = 0, 1, 2, \ldots \tag{1.3}$$

$$a_{4m+2} = \frac{1}{z}, \quad b_{4m+2} = \mu_{4m+2}, \quad m = 0, 1, 2, \ldots \tag{1.4}$$

$$a_{4m+3} = \lambda_{4m+3} z, \quad b_{4m+3} = \mu_{4m+3} z, \quad m = 0, 1, 2, \ldots \tag{1.5}$$

with λ_n, μ_n constants, $\mu_1 = 1$. The constants λ_n, μ_n are required to satisfy the relation

$$\lambda_{2n+1} = 1 - \mu_{2n}\mu_{2n+1}, \quad n = 1, 2, \ldots . \tag{1.6}$$

(A modified PC-fraction is related to a standard PC-fraction through a simple equivalence transformation. The coefficients λ_n, μ_n have usually been called α_n, β_n, but we need especially the notation α_n for other purposes.)

Let A_n and B_n denote the canonical numerators and denominators of the modified PC-fraction (1.1). These functions are of the form

$$A_{4m} = \frac{\tilde{p}_{2m}}{z^m}, \quad B_{4m} = \frac{p_{2m}}{z^m}, \quad m = 1, 2, \ldots \tag{1.7}$$

$$A_{4m+1} = \frac{\tilde{q}_{2m}}{z^m}, \quad B_{4m+1} = \frac{q_{2m}}{z^m}, \quad m = 0, 1, 2, \ldots \tag{1.8}$$

$$A_{4m+2} = \frac{\tilde{r}_{2m+1}}{z^{m+1}}, \quad B_{4m+2} = \frac{r_{2m+1}}{z^{m+1}}, \quad m = 0, 1, 2, \ldots \tag{1.9}$$

$$A_{4m+3} = \frac{\tilde{s}_{2m+1}}{z^{m+1}}, \quad B_{4m+3} = \frac{s_{2m+1}}{z^{m+1}}, \quad m = 0, 1, 2, \ldots \tag{1.10}$$

where $p_n, q_n, r_n, s_n, \tilde{p}_n, \tilde{q}_n, \tilde{r}_n, \tilde{s}_n$ are polynomials of degree n.

The modified PC-fraction determines two formal power series

$$L_0 = c_0^{(0)} + \sum_{k=1}^{\infty} c_k z^k \tag{1.11}$$

$$L_\infty = -c_0^{(\infty)} - \sum_{k=1}^{\infty} c_{-k} z^{-k} \tag{1.12}$$

which are related to the continued fraction in the following way: The even approximants (partial fractions) $\frac{A_{2m}}{B_{2m}}$ are weak (m,m) two-point Padé approximants for the pair (L_0, L_∞) of order $(m+1, m)$, and the odd approximants $\frac{A_{2m+1}}{B_{2m+1}}$ are weak (m,m) two-point Padé approximants for the pair (L_0, L_∞) of order $(m, m+1)$. This means that

$$A_{2m} - L_0 B_{2m} = O(z^{m+1}) \quad \text{as} \quad z \to 0, \quad m = 1, 2, \ldots \tag{1.13}$$

$$A_{2m} - L_\infty B_{2m} = O(1) \quad \text{as} \quad z \to \infty, \quad m = 1, 2, \ldots \tag{1.14}$$

$$A_{2m+1} - L_0 B_{2m+1} = O(z^m) \quad \text{as} \quad z \to 0, \quad m = 0, 1, 2, \ldots \tag{1.15}$$

$$A_{2m+1} - L_\infty B_{2m+1} = O\left(\frac{1}{z}\right) \quad \text{as} \quad z \to \infty, \quad m = 0, 1, 2, \ldots \ . \tag{1.16}$$

When $\mu_{2m} \neq 0$ for all m, the even contraction of (1.1) exists (with $\frac{A_{2m}}{B_{2m}}$ as approximants). This continued fraction

$$\kappa_0 + K_{n=1}^{\infty} \frac{\theta_n}{\kappa_n} \tag{1.17}$$

is a *modified T-fraction* with elements

$$\theta_{2m} = -F_{2m}, \quad \kappa_{2m} = 1 - G_{2m} z, \quad m = 1, 2, \ldots \tag{1.18}$$

$$\theta_{2m+1} = -F_{2m+1}, \quad \kappa_{2m+1} = \frac{1}{z} - G_{2m+1}, \quad m = 0, 1, 2, \ldots \tag{1.19}$$

with F_n, G_n constants. The canonical denominators (which are the functions B_{2m}) are orthogonal Laurent polynomials obtained by orthogonalization of the basis $\{1, z^{-1}, z, z^{-2}, z^2, \ldots\}$ with respect to a given measure or linear functional. (Cf. Section 2).

When $\mu_{2m+1} \neq 0$ for all m, the odd contraction of (1.1) exists (with $\frac{A_{2m+1}}{B_{2m+1}}$ as approximants). This continued fraction

$$\kappa_0' + K_{n=1}^{\infty} \frac{\rho_n}{\sigma_n} \tag{1.20}$$

is a *modified M-fraction* with elements

$$\rho_{2m} = -U_{2m}, \quad \sigma_{2m} = (1 - V_{2m} \cdot \frac{1}{z}), \quad m = 1, 2, \ldots \tag{1.21}$$

$$\rho_{2m+1} = -U_{2m+1}, \quad \sigma_{2m+1} = (z - V_{2m+1}), \quad m = 0, 1, 2, \ldots \tag{1.22}$$

with U_n, V_n constants. The canonical denominators (which are the functions B_{2m+1}) are orthogonal Laurent polynomials obtained by orthogonalization of the basis $\{1, z, z^{-1}, z^2, z^{-2}, \ldots\}$.

We refer to [1,2,3,16,17,18,19,20,22,23] for more details on two-point Padé approximants and related continued fractions. Especially relevant for the discussion in this section are [2,18,19].

An analogous situation arises when interpolation at $\{0, \infty\}$ is replaced by interpolation tables determined by a general sequence of basic points $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ on the real line. It is known that regular orthogonal rational functions with poles in the set $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ are canonical denominators of a continued fraction called an *MP*-fraction. And vice versa: The

canonical denominators of an *MP*-fraction are orthogonal rational functions with respect to a functional. The theory of such orthogonal rational functions and the associated continued fractions is briefly reviewed in Section 2.

The main content of this paper is the following:

I. We show how to extend an *MP*-fraction to a continued fraction of a new type, an *EMP*-fraction, which has the *MP*-fraction as its even contraction. (Section 3).

II. We formally introduce *EMP*-fractions and show that the canonical numerators and denominators of an *EMP*-fraction has a structure analogous to that of the numerators and denominators of a modified *PC*-fraction. (Section 4).

III. We show that an *EMP*-fraction determines a Newton series Γ associated with the interpolation table $\{\infty, 0, \alpha_1, \alpha_1, \alpha_2, \alpha_2, \alpha_3, \alpha_3, \dots\}$ and a Newton series Δ associated with the interpolation table $\{\infty, 0, \alpha_2, \alpha_2, \alpha_1, \alpha_1, \alpha_4, \alpha_4, \dots\}$, such that the even approximants of the *EMP*-fraction are multi-point Padé approximants to Γ and the odd approximants are multi-point Padé approximants to Δ. (Section 5.)

IV. We show that the odd contraction (when it exists) of an *EMP*-fraction is equivalent to an *MP*-fraction, whose canonical denominators are orthogonal rational functions associated with the sequence of basic points $\{\alpha_2, \alpha_1, \alpha_4, \alpha_3, \dots\}$. (Section 6.)

2 MP-fractions

Let $\{\alpha_n : n = 1, 2, \dots\}$ be a sequence of not necessarily distinct points on the (finite) real line $\boldsymbol{R}$. We call these points *basic points.*

We define the functions ω_n by

$$\omega_0 = 1, \quad \omega_n(z) = \prod_{k=1}^{n}(z - \alpha_k), \quad n = 1, 2, \ldots \tag{2.1}$$

and the spaces $\mathcal{L}_n$ and $\mathcal{L}$ by

$$\mathcal{L}_n = \text{Span}\left\{\frac{1}{\omega_0}, \frac{1}{\omega_1}, \ldots, \frac{1}{\omega_n}\right\}, \quad \mathcal{L} = \bigcup_{n=0}^{\infty} \mathcal{L}_n. \tag{2.2}$$

The elements of $\mathcal{L}_n$ are exactly the functions that can be written in the form

$$L(z) = \frac{p(z)}{\omega_n(z)}, \quad p \in \mathcal{P}_n \tag{2.3}$$

where $\mathcal{P}_n$ denotes the space of polynomials of degree at most n.

Let M be a given linear functional on $\mathcal{L} \cdot \mathcal{L}$. We define the form $\langle , \rangle$ on $\mathcal{L} \times \mathcal{L}$ by

$$\langle K, L \rangle = M[K \cdot \bar{L}] \tag{2.4}$$

We assume that M is real for all functions in $\mathcal{L} \cdot \mathcal{L}$ that are real on $\boldsymbol{R}$, and also that M is definite, i.e., that

$$\langle L, L \rangle \neq 0 \quad \text{for} \quad L \in \mathcal{L}, \quad L \not\equiv 0. \tag{2.5}$$

Let $\{\Phi_n : n = 0, 1, 2, \ldots\}$ be the essentially unique orthogonal sequence obtained by the Gram-Schmidt process from the sequence $\left\{\frac{1}{\omega_n} : n = 0, 1, 2, \ldots\right\}$. Each Φ_n can be written in the form

$$\Phi_n(z) = \frac{K_n(z)}{\omega_n(z)} \tag{2.6}$$

where $K_n \in \mathcal{P}_n$. It follows from the construction that

$$K_n(\alpha_n) \neq 0, \quad n = 0, 1, 2, \ldots \tag{2.7}$$

The function Φ_n is called *regular* if $K_n(\alpha_{n-1}) \neq 0$.

We shall in this and the next section assume that the sequence $\{\Phi_n\}$ *is regular, i.e.,*

$$K_n(\alpha_{n-1}) \neq 0, \quad n = 1, 2, \ldots . \tag{2.8}$$

To avoid considering exceptional cases in some formulas, we shall use the following conventions:

$$z - \alpha_{-1} \quad \text{means} \quad z \tag{2.9}$$

$$z - \alpha_0 \quad \text{means} \quad 1. \tag{2.10}$$

The sequence $\{\Phi_n\}$ satisfies a recurrence relation of the form

$$\Phi_n(z) = \left(h_n + \frac{g_n}{z - \alpha_n}\right) \Phi_{n-1}(z) + F_n \frac{z - \alpha_{n-2}}{z - \alpha_n} \Phi_{n-2}(z), \quad n = 1, 2, \ldots . \tag{2.11}$$

with initial conditions

$$\Phi_0 = 1, \quad \Phi_{-1} = 0. \tag{2.12}$$

The coefficients satisfy the inequalities

$$F_n \neq 0, \quad g_n + h_n(\alpha_{n-1} - \alpha_n) \neq 0, \quad n = 1, 2, \ldots . \tag{2.13}$$

We note that the sum $h_n + \dfrac{g_n}{z - \alpha_n}$ may be written in various ways:

$$h_n + \frac{g_n}{z - \alpha_n} = \frac{x_n \zeta_n(z) + y_n \tau_n(z)}{z - \alpha_n}, \tag{2.14}$$

where $\{\zeta_n, \tau_n\}$ is an arbitrary basis for the space $\mathcal{P}_1$.

The functions Φ_n are canonical denominators of a continued fraction

$$\kappa_0 + K_{n=1}^{\infty} \frac{\theta_n}{\kappa_n} \tag{2.15}$$

where

$$\theta_n = F_n \frac{z - \alpha_{n-2}}{z - \alpha_n}, \quad n = 1, 2, \ldots \tag{2.16}$$

$$\kappa_n = h_n + \frac{g_n}{z - \alpha_n}, \quad n = 1, 2, \ldots \tag{2.17}$$

We denote by Ψ_n the canonical numerators of the continued fraction, i.e., the solution of the three-term recurrence relation with elements θ_n, κ_n and initial values

$$\Psi_0 = \kappa_0, \quad \Psi_{-1} = 1, \quad \kappa_0 = M[1]^{-\frac{1}{2}}. \tag{2.18}$$

The functions Ψ_n, Φ_n thus satisfy the relation

$$\begin{bmatrix} \Psi_n \\ \Phi_n \end{bmatrix} = \kappa_n \begin{bmatrix} \Psi_{n-1} \\ \Phi_{n-1} \end{bmatrix} + \theta_n \begin{bmatrix} \Psi_{n-2} \\ \Phi_{n-2} \end{bmatrix}, \quad n = 1, 2, \ldots \tag{2.19}$$

with initial conditions

$$\begin{bmatrix} \Psi_0 \\ \Phi_0 \end{bmatrix} = \begin{bmatrix} \kappa_0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \Psi_{-1} \\ \Phi_{-1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{2.20}$$

Continued fractions of the kind desribed above are called *MP-fractions (Multi-point Padé continued fractions).*

We have seen that a regular sequence of orthogonal functions in $\mathcal{L}$ determines an *MP*-fraction, the orthogonal functions being the canonical denominators of the continued fraction. On the other hand, the sequence of canonical denominators of an *MP*-fraction is a

regular sequence of orthogonal functions in $\mathcal{L}$ determined by a suitable functional. Thus there is a one-to-one correspondence between functionals M giving rise to regular sequences of orthogonal functions and *MP*-fractions. Properties tied to such a functional M may then be considered as properties tied to the corresponding continued fraction, and vice versa.

The sequence of approximants $\left\{\frac{\Psi_n}{\Phi_n}\right\}$ determines a sequence of interpolation values with respect to the interpolation table $\{\infty, 0, \alpha_1, \alpha_1, \alpha_2, \alpha_2, \alpha_3, \alpha_3, \dots\}$. We shall give a detailed discussion of this relationship in the course of the discussions in Section 5.

For more information on orthogonal functions in the spaces $\mathcal{L}$, corresponding continued fractions, and related topics, we refer to [4, 5, 6, 7, 8, 9, 10, 13, 14, 15, 24, 25].

3 Extension of MP-fractions

We shall construct a continued fraction

$$\kappa_0 + K_{n=1}^{\infty} \frac{a_n}{b_n} \tag{3.1}$$

which has (2.15)-(2.17) as its even contraction and with the property that the approximants (partial fractions) represent multi-point Padé approximants to Newton series determined by the functional M and the sequence $\{\alpha_n : n = 1, 2, \dots\}$ (or alternatively, by the continued fraction) according to certain patterns.

$$\textit{We shall in this section assume that } \alpha_{2m} \neq \alpha_{2m-1}, \quad m = 1, 2, \dots \tag{3.2}$$

We note that this condition is in particular satisfied when $\{\alpha_n\}$ is obtained by cyclic repetition of a finite number of distinct points $\alpha_1, \dots, \alpha_p \quad (p > 1)$.

We make use of the following observation: When (3.2) is satisfied, $\{z-\alpha_{2m-1}, z-\alpha_{2m}\}$ is a base for $\mathcal{P}_1$. The elements (2.17) of (2.15) may therefore be represented in the form

$$\kappa_{2m} = H_{2m} + G_{2m}\frac{z-\alpha_{2m-1}}{z-\alpha_{2m}}, \quad m = 1, 2, \ldots \tag{3.3}$$

$$\kappa_{2m+1} = H_{2m+1}\frac{z-\alpha_{2m}}{z-\alpha_{2m+1}} + G_{2m+1}\frac{z-\alpha_{2m-1}}{z-\alpha_{2m+1}}, \quad m = 0, 1, 2, \ldots. \tag{3.4}$$

(If (3.2) is not satisfied, the continued fraction (2.15) with elements (2.16), (3.3)-(3.4) is not associated with a sequence of orthogonal rational functions.) Expressions for the new coefficients are

$$G_{2m} = \frac{g_{2m}}{\alpha_{2m}-\alpha_{2m-1}}, \quad m = 1, 2, \ldots \tag{3.5}$$

$$H_{2m} = h_{2m} - \frac{g_{2m}}{\alpha_{2m}-\alpha_{2m-1}}, \quad m = 1, 2, \ldots \tag{3.6}$$

$$G_{2m+1} = \frac{g_{2m+1}}{\alpha_{2m}-\alpha_{2m-1}} - h_{2m+1}\frac{\alpha_{2m+1}-\alpha_{2m}}{\alpha_{2m}-\alpha_{2m-1}}, \quad m = 1, 2, \ldots \tag{3.7}$$

$$H_{2m+1} = \frac{-g_{2m+1}}{\alpha_{2m}-\alpha_{2m-1}} + h_{2m+1}\frac{\alpha_{2m+1}-\alpha_{2m-1}}{\alpha_{2m}-\alpha_{2m-1}}, \quad m = 1, 2, \ldots \tag{3.8}$$

$$G_1 = h_1, \quad H_1 = g_1 - \alpha_1 h_1. \tag{3.9}$$

It follows from (2.13) that

$$G_{2m+1} \neq 0, \ m = 0, 1, 2, \ldots, \quad H_{2m} \neq 0, \ m = 1, 2, \ldots \tag{3.10}$$

We shall in this section assume that the sequence $\{\Phi_n\}$
is strongly regular, i.e., that also (3.11)
$G_{2m} \neq 0, \ m = 1, 2, \ldots, \quad H_{2m+1} \neq 0, \ m = 0, 1, 2, \ldots$

(This is always the case if $\alpha_{2m+1} = \alpha_{2m-1}$ for all m, cf. the situation where $\alpha_{2m} = \infty$, $\alpha_{2m-1} = 0$ for all m.)

We shall assume that the functions Φ_n are normalized such that

$$H_n = 1, \quad n = 1, 2, \ldots . \tag{3.12}$$

We may then write

$$\kappa_{2m} = 1 + G_{2m} \frac{z - \alpha_{2m-1}}{z - \alpha_{2m}}, \quad m = 1, 2, \ldots, \tag{3.13}$$

$$\kappa_{2m+1} = \frac{z - \alpha_{2m}}{z - \alpha_{2m+1}} + G_{2m+1} \frac{z - \alpha_{2m-1}}{z - \alpha_{2m+1}}, \quad m = 0, 1, 2, \ldots . \tag{3.14}$$

From general formulas for the even contraction of a continued fraction (see e.g. [3, 16, 22, 26]) we find that $\{a_n\}$, $\{b_n\}$ in the extension (3.1) must satisfy the equations

$$b_2 a_1 = \theta_1 \tag{3.15}$$

$$\frac{b_{2n} a_{2n-1} a_{2n-2}}{b_{2n-2}} = -\theta_n, \quad n = 2, 3, \ldots \tag{3.16}$$

$$a_2 + b_1 b_2 = \kappa_1 \tag{3.17}$$

$$a_{2n} + b_{2n} b_{2n-1} + a_{2n-1} \frac{b_{2n}}{b_{2n-2}} = \kappa_n, \quad n = 2, 3, \ldots \tag{3.18}$$

By taking into account (2.16), (3.12)-(3.13) we may write these equations in the following form:

$$b_2 a_1 = \frac{F_1 z}{z - \alpha_1} \tag{3.19}$$

$$\frac{b_{4m}a_{4m-1}a_{4m-2}}{b_{4m-2}} = -F_{2m}\frac{z-\alpha_{2m-2}}{z-\alpha_{2m}}, \quad m = 1, 2, \ldots \tag{3.20}$$

$$\frac{b_{4m+2}a_{4m+1}a_{4m}}{b_{4m}} = -F_{2m+1}\frac{z-\alpha_{2m-1}}{z-\alpha_{2m+1}}, \quad m = 1, 2, \ldots \tag{3.21}$$

$$a_2 + b_1 b_2 = \frac{1}{z-\alpha_1} + G_1\frac{z}{z-\alpha_1} \tag{3.22}$$

$$a_{4m} + b_{4m}b_{4m-1} + a_{4m-1}\frac{b_{4m}}{b_{4m-2}} = 1 + G_{2m}\frac{z-\alpha_{2m-1}}{z-\alpha_{2m}}, \quad m = 1, 2, \ldots \tag{3.23}$$

$$a_{4m+2} + b_{4m+2}b_{4m+1} + a_{4m+1}\frac{b_{4m+2}}{b_{4m}} = \frac{z-\alpha_{2m}}{z-\alpha_{2m+1}} + G_{2m+1}\frac{z-\alpha_{2m-1}}{z-\alpha_{2m+1}}, \quad m = 1, 2, \ldots \tag{3.24}$$

Our first task is to find an appropriate solution of these functional equations. We introduce the notation

$$\lambda_1 = \frac{F_1}{G_1}, \quad \mu_1 = 1 \tag{3.25}$$

$$\mu_{2n} = G_1 G_2 \cdots G_{n-1} G_n, \quad n = 1, 2, \ldots \tag{3.26}$$

and define

$$a_1 = \frac{\lambda_1 z}{z-\alpha_1}, \quad b_1 = \frac{\mu_1 z}{z-\alpha_1} = \frac{z}{z-\alpha_1} \tag{3.27}$$

$$a_{4m} = 1, \quad m = 1, 2, \ldots \tag{3.28}$$

$$a_{4m+2} = \frac{z-\alpha_{2m}}{z-\alpha_{2m+1}}, \quad m = 0, 1, 2, \ldots \tag{3.29}$$

$$b_{2n} = \mu_{2n}, \quad n = 1, 2, \ldots \tag{3.30}$$

We observe that with these definitions, (3.19) and (3.22) are satisfied.

It follows from (3.20)-(3.21) and the definitions above that

$$a_{4m-1} = \lambda_{4m-1} \frac{z - \alpha_{2m-1}}{z - \alpha_{2m}}, \quad m = 1, 2, \ldots \tag{3.31}$$

$$a_{4m+1} = \lambda_{4m+1} \frac{z - \alpha_{2m-1}}{z - \alpha_{2m+1}}, \quad m = 1, 2, \ldots \tag{3.32}$$

where

$$\lambda_{2n+1} = -\frac{F_{n+1}}{G_{n+1}}, \quad n = 1, 2, \ldots \tag{3.33}$$

We note that by (2.13),

$$\lambda_{2n+1} \neq 0, \quad n = 0, 1, 2, \ldots \tag{3.34}$$

Similarly it follows from (3.23)-(3.24) and the definitions above that

$$b_{4m-1} = \mu_{4m-1} \frac{z - \alpha_{2m-1}}{z - \alpha_{2m}}, \quad m = 1, 2, \ldots \tag{3.35}$$

$$b_{4m+1} = \mu_{4m+1} \frac{z - \alpha_{2m-1}}{z - \alpha_{2m+1}}, \quad m = 1, 2, \ldots \tag{3.36}$$

where

$$\mu_{2n+1} = \frac{G_{n+1}}{\mu_{2n+2}} - \frac{\lambda_{2n+1}}{\mu_{2n}} = \frac{G_{n+1} + F_{n+1}}{\mu_{2n+2}}, \quad n = 1, 2, \ldots \tag{3.37}$$

In this way we have determined a continued fraction (3.1) whose even contraction is the continued fraction (2.15)-(2.16), (3.13)-(3.14). For convenience we repeat the form of

the elements:

$$a_{4m-1} = \lambda_{4m-1}\frac{z-\alpha_{2m-1}}{z-\alpha_{2m}}, \quad b_{4m-1} = \mu_{4m-1}\frac{z-\alpha_{2m-1}}{z-\alpha_{2m}}, \quad m = 1, 2, \ldots \tag{3.38}$$

$$a_{4m} = 1, \quad b_{4m} = \mu_{4m}, \quad m = 1, 2, \ldots \tag{3.39}$$

$$a_{4m+1} = \lambda_{4m+1}\frac{z-\alpha_{2m-1}}{z-\alpha_{2m+1}}, \quad b_{4m+1} = \mu_{4m+1}\frac{z-\alpha_{2m-1}}{z-\alpha_{2m+1}}, \quad m = 0, 1, 2, \ldots \tag{3.40}$$

$$a_{4m+2} = \frac{z-\alpha_{2m}}{z-\alpha_{2m+1}}, \quad b_{4m+2} = \mu_{4m+2}, \quad m = 0, 1, 2, \ldots \tag{3.41}$$

where the constants λ_n, μ_n are given by formulas (3.25)-(3.26), (3.33), (3.37).

We find that the following relationship holds:

$$\lambda_{2n+1} + \mu_{2n}\mu_{2n+1} = 1, \quad n = 1, 2, \ldots \tag{3.42}$$

These continued fractions belong to a class of continued fractions that we shall call *EMP-fractions (Extended multi-point Padé continued fractions)*. We shall give a formal definition and general discussion of *EMP*-fractions in Section 4.

We formulate the main extension result as a theorem.

Theorem 1. *The continued fraction* $\kappa_0 + K_{n=1}^{\infty}\frac{a_n}{b_n}$ *with* a_n, b_n *given by formulas (3.25)-(3.26), (3.33), (3.37)-(3.41), has as its even contraction the MP-fraction* $\kappa_0 + K_{n=1}^{\infty}\frac{\theta_n}{\kappa_n}$, *with* θ_n, κ_n *given by formulas (2.16), (3.13)-(3.14).*

4 EMP-fractions

We define an *EMP-fraction (Extended multi-point Padé continued fraction)* to be a continued fraction

$$\kappa_0 + K_{n=1}^{\infty} \frac{a_n}{b_n} \tag{4.1}$$

where the elements are given by (cf. (3.38)-(3.41))

$$a_{4m-1} = \lambda_{4m-1} \frac{z - \alpha_{2m-1}}{z - \alpha_{2m}}, \quad b_{4m-1} = \mu_{4m-1} \frac{z - \alpha_{2m-1}}{z - \alpha_{2m}}, \quad m = 1, 2, \ldots \tag{4.2}$$

$$a_{4m} = 1, \quad b_{4m} = \mu_{4m}, \quad m = 1, 2, \ldots \tag{4.3}$$

$$a_{4m+1} = \lambda_{4m+1} \frac{z - \alpha_{2m-1}}{z - \alpha_{2m+1}}, \quad b_{4m+1} = \mu_{4m+1} \frac{z - \alpha_{2m-1}}{z - \alpha_{2m+1}}, \quad m = 0, 1, 2, \ldots \tag{4.4}$$

$$a_{4m+2} = \frac{z - \alpha_{2m}}{z - \alpha_{2m+1}}, \quad b_{4m+2} = \mu_{4m+2}, \quad m = 0, 1, 2, \ldots \tag{4.5}$$

where λ_n, μ_n are constants,

$$\mu_1 = 1 \tag{4.6}$$

$$\lambda_{2n+1} \neq 0, \quad n = 0, 1, 2, \ldots \tag{4.7}$$

and where

$$\lambda_{2n+1} + \mu_{2n}\mu_{2n+1} = 1, \quad n = 1, 2, \ldots \tag{4.8}$$

Let A_n and B_n denote the canonical numerators and denominators of the *EMP*-fraction (4.1). We write out explicitely the recurrence formulas for $\{A_n\}$, $\{B_n\}$:

$$\begin{bmatrix} A_{4m} \\ B_{4m} \end{bmatrix} = \mu_{4m} \begin{bmatrix} A_{4m-1} \\ B_{4m-1} \end{bmatrix} + \begin{bmatrix} A_{4m-2} \\ B_{4m-2} \end{bmatrix}, \quad m = 1, 2, \ldots \tag{4.9}$$

$$\begin{bmatrix} A_{4m+1} \\ B_{4m+1} \end{bmatrix} = \mu_{4m+1} \frac{z - \alpha_{2m-1}}{z - \alpha_{2m+1}} \begin{bmatrix} A_{4m} \\ B_{4m} \end{bmatrix} + \lambda_{4m+1} \frac{z - \alpha_{2m-1}}{z - \alpha_{2m+1}} \begin{bmatrix} A_{4m-1} \\ B_{4m-1} \end{bmatrix}, \quad m = 0, 1, 2, \ldots \tag{4.10}$$

$$\begin{bmatrix} A_{4m+2} \\ B_{4m+2} \end{bmatrix} = \mu_{4m+2} \begin{bmatrix} A_{4m+1} \\ B_{4m+1} \end{bmatrix} + \frac{z - \alpha_{2m}}{z - \alpha_{2m+1}} \begin{bmatrix} A_{4m} \\ B_{4m} \end{bmatrix}, \quad m = 0, 1, 2, \ldots \tag{4.11}$$

$$\begin{bmatrix} A_{4m+3} \\ B_{4m+3} \end{bmatrix} = \mu_{4m+3} \frac{z - \alpha_{2m+1}}{z - \alpha_{2m+2}} \begin{bmatrix} A_{4m+2} \\ B_{4m+2} \end{bmatrix} + \lambda_{4m+3} \frac{z - \alpha_{2m+1}}{z - \alpha_{2m+2}} \begin{bmatrix} A_{4m+1} \\ B_{4m+1} \end{bmatrix}, \quad m = 0, 1, 2, \ldots \tag{4.12}$$

$$\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} \kappa_0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} A_{-1} \\ B_{-1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{4.13}$$

We shall describe the structure of A_n and B_n. In the following, $p_n, \tilde{p}_n, q_n, \tilde{q}_n, r_n, \tilde{r}_n, s_n, \tilde{s}_n$ denote polynomials in $\mathcal{P}_n$.

We find from (4.6), (4.10) that

$$A_1 = (\kappa_0 + \lambda_1) \frac{z}{z - \alpha_1} = \frac{\tilde{q}_1}{\omega_1}, \tag{4.14}$$

$$B_1 = \frac{z}{z-\alpha_1} = \frac{q_1}{\omega_1}. \tag{4.15}$$

Similarly we find from (4.11) that

$$A_2 = \mu_2 \frac{\tilde{q}_1}{\omega_1} + \frac{1}{z-\alpha_1} \cdot \kappa_0 = \frac{\tilde{r}_1}{\omega_1}, \tag{4.16}$$

$$B_2 = \mu_2 \frac{q_1}{\omega_1} + \frac{1}{z-\alpha_1} \cdot 1 = \frac{r_1}{\omega_1}, \tag{4.17}$$

and from (4.12) that

$$A_3 = \mu_3 \frac{z-\alpha_1}{z-\alpha_2} \cdot \frac{\tilde{r}_1}{z-\alpha_1} + \lambda_3 \frac{z-\alpha_1}{z-\alpha_2} \cdot \frac{\tilde{q}_1}{z-\alpha_1} = \frac{\tilde{s}_1}{z-\alpha_2}, \tag{4.18}$$

$$B_3 = \mu_3 \frac{z-\alpha_1}{z-\alpha_2} \cdot \frac{r_1}{z-\alpha_1} + \lambda_3 \frac{z-\alpha_1}{z-\alpha_2} \cdot \frac{q_1}{z-\alpha_1} = \frac{s_1}{z-\alpha_2}. \tag{4.19}$$

From here it follows by straightforward induction that

$$A_{4m}(z) = \frac{\tilde{p}_{2m}(z)}{\omega_{2m}(z)}, \quad B_{4m}(z) = \frac{p_{2m}(z)}{\omega_{2m}(z)}, \quad m = 1, 2, \ldots \tag{4.20}$$

$$\begin{aligned} A_{4m+1}(z) &= \frac{\tilde{q}_{2m}(z)}{(z-\alpha_{2m+1})(z-\alpha_{2m})\omega_{2m-2}(z)}, \\ B_{4m+1}(z) &= \frac{q_{2m}(z)}{(z-\alpha_{2m+1})(z-\alpha_{2m})\omega_{2m-2}(z)}, \quad m = 1, 2, \ldots \end{aligned} \tag{4.21}$$

$$A_{4m+2}(z) = \frac{\tilde{r}_{2m+1}(z)}{\omega_{2m+1}(z)}, \quad B_{2m+2}(z) = \frac{r_{2m+1}(z)}{\omega_{2m+1}(z)}, \quad m = 0, 1, 2, \ldots \tag{4.22}$$

$$A_{4m+3}(z) = \frac{\tilde{s}_{2m+1}(z)}{(z-\alpha_{2m+2})\omega_{2m}(z)}, \quad B_{4m+3}(z) = \frac{s_{2m+1}(z)}{(z-\alpha_{2m+2})\omega_{2m}(z)}, \quad m = 0, 1, 2, \ldots \tag{4.23}$$

We shall also make use of the following notation for the numerator polynomials in the canonical numerators and denominators:

$$P_0 = \kappa_0, \quad Q_0 = 1, \quad P_1 = \tilde{q}_1, \quad Q_1 = q_1 \tag{4.24}$$

$$P_{4m} = \tilde{p}_{2m}, \quad Q_{4m} = p_{2m}, \quad m = 1, 2, \ldots \tag{4.25}$$

$$P_{4m+1} = \tilde{q}_{2m}, \quad Q_{4m+1} = q_{2m}, \quad m = 1, 2, \ldots \tag{4.26}$$

$$P_{4m+2} = \tilde{r}_{2m+1}, \quad Q_{4m+2} = r_{2m+1}, \quad m = 0, 1, 2, \ldots \tag{4.27}$$

$$P_{4m+3} = \tilde{s}_{2m+1}, \quad Q_{4m+3} = s_{2m+1}, \quad m = 0, 1, 2, \ldots \tag{4.28}$$

We formulate the main result on the structure of the canonical numerators and denominators in a theorem.

Theorem 2 *The canonical numerators and denominators A_n, B_n of the EMP-fraction $\kappa_0 + K_{n=1}^{\infty} \frac{a_n}{b_n}$ determined by (4.2)-(4.8) are of the form given by formulas (4.14)-(4.15), (4.20)-(4.23).*

5 Interpolation

We shall now show how the approximants of the *EMP*-fraction (4.1)-(4.8) determine Newton series for which these approximants are multi-point Padé approximants. (Thus in particular a functional M which gives rise to an *MP*-fraction (2.15)-(2.16), (3.13)-(3.14)

and hence to an *EMP*-fraction by extension, also may be said to determine such a series). For basic information on Newton series and multi-point Padé approximation, see e.g. [11,12,20,21,26,27]. We briefly recall that a Newton series with respect to a sequence $\{\beta_n : n = 1, 2, \dots\}$ of (not necessarily distinct) interpolation points is a formal series

$$c_0 + c_1(z-\beta_1) + c_2(z-\beta_1)(z-\beta_2) + \cdots + c_n(z-\beta_1)\cdots(z-\beta_n) + \cdots \tag{5.1}$$

and that the (formal) Newton series expansion of the function φ is the unique Newton series (5.1) with the property

$$\varphi(z) - [c_0 + c_1(z-\beta_1) + \cdots + c_n(z-\beta_1)\cdots(z-\beta_n)] = O[(z-\beta_1)\cdots(z-\beta_{n+1})], \quad n = 1, 2, \dots \tag{5.2}$$

This means that in the formal Newton series expansion of $\varphi(z) - [c_0 + c_1(z-\beta_1) + \cdots + c_n(z-\beta_1)\cdots(z-\beta_n)]$, the coefficients of $1,\ (z-\beta_1), \cdots (z-\beta_1)\cdots(z-\beta_n)$ are zero. We refer especially to [7, 9] for related discussions of the situation when the functional M is positive definite.

We shall call the continued fraction (4.1)–(4.8) *normal* if the following conditions are satisfied: The polynomials p_{2m}, q_{2m}, r_{2m+1} and s_{2m+1} are of full degree and do not vanish at the origin, p_{2m} and q_{2m} do not vanish at any of the points $\alpha_1, \dots, \alpha_{2m}$, r_{2m+1} does not vanish at any of the points $\alpha_1, \dots, \alpha_{2m+1}$, and s_{2m+1} does not vanish at any of the points $\alpha_1, \dots, \alpha_{2m}, \alpha_{2m+2}$. (When $\mu_n \neq 0$ for all n, so that the even and the odd contraction of the continued fraction exists, cf. Section 6, this requirement means that 0 is a regular value for all the orthogonal functions B_{2m} and B_{2m+1}. For this concept of a regular value, see [6].)

We shall in this section assume that the EMP-fraction (4.1)–(4.8) is normal.

For later use we need expressions for the products $a_1a_2a_3\cdots a_n$. We find from (3.38)-(3.41):

$$a_1a_2\cdots a_{4m} = \frac{\Lambda_{4m}z}{(z-\alpha_{2m-1})(z-\alpha_{2m})}, \quad m = 1,2,\ldots \tag{5.3}$$

$$a_1a_2\cdots a_{4m+1} = \frac{\Lambda_{4m+1}z}{(z-\alpha_{2m})(z-\alpha_{2m+1})}, \quad m = 0,1,2\ldots \tag{5.4}$$

$$a_1a_2\cdots a_{4m+2} = \frac{\Lambda_{4m+2}z}{(z-\alpha_{2m+1})^2}, \quad m = 0,1,2\ldots \tag{5.5}$$

$$a_1a_2\cdots a_{4m+3} = \frac{\Lambda_{4m+3}z}{(z-\alpha_{2m+1})(z-\alpha_{2m+2})}, \quad m = 0,1,2\ldots \tag{5.6}$$

where the Λ_n are constants.

We shall make use of the following formulas connecting different numerators and denominators (see e.g. [3, 16, 22, 26]):

$$A_nB_{n-1} - A_{n-1}B_n = (-1)^{n-1}a_1a_2\cdots a_n, \quad n = 1,2,\ldots \tag{5.7}$$

$$A_{n+1}B_{n-1} - A_{n-1}B_{n+1} = (-1)^{n-1}a_1a_2\cdots a_nb_{n+1}, \quad n = 1,2,\ldots \tag{5.8}$$

By substituting from (4.2)-(4.5) and (4.20)-(4.27) in (5.7)-(5.8) we obtain the following formulas (e_n and ε_n are constants):

$$\frac{P_{4m+1}}{Q_{4m+1}} - \frac{P_{4m}}{Q_{4m}} = e_{4m}\frac{z\omega_{2m-2}\omega_{2m}}{p_{2m}q_{2m}}, \quad m = 1,2,\ldots \tag{5.9}$$

$$\frac{P_{4m+2}}{Q_{4m+2}} - \frac{P_{4m+1}}{Q_{4m+1}} = e_{4m+1}\frac{z(z-\alpha_{2m})\omega_{2m-2}\omega_{2m}}{q_{2m}r_{2m+1}}, \quad m = 1,2,\ldots \tag{5.10}$$

$$\frac{P_{4m+3}}{Q_{4m+3}} - \frac{P_{4m+2}}{Q_{4m+2}} = e_{4m+2}\frac{z\omega_{2m}\omega_{2m}}{r_{2m+1}s_{2m+1}}, \quad m = 0, 1, 2, \ldots \tag{5.11}$$

$$\frac{P_{4m+4}}{Q_{4m+4}} - \frac{P_{4m+3}}{Q_{4m+3}} = e_{4m+3}\frac{z(z-\alpha_{2m+2})\omega_{2m}\omega_{2m}}{s_{2m+1}p_{2m+2}}, \quad m = 0, 1, 2, \ldots \tag{5.12}$$

$$\frac{P_{4m+1}}{Q_{4m+1}} - \frac{P_{4m-1}}{Q_{4m-1}} = \varepsilon_{4m-1}\frac{z(z-\alpha_{2m})\omega_{2m-2}\omega_{2m-2}}{s_{2m-1}q_{2m}}, \quad m = 1, 2, \ldots \tag{5.13}$$

$$\frac{P_{4m+2}}{Q_{4m+2}} - \frac{P_{4m}}{Q_{4m}} = \varepsilon_{4m}\frac{z\omega_{2m-1}\omega_{2m}}{p_{2m}r_{2m+1}}, \quad m = 1, 2, \ldots \tag{5.14}$$

$$\frac{P_{4m+3}}{Q_{4m+3}} - \frac{P_{4m+1}}{Q_{4m+1}} = \varepsilon_{4m+1}\frac{z(z-\alpha_{2m})\omega_{2m-2}\omega_{2m}}{q_{2m}s_{2m+1}}, \quad m = 1, 2, \ldots \tag{5.15}$$

$$\frac{P_{4m+4}}{Q_{4m+4}} - \frac{P_{4m+2}}{Q_{4m+2}} = \varepsilon_{4m+2}\frac{z\omega_{2m}\omega_{2m+1}}{r_{2m+1}p_{2m+2}}, \quad m = 1, 2, \ldots \tag{5.16}$$

We note that

$$\frac{P_{n+1}}{Q_{n+1}} - \frac{P_n}{Q_n} = O\left(\frac{1}{z}\right) \quad \text{as} \quad z \to \infty, \quad n = 1, 2, \ldots \tag{5.17}$$

We shall consider Newton series with respect to the interpolation polynomial sequences

$$\begin{gathered}\{1, z, z(z-\alpha_1), \ldots, \quad z\omega_{2m-2}^2, \quad z(z-\alpha_{2m-1})\omega_{2m-2}^2, \\ z\omega_{2m-1}^2, z(z-\alpha_{2m})\omega_{2m-1}^2, \quad z\omega_{2m}^2, \ldots\}\end{gathered} \tag{5.18}$$

and

$$\begin{gathered}\{1, z, z(z-\alpha_2), \ldots, \quad z\omega_{2m-2}^2, \quad z(z-\alpha_{2m})\omega_{2m-2}^2, \\ z(z-\alpha_{2m})^2\omega_{2m-2}^2, \quad z(z-\alpha_{2m})^2(z-\alpha_{2m-1})\omega_{2m-2}^2, \quad z\omega_{2m}^2 \ldots\}\end{gathered} \tag{5.19}$$

i.e., with respect to the interpolation tables

$$\{\infty, 0, \alpha_1, \alpha_1, \alpha_2, \alpha_2, \alpha_3, \alpha_3, \alpha_4, \alpha_4, \ldots\} \tag{5.20}$$

and

$$\{\infty, 0, \alpha_2, \alpha_2, \alpha_1, \alpha_1, \alpha_4, \alpha_4, \alpha_3, \alpha_3, \ldots\} \tag{5.21}$$

where the interpolation at ∞ will be written out separately.

These series have the form

$$\begin{gathered}\gamma_0 + \gamma_1 z + \gamma_2 z(z-\alpha_1) + \gamma_3 z\omega_1^2 + \gamma_4(z-\alpha_2)\omega_1^2 + \cdots \\ \cdots + \gamma_{4m-3} z\omega_{2m-2}^2 + \gamma_{4m-2} z(z-\alpha_{2m-1})\omega_{2m-2}^2 + \\ \gamma_{4m-1} z\omega_{2m-1}^2 + \gamma_{4m} z(z-\alpha_{2m})\omega_{2m-1}^2 + \cdots\end{gathered} \tag{5.22}$$

and

$$\begin{gathered}\delta_0 + \delta_1 z + \delta_2 z(z-\alpha_2) + \delta_3 z(z-\alpha_2)^2 + \delta_4 z(z-\alpha_1)(z-\alpha_2)^2 + \cdots \\ \cdots + \delta_{4m-3} z\omega_{2m-2}^2 + \delta_{4m-2} z(z-\alpha_{2m})\omega_{2m-2}^2 \\ + \delta_{4m-1} z(z-\alpha_{2m})^2\omega_{2m-2}^2 \\ + \delta_{4m} z(z-\alpha_{2m-1})(z-\alpha_{2m})^2\omega_{2m-2}^2 + \cdots\end{gathered} \tag{5.23}$$

Let

$$\begin{gathered}\gamma_0^{(4m)} + \gamma_1^{(4m)} z + \gamma_2^{(4m)} z(z-\alpha_1) + \gamma_3^{(4m)} z(z-\alpha_1)^2 + \cdots \\ + \gamma_{4m-3}^{(4m)} z\omega_{2m-2}^2 + \gamma_{4m-2}^{(4m)} z(z-\alpha_{2m-1})\omega_{2m-2}^2 \\ + \gamma_{4m-1}^{(4m)} z\omega_{2m-1}^2 + \gamma_{4m}^{(4m)} z(z-\alpha_{2m})\omega_{2m-1}^2 + \cdots\end{gathered} \tag{5.24}$$

and

$$\begin{gathered}\gamma_0^{(4m+2)} + \gamma_1^{(4m+2)} z + \gamma_2^{(4m+2)} z(z-\alpha_1) + \gamma_3^{(4m+2)} z(z-\alpha_1)^2 + \cdots \\ + \gamma_{4m-3}^{(4m+2)} z\omega_{2m-2}^2 + \gamma_{4m-2}^{(4m+2)} z(z-\alpha_{2m-1})\omega_{2m-2}^2 \\ + \gamma_{4m-1}^{(4m+2)} z\omega_{2m-1}^2 + \gamma_{4m}^{(4m+2)} z(z-\alpha_{2m})\omega_{2m-1}^2 \\ + \gamma_{4m+1}^{(4m+2)} z\omega_{2m}^2 + \gamma_{4m+2}^{(4m+2)} z(z-\alpha_{2m+1})\omega_{2m}^2 + \cdots\end{gathered} \tag{5.25}$$

be the formal Newton series expansions of $\frac{P_{4m}}{Q_{4m}}$ and $\frac{P_{4m+2}}{Q_{4m+2}}$ with respect to the sequence (5.18). It follows from (5.14) and (5.16) that

$$\gamma_k^{(4m)} = \gamma_k^{(4m+2)}, \quad k = 0, 1, \ldots, \quad 4m-1, \tag{5.26}$$

$$\gamma_k^{(4m+2)} = \gamma_k^{(4m+4)}, \quad k = 0, 1, \ldots, \quad 4m+1. \tag{5.27}$$

The formulas (5.26)-(5.27) imply that $\gamma_k^{(2n)}$ does not depend on n for $k < 2n$. Hence if n is large enough, then the numbers $\gamma_k = \gamma_k^{(2n)}$ are well defined and independent of n:

$$\gamma_k = \gamma_k^{(2n)}, \quad 2n \geq k+1. \tag{5.28}$$

Thus the *EMP*-fraction (4.1)-(4.5) determines a unique formal Newton series (5.22) via the Newton series expansions of the even approximants. (Hence the functional M also determines such a formal Newton series through the corresponding *MP*-fraction and then the *EMP*-fraction obtained by extension.)

Similarly let

$$\begin{gathered}\delta_0^{(4m+1)} + \delta_1^{(4m+1)} z + \delta_2^{(4m+1)} z(z-\alpha_2) + \delta_3^{(4m+1)} z(z-\alpha_2)^2 \\ + \cdots + \delta_{4m-3}^{(4m+1)} z\omega_{2m-2}^2 + \delta_{4m-2}^{(4m+1)} z(z-\alpha_{2m})\omega_{2m-2}^2 \\ + \delta_{4m-1}^{(4m+1)} z(z-\alpha_{2m})^2\omega_{2m-2}^2 + \delta_{4m}^{(4m+1)} z(z-\alpha_{2m})\omega_{2m-2}\omega_{2m} + \cdots\end{gathered} \tag{5.29}$$

and

$$\begin{gathered}\delta_0^{(4m+3)} + \delta_1^{(4m+3)} z + \delta_2^{(4m+3)} z(z-\alpha_2) + \delta_3^{(4m+3)} z(z-\alpha_2)^2 \\ + \cdots + \delta_{4m-3}^{(4m+3)} z\omega_{2m-2}^2 + \delta_{4m-2}^{(4m+3)} z(z-\alpha_{2m})\omega_{2m-2}^2 \\ + \delta_{4m-1}^{(4m+3)} z(z-\alpha_{2m})^2\omega_{2m-2}^2 + \delta_{4m}^{(4m+3)} z(z-\alpha_{2m-1})(z-\alpha_{2m})^2\omega_{2m-2}^2 \\ + \delta_{4m+1}^{(4m+3)} z\omega_{2m}^2 + \delta_{4m+2}^{(4m+3)} z(z-\alpha_{2m+2})\omega_{2m}^2 + \cdots\end{gathered} \tag{5.30}$$

be the formal Newton series expansions of $\frac{P_{4m+1}}{Q_{4m+1}}$ and of $\frac{P_{4m+3}}{Q_{4m+3}}$ with respect to the sequence (5.19). It follows from (5.13) and (5.15) that

$$\delta_k^{(4m+1)} = \delta_k^{(4m+3)}, \quad k = 0, 1, \ldots \quad 4m-1, \tag{5.31}$$

$$\delta_k^{(4m+3)} = \delta_k^{(4m+5)}, \quad k = 0, 1, \ldots\ldots \quad 4m+1. \tag{5.32}$$

Thus we see as above that the *EMP*-fraction (4.1)-(4.5) determines a unique formal Newton series (5.23) via the Newton series expansions of the odd approximants:

$$\delta_k = \delta_k^{(2n+1)}, \quad 2n \geq k+1. \tag{5.33}$$

Clearly $\delta_0 = \gamma_0 = \kappa_0$. From (5.17) we see that there is a unique well defined value γ_∞ given by

$$\gamma_\infty = \lim_{z\to\infty} \frac{P_n(z)}{Q_n(z)} = \lambda_1 + \kappa_0, \quad n = 1, 2, \ldots \tag{5.34}$$

Let Γ and Δ denote the formal Newton series (5.22) and (5.23) determined by (5.28) and (5.33), and let P_n, Q_n, P_n/Q_n in addition to having their basic meaning also denote Newton series expansions of these functions with respect to the sequences (5.18) and (5.19).

It follows from the preceeding arguments that

$$\begin{aligned} \frac{P_{4m}}{Q_{4m}} &= \gamma_0 + \gamma_1 z + \gamma_2 z(z-\alpha_1) + \cdots \\ &+\gamma_{4m-1} z\omega_{2m-1}^2 + \gamma_{4m}^{(4m)} z(z-\alpha_{2m})\omega_{2m-1}^2 + \cdots, \quad m = 1, 2, \ldots \end{aligned} \tag{5.35}$$

$$\begin{aligned} \frac{P_{4m+2}}{Q_{4m+2}} &= \gamma_0 + \gamma_1 z + \gamma_2 z(z-\alpha_1) + \cdots \\ &+\gamma_{4m+1} z\omega_{2m}^2 + \gamma_{4m+2}^{(4m+2)} z(z-\alpha_{2m+1})\omega_{2m}^2 + \cdots, \quad m = 0, 1, 2, \ldots \end{aligned} \tag{5.36}$$

$$\frac{P_{4m+1}}{Q_{4m+1}} = \gamma_0 + \delta_1 z + \delta_2 z(z-\alpha_2) + \cdots$$
$$+\delta_{4m-1} z(z-\alpha_{2m})^2 \omega_{2m-2}^2 + \delta_{4m}^{(4m+1)}(z-\alpha_{2m-1})(z-\alpha_{2m})^2\omega_{2m-2}^2 + \cdots, \quad m = 1,2,\ldots \tag{5.37}$$

$$\frac{P_{4m+3}}{Q_{4m+3}} = \gamma_0 + \delta_1 z + \delta_2 z(z-\alpha_2) + \cdots$$
$$+\delta_{4m+1} z\omega_{2m}^2 + \delta_{4m+2}^{(4m+3)} z(z-\alpha_{2m+2})\omega_{2m}^2 + \cdots, \quad m = 0,1,2,\ldots \tag{5.38}$$

$$\frac{P_n}{Q_n} - \gamma_\infty = O\left(\frac{1}{z}\right) \quad \text{as} \quad z \to \infty, \quad n = 1,2,\ldots \tag{5.39}$$

Furthermore (with x_n and y_n constants)

$$P_{4m} - \Gamma Q_{4m} = x_{4m} z\omega_{2m-1}\omega_{2m} + \cdots, \quad m = 1,2,\ldots \tag{5.40}$$

$$P_{4m+2} - \Gamma Q_{4m+2} = x_{4m+2} z\omega_{2m}\omega_{2m+1} + \cdots, \quad m = 0,1,2,\ldots \tag{5.41}$$

$$P_{4m+1} - \Delta Q_{4m+1} = x_{4m+1} z(z-\alpha_{2m})\omega_{2m-2}\omega_{2m} + \cdots, \quad m = 1,2,\ldots \tag{5.42}$$

$$P_{4m+3} - \Delta Q_{4m+3} = x_{4m+3} z(z-\alpha_{2m+2})\omega_{2m}^2 + \cdots, \quad m = 0,1,2,\ldots \tag{5.43}$$

By (5.9) and (5.42) it follows that $P_{4m} - \Delta Q_{4m}$ is of the form (5.23) with $\delta_k = 0$ for $k = 0,1,\ldots,\ 4m-1$. Hence

$$P_{4m} - \Delta Q_{4m} = y_{4m} z(z-\alpha_{2m})\omega_{2m-2}^2 + \cdots, \quad m = 1,2,\ldots \tag{5.44}$$

Similarly we find by using (5.11) and (5.43), (5.10) and (5.41), and (5.12) and (5.40) that

$$P_{4m+2} - \Delta Q_{4m+2} = y_{4m+2} z\omega_{2m}^2 + \cdots, \quad m = 0, 1, 2, \ldots \tag{5.45}$$

$$P_{4m+1} - \Gamma Q_{4m+1} = y_{4m+1} z\omega_{2m-2}\omega_{2m-1} + \cdots, \quad m = 1, 2, \ldots \tag{5.46}$$

$$P_{4m+3} - \Gamma Q_{4m+3} = y_{4m+3} z\omega_{2m}^2 + \cdots, \quad m = 0, 1, 2, \ldots \tag{5.47}$$

Since P_{4m} and Q_{4m} are polynomials of degree $2m$, the quotient $\frac{P_{4m}}{Q_{4m}}$ has $4m+1$ free parameters. It follows from (5.35) (or (5.40)) and (5.39) that $\frac{P_{4m}}{Q_{4m}}$ satisfies $4m+1$ interpolation conditions (at $\infty, 0, \alpha_1, \alpha_1, \ldots \alpha_{2m-1}, \alpha_{2m-1}, \alpha_{2m}$) determining the coefficients $\gamma_\infty, \gamma_0, \cdots, \gamma_{4m-1}$. We conclude that $\frac{P_{4m}}{Q_{4m}}$ has maximal degree of interpolation to γ_∞ at ∞ and the series Γ. Thus $\frac{P_{4m}}{Q_{4m}}$ is the $(2m, 2m)$ multi-point Padé appoximant to γ_∞ and Γ. Similarly we find that $\frac{P_{4m+2}}{Q_{4m+2}}$ is the $(2m+1, 2m+1)$ multi-point Padé approximant to γ_∞ and Γ, that $\frac{P_{4m+1}}{Q_{4m+1}}$ is the $(2m, 2m)$ multi-point Padé approximant to γ_∞ and Δ, and that $\frac{P_{4m+3}}{Q_{4m+3}}$ is the $(2m+1, 2m+1)$ multi-point Padé approximant to γ_∞ and Δ. — Multi-point Padé approximants to a Newton series, i.e. to interpolation data given with respect to a fixed sequence of interpolation points, are also called Newton-Padé approximants.

We also note that $\frac{P_{4m}}{Q_{4m}}$ and $\frac{P_{4m+2}}{Q_{4m+2}}$ interpolate γ_∞ and Δ with order 2 and 1 less than maximal, and that $\frac{P_{4m+1}}{Q_{4m+1}}$ and $\frac{P_{4m+3}}{Q_{4m+3}}$ interpolate γ_∞ and Γ with order 2 and 1 less than maximal.

We formulate the main interpolation result in a theorem.

Theorem 3. *The EMP-fraction (4.1)-(4.8) determines a Newton series Γ with respect to the sequence (5.18) and a Newton series Δ with respect to the sequence (5.19), and a*

value γ_∞ *at* ∞. *The even approximants* $\frac{P_{2n}}{Q_{2n}}$ *of the EMP-fraction are* (n,n) *multi-point Padé approximants to* γ_∞ *and* Γ, *the odd approximants* $\frac{P_{2n+1}}{Q_{2n+1}}$ *of the EMP-fraction are* (n,n) *multi-point Padé approximants to* γ_∞ *and* Δ. *The even approximants interpolate* γ_∞ *and* Δ *with order less than maximal, and the odd approximants interpolate* γ_∞ *and* Γ *with order less than maximal.*

Remark. The interpolation results above are proved with reference to the *EMP*-fraction (4.1)-(4.8). When $\alpha_{2m} \neq \alpha_{2m-1}$ for all m, the *MP*-fraction (2.15)-(2.16), (3.13)-(3.14) is the even contraction of an *EMP*-fraction, and thus interpolation to γ_∞ and Γ of approximants of this *MP*-fraction are also proved. Even if $\alpha_{2m} \neq \alpha_{2m-1}$ is not satisfied for all m, the formulas (5.14) and (5.16) can still be proved directly from the recurrence relation of the *MP*-fraction. Hence interpolation to γ_∞ and Γ of the approximants of the *MP*-fraction can be proved in the general case (as long as the sequence $\{\Phi_n\}$ of orthogonal functions is regular). — Similar considerations concerning interpolation properties connected with the odd approximants of the *EMP*-fraction and γ_∞, Δ can be made when the nature of these approximants has been studied in Section 6.

6 Contraction of EMP-fractions

The *EMP*-fraction discussed in Section 3 has by the very construction the *MP*-fraction (2.15)-(2.16), (3.13)-(3.14) as its even contraction. And it follows from (3.25)-(3.26) and (3.33) that

$$F_1 = \lambda_1 \mu_2, \quad G_1 = \mu_2 \tag{6.1}$$

$$F_n = -\lambda_{2n-1}\frac{\mu_{2n}}{\mu_{2n-2}}, \quad n = 2,3,\ldots \tag{6.2}$$

$$G_n = \frac{\mu_{2n}}{\mu_{2n-2}}, \quad n = 2,3,\ldots . \tag{6.3}$$

Conversely, assume that

$$\mu_{2n} \neq 0, \quad n = 1,2,\ldots. \tag{6.4}$$

in the *EMP*-fraction (4.1)-(4.8). Then this *EMP*-fraction has an even contraction

$$\kappa_0 + K_{n=1}^{\infty}\frac{\theta_n}{\kappa_n} \tag{6.5}$$

determined by the equations (3.15)-(3.18). Substituting from (4.2)-(4.5) and taking into account (4.7)-(4.8) we get

$$\theta_n = F_n\frac{z-\alpha_{n-2}}{z-\alpha_n}, \quad n = 1,2,\ldots \tag{6.6}$$

$$\kappa_{2m} = 1 + G_{2m}\frac{z-\alpha_{2m-1}}{z-\alpha_{2m}}, \quad m = 1,2,\ldots \tag{6.7}$$

$$\kappa_{2m+1} = \frac{z-\alpha_{2m}}{z-\alpha_{2m+1}} + G_{2m+1}\frac{z-\alpha_{2m-1}}{z-\alpha_{2m+1}}, \quad m = 0,1,2,\ldots \tag{6.8}$$

where F_n and G_n are defined by(6.1)-(6.3). Thus the even contraction of the *EMP*-fraction (which exists when (6.4) is satisfied) is an *MP*-fraction of the form (2.15)-(2.16), (3.13)-(3.14).

We note that if the *EMP*-fraction (4.1)–(4.8) is normal then the canonical denominators $K_{2m} = B_{4m}$, $K_{2m+1} = B_{4m+2}$, are completely regular in the sense that the numerator

polynomials p_{2m}, r_{2m+1} have no common factors with the corresponding denominator polynomials w_{2m}, w_{2m+1}.

We shall now investigate the odd contraction of the *EMP*-fraction (4.1)-(4.8). This odd contraction is by definition (when it exists) the continued fraction

$$\kappa_0' + K_{n=1}^{\infty} \frac{\xi_n}{\eta_n} \tag{6.9}$$

whose nth canonical numerator and denominator are the $(2n+1)$th canonical numerator and denominator of the *EMP*-fraction. Thus if C_n and D_n denote the canonical numerator and denominator of (6.9) we have

$$C_n = A_{2n+1}, \quad D_n = B_{2n+1}, \quad n = 0, 1, 2, \ldots \tag{6.10}$$

We shall in the rest of this section assume that

$$\mu_{2n+1} \neq 0, \quad n = 0, 1, 2, \ldots \tag{6.11}$$

From general formulas for the odd contraction of a continued fraction (see e.g. [3, 16, 22, 26]) we find that κ_0', $\{\xi_n\}$, $\{\eta_n\}$ are given by

$$\kappa_0' = \frac{A_1}{B_1} = \kappa_0 + \lambda_1 \tag{6.12}$$

$$\xi_n = -a_{2n-1} a_{2n} \frac{b_{2n+1}}{b_{2n-1}}, \quad n = 1, 2, \ldots \tag{6.13}$$

$$\eta_n = a_{2n+1} + b_{2n} b_{2n+1} + a_{2n} \frac{b_{2n+1}}{b_{2n-1}}, \quad n = 1, 2, \ldots . \tag{6.14}$$

Substituting from (4.1)-(4.5), taking into account (4.7)-(4.8), we get

$$\xi_{2m} = -\lambda_{4m-1} \frac{\mu_{4m+1}}{\mu_{4m-1}} \cdot \frac{z - \alpha_{2m-1}}{z - \alpha_{2m+1}}, \quad m = 1, 2, \ldots \tag{6.15}$$

$$\eta_{2m} = \frac{z-\alpha_{2m-1}}{z-\alpha_{2m+1}} + \frac{\mu_{4m+1}}{\mu_{4m-1}} \cdot \frac{z-\alpha_{2m}}{z-\alpha_{2m+1}}, \quad m = 1, 2, \ldots \tag{6.16}$$

$$\xi_{2m+1} = -\lambda_{4m+1} \frac{\mu_{4m+3}}{\mu_{4m+1}} \cdot \frac{z-\alpha_{2m}}{z-\alpha_{2m+2}}, \quad m = 0, 1, 2, \ldots \tag{6.17}$$

$$\eta_{2m+1} = \frac{z-\alpha_{2m+1}}{z-\alpha_{2m+2}} + \frac{\mu_{4m+3}}{\mu_{4m+1}} \cdot \frac{(z-\alpha_{2m})(z-\alpha_{2m+1})}{(z-\alpha_{2m-1})(z-\alpha_{2m+2})}, \quad m = 0, 1, 2, \ldots \tag{6.18}$$

By an equivalence transformation we obtain a continued fraction

$$\kappa_0' + K_{n=1}^{\infty} \frac{\rho_n}{\sigma_n} \tag{6.19}$$

whose canonical numerators Π_n and denominators Ω_n are given by

$$\Pi_{2m} = \frac{z-\alpha_{2m+1}}{z-\alpha_{2m-1}} C_{2m} = \frac{z-\alpha_{2m+1}}{z-\alpha_{2m-1}} A_{4m+1}, \quad m = 0, 1, 2, \ldots \tag{6.20}$$

$$\Omega_{2m} = \frac{z-\alpha_{2m+1}}{z-\alpha_{2m-1}} D_{2m} = \frac{z-\alpha_{2m+1}}{z-\alpha_{2m-1}} B_{4m+1}, \quad m = 1, 2, \ldots \tag{6.21}$$

$$\Pi_{2m+1} = C_{2m+1} = A_{4m+3}, \quad m = 0, 1, 2, \ldots \tag{6.22}$$

$$\Omega_{2m+1} = D_{2m+1} = B_{4m+3}, \quad m = 0, 1, 2, \ldots \tag{6.23}$$

We observe that Π_{2m}, Ω_{2m} belong to Span $\left\{1, \ldots, \frac{1}{\omega_{2m-2}}, \frac{1}{(z-\alpha_{2m})\omega_{2m-2}}, \frac{1}{\omega_{2m}}\right\} = \mathcal{L}_{2m}$ and Π_{2m+1}, Ω_{2m+1} belong to Span $\left\{1, \ldots, \frac{1}{\omega_{2m-2}}, \frac{1}{(z-\alpha_{2m})\omega_{2m-2}}, \frac{1}{\omega_{2m}}, \frac{1}{(z-\alpha_{2m+1})\omega_{2m}}\right\}$. We also note that the continued fractions (6.9) and (6.19) are identical if $\alpha_{2m-1} = \alpha_{2m+1}$ for all m.

By substituting from (6.20)-(6.23) in the recurrence relation

$$\begin{bmatrix} C_n \\ D_n \end{bmatrix} = \eta_n \begin{bmatrix} C_{n-1} \\ D_{n-1} \end{bmatrix} + \xi_n \begin{bmatrix} C_{n-2} \\ D_{n-2} \end{bmatrix}, \quad n = 1, 2, \ldots \tag{6.24}$$

or by using general formulas expressing relationships between equivalent continued fractions (see e.g. [3, 16, 22, 26]) we find that

$$\rho_{2m} = U_{2m} \frac{z - \alpha_{2m-3}}{z - \alpha_{2m-1}}, \quad m = 1, 2, \ldots \tag{6.25}$$

$$\sigma_{2m} = 1 + V_{2m} \frac{z - \alpha_{2m}}{z - \alpha_{2m-1}}, \quad m = 1, 2, \ldots \tag{6.26}$$

$$\rho_{2m+1} = U_{2m+1} \frac{z - \alpha_{2m}}{z - \alpha_{2m+2}}, \quad m = 0, 1, 2, \ldots \tag{6.27}$$

$$\sigma_{2m+1} = \frac{z - \alpha_{2m-1}}{z - \alpha_{2m+2}} + V_{2m+1} \frac{z - \alpha_{2m}}{z - \alpha_{2m+2}}, \quad m = 0, 1, 2, \ldots . \tag{6.28}$$

where

$$U_{2m} = -\lambda_{4m-1} \frac{\mu_{4m+1}}{\mu_{4m-1}}, \quad m = 1, 2, \ldots \tag{6.29}$$

$$V_{2m} = \frac{\mu_{4m+1}}{\mu_{4m-1}}, \quad m = 1, 2, \ldots \tag{6.30}$$

$$U_{2m+1} = -\lambda_{4m+1} \frac{\mu_{4m+3}}{\mu_{4m+1}}, \quad m = 0, 1, 2, \ldots \tag{6.31}$$

$$V_{2m+1} = \frac{\mu_{4m+3}}{\mu_{4m+1}}, \quad m = 0, 1, 2, \ldots \tag{6.32}$$

This shows that the continued fraction (6.19) is a continued fraction of the special type (2.15)-(2.16), (3.13)-(3.14) determined by the basic sequence $\{\beta_n : n = 1, 2, \dots\}$, where

$$\beta_{2m} = \alpha_{2m-1}, \quad m = 1, 2, \dots, \quad \beta_{2m+1} = \alpha_{2m+2}, \quad m = 0, 1, 2, \dots . \tag{6.33}$$

Thus if (3.2) is satisfied, the canonical denominators Ω_n are orthogonal rational functions and the canonical numerators are associated functions of the second kind corresponding to the basic sequence $\{\alpha_2, \alpha_1, \alpha_4, \alpha_3, \dots\}$. Because of the one-to-one correspondence between (certain) functionals and *EMP*-fractions satisfying (6.4) and (6.11), it follows that when these conditions are satisfied, the canonical denominators of the even contraction of the *EMP*-fraction and the canonical denominators of the odd contraction of the *EMP*-fraction are orthogonal functions with respect to the same functional M (but with different orderings of the basic points).

We note that if the *EMP*-fraction (4.1)–(4.8) is normal, then the canonical denominators $\Omega_{2m} = \dfrac{z - \alpha_{2m+1}}{z - \alpha_{2m-1}} B_{4m+1}$, $\Omega_{2m+1} = B_{4m+3}$, are completely regular in the sense that the numerator polynomials q_{2m}, s_{2m+1} have no common factors with the corresponding denominator polynomials w_{2m}, w_{2m+1}.

We formulate the main result on the odd contraction of an *EMP*-fraction as a theorem.

Theorem 4. *Let the EMP-fraction* $\kappa_0 + K_{n=1}^{\infty} \dfrac{a_n}{b_n}$ *(associated with a basic sequence* $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \dots\}$*) have an odd contraction* $\kappa_0' + K_{n=1}^{\infty} \dfrac{\xi_n}{\eta_n}$. *Then this odd contraction is equivalent to an MP-fraction* $\kappa_0' + K_{n=1}^{\infty} \dfrac{\rho_n}{\sigma_n}$ *associated with the basic sequence* $\{\alpha_2, \alpha_1, \alpha_4, \alpha_3, \dots\}$. *When the EMP-fraction has both an even and an odd contraction, the canonical denominators of both MP-fractions are orthogonal functions with respect to the same functional.*

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