An extended relation between orthogonal rational functions on the unit circle and the interval $[-1, 1]$.*

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Abstract

The relation between orthogonal polynomials on the unit circle and the interval has been given by Szegő. In this paper we generalise this relation to orthogonal rational functions with prescribed *complex* poles. As an application we obtain explicit expressions for the orthogonal rational functions with respect to the weight functions $1 \pm \cos\theta$ on the unit circle, respectively $\{(1+x)/(1-x)\}^{\pm 1/2}$ on the interval and use one of them to illustrate the relation.

1 Introduction

The relation $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ gives a relation between a point $x = \cos\theta \in I = [-1, 1]$ and a point $z = e^{i\theta}$ on the unit circle $\mathbb{T}$. In general, $x = J(z)$, $J(z) = \frac{1}{2}(z + z^{-1})$ is called the Joukowski transform. Because

$$\int_I f(x)u(x)dx = \frac{1}{2}\int_{-\pi}^{\pi} f(\cos\theta)w(\theta)d\theta$$

with $w(\theta) = u(\cos(\theta))|\sin(\theta)|$, we may expect a relation between orthogonal polynomials (OP) on the unit circle with respect to the weight $w(\theta)$ and OP on the interval I with weight $u(x)$. Szegő [6, p. 294] has given such a relation. In fact, since $f(x) = f(\cos\theta) = f(\frac{1}{2}(z+z^{-1}))$ transforms a polynomial in x into a trigonometric polynomial in z, both the OP $\phi_n(z)$ on the unit circle and their reciprocals $z^n\phi_n(1/z)$ are needed to describe an OP on I. Conversely, also two OP sequences on I are necessary to describe the OP on the unit circle: one is orthogonal with respect to $u(x)$ and the other with respect to $(1-x^2)u(x)$. For example when ϕ_n is orthonormal on $\mathbb{T}$ with respect to w and p_n and q_n are orthonormal on I with respect to $u(x)$ and $(1-x^2)u(x)$ respectively, then Szegő proved relations like

$$\begin{aligned} p_n(x) &= C_n[z^{-n}\phi_{2n}(z) + z^n\phi_{2n}(z^{-1})] \\ q_n(x) &= D_n\frac{z^{-n}\phi_{2n+1}(z) - z^n\phi_{2n+1}(z^{-1})}{z - x^{-1}} \end{aligned}$$

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with C_n and D_n normalizing constants.

Orthogonal polynomials on the unit circle have been generalised to orthogonal rational functions (ORF) with prescribed poles outside the closed unit disk [3]. The OP on the unit circle are a special case that is obtained by placing all the poles at infinity. Using an appropriate branch for the inverse Joukowski transform, ORF on I with poles outside I are linked with ORF on $\mathbb{T}$ with poles outside the closed unit disk. Most of the classical results of orthogonal polynomials have been extended like the recurrence relations, the Christoffel-Darboux relation, properties of their zeros etc.

The previously mentioned Szegő relation between OPs on the unit circle and the interval has been generalised to ORFs in [9, Thm. 4.1]. But the relation was only given for *real* poles outside the interval $[-1, 1]$. This paper will extend this relation to ORFs with arbitrary *complex* poles outside the interval.

What is true for OPs should obviously also hold for ORFS. In order to obtain a real x satisfying $x = \frac{1}{2}(z + z^{-1})$ one obviously needs positive as well as negative powers of z. That explains why we need polynomials in z but also polynomials in x^{-1} in our construction. The involvement of the reciprocal polynomials on the unit circle can also be formulated in terms of (orthogonal) Laurent polynomials on the unit circle. Such Laurent polynomials have been studied for example by Thron [7, Thm. 1]) and subsequently by many others. While for polynomials it is clear what the ordering of the polynomials should be: they are ordered with increasing degree, the ordering of (orthogonal) Laurent polynomials is not unique. Increasing their degree can be on the side of the positive powers or on the side of the negative powers of z. The most obvious is to choose for an alternating extension, but even then, there are two possible choices, depending on what side one starts the extension. Considering Laurent polynomials as a special case of rational functions having only poles at 0 and ∞, this ordering corresponds to the order in which the poles are introduced. Thus Gram Schmidt orthogonalization can be applied subsequently to $z, z^{-1}, z^2, z^{-2}, \ldots$ or to $z^{-1}, z, z^{-2}, z^2, \ldots$, which introduce the poles in the order $\infty, 0, \infty, 0, \ldots$ or in the order $0, \infty, 0, \infty, \ldots$, giving rise to two different orthogonal Laurent polynomials. There orthogonal Laurent polynomials are in fact the ones that appear in the right hand sides of the above Szegő relations. Both of them can be expressed in terms of the OPs and their reciprocals.

For ORF, which have more general poles than Laurent polynomials, this idea is similar and one can consider again two different ORF sequences based on introducing poles in the order $\alpha_1, 1/\overline{\alpha}_1, \alpha_2, 1/\overline{\alpha}_2, \ldots$ or $1/\overline{\alpha}_1, \alpha_1, 1/\overline{\alpha}_2, \alpha_2, \ldots$ as was done in [2, p. 168–171]. Using a generalisation of the reciprocals of a polynomial, both ORF sequences for the different nestings can be obtained from an ORF sequence where the poles are introduced like $\alpha_1, \alpha_1, \alpha_2, \alpha_2, \ldots$. To guarantee real ORF on the interval, the α_k had to be real. And this was the condition adopted in [9]. However, if we allow complex poles, we can remove this restriction but we should work with α_k and $1/\alpha_k$ instead of α_k and $1/\overline{\alpha}_k$ and adapt the definition of the reciprocal accordingly (see [4]).

The outline of the paper is as follows. In the next section we give the necessary theoretical preliminaries. In Section 3 we are concerned with the construction of two distinct orthogonal bases for the space ofrational functions with poles among $\{\beta_k, 1/\beta_k\}_{k=1}^{\infty}$ (similar as in [2, p. 168–171] for $\mathcal{R}$). Section 4 then contains the main result. Next, in Section 5 we derive explicit expressions for the ORF on the unit circle with respect to the weight functions $w(\theta) = 1 \pm \cos\theta$. We conclude the article with an illustration of the relation for the weight functions $u(x) = \{(1-x)/(1+x)\}^{1/2}$ on the interval, respectively $w(\theta) = 1 - \cos\theta$ on the unit circle.

2 Preliminaries

The field of complex numbers will be denoted by $\mathbb{C}$ and the Riemann sphere by $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let $a \in \mathbb{C}$, then $\Re(a)$ refers to the real part of a. The unit circle and the open unit disc will be represented by respectively $\mathbb{T} = \{z : |z| = 1\}$ and $\mathbb{D} = \{z : |z| < 1\}$. We will study ORF on the unit circle $\mathbb{T}$ and on the real interval $I = [-1, 1]$. The complement of this interval with respect to a set X will be given by X^I, e.g. $\overline{\mathbb{C}}^I = \overline{\mathbb{C}} \setminus I$.

With $\mathcal{P}_n$ we denote the space of polynomials of degree less than or equal to n while $\mathcal{P}$ represents the space of all polynomials. For any complex function f, we define the involution operation or substar conjugate by $f_*(z) = \overline{f(1/\overline{z})}$. Furthermore, we define f^c by $f^c(z) = \overline{f(\overline{z})}$, and consequently f_*^c by $f_*^c(z) = f(1/z)$. Let ν be a positive measure on an infinite set Ω (e.g. $\Omega = \mathbb{T}$). Then we can define a positive definite inner product on a space Ψ (e.g. $\Psi = \mathcal{P}$) by

$$\langle f, g \rangle_\Omega = \int_\Omega f(t)\overline{g(t)} d\nu(t). \tag{2.1}$$

Note that for $t \in \mathbb{T}$ we have that $\overline{g(t)} = g_*(t)$, while for $t \in I$ we have that $\overline{g(t)} = g^c(t)$. In the remainder of this paper we shall omit the subscript $_\Omega$ if $\Omega = \mathbb{T}$, and hence $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_\mathbb{T}$.

Suppose a sequence of poles $\mathcal{A} = \{\alpha_0 = \infty, \alpha_1, \alpha_2, \ldots\} \subset \overline{\mathbb{C}}^I$ is given and define the factors

$$Z_k(z) = \frac{z}{1 - z/\alpha_k} \quad k = 1, 2, \ldots \tag{2.2}$$

and the basis functions

$$b_0 = 1, \quad b_k(z) = b_{k-1}(z) Z_k(z) \quad k = 1, 2, \ldots \quad . \tag{2.3}$$

These basis functions generate the spaces of rational functions with poles in $\mathcal{A}$ defined by $\mathcal{K}_n = \text{span}\{b_0, \ldots, b_n\}$ and $\mathcal{K} = \cup_{n=0}^\infty \mathcal{K}_n$. If we orthogonalise these basis functions for $\Psi = \mathcal{K}_n$ with respect to the inner product on the set $\Omega = I$, then we obtain a basis of ORF $\{\varphi_0, \ldots, \varphi_n\}$ on the interval I. Note that $\mathcal{K}_n$ and $\mathcal{K}$ are rational generalisations of $\mathcal{P}_n$ and $\mathcal{P}$. Indeed, if all $\alpha_k = \infty$, the expression in (2.2) becomes $Z_k(z) = z$ and the expression in (2.3) becomes $b_k(z) = z^k$.

Another sequence of basis functions will be used for the unit circle. Given a sequence of complex numbers $\mathcal{B} = \{\beta_0 = 0, \beta_1, \beta_2, \ldots\} \subset \mathbb{D}$, we define the Blaschke factors for $\mathcal{B}$

$$\zeta_k(z) = \frac{z - \beta_k}{1 - \overline{\beta}_k z}, \quad k = 1, 2, \ldots, \tag{2.4}$$

and the corresponding Blaschke products for $\mathcal{B}$

$$B_0 = 1, \quad B_k(z) = B_{k-1}(z)\zeta_k(z) \quad k = 1, 2, \ldots \quad . \tag{2.5}$$

These Blaschke products now generate the spaces of rational functions $\mathcal{L}_n = \text{span}\{B_0, \ldots, B_n\}$ and $\mathcal{L} = \cup_{n=0}^\infty \mathcal{L}_n$. If we orthogonalise these basis functions for $\Psi = \mathcal{L}_n$ with respect to the inner product on the set $\Omega = \mathbb{T}$, then we obtain a basis of ORF $\{\phi_0, \ldots, \phi_n\}$ on the unit circle. Note that in this case for all $\beta_k = 0$, again $\mathcal{L}_n = \mathcal{P}_n$ and $\mathcal{L} = \mathcal{P}$.

With the definition of the substar conjugate and the superscript ^{c} we can define $\mathcal{L}_{n*} = \{f : f_* \in \mathcal{L}_n\}$, $\mathcal{L}_n^c = \{f : f^c \in \mathcal{L}_n\}$ and $\mathcal{L}_{n*}^c = \{f : f_*^c \in \mathcal{L}_n\}$. Note that, if f has a pole in $z = p$, then f_* (respectively f^c and f_*^c) has a pole in $z = 1/\overline{p}$ (respectively $z = \overline{p}$ and $z = 1/p$).

Furthermore, for the unit circle we define the superstar conjugate respectively n-superstar conjugate of a complex function $f_k \in (\mathcal{L}_k \backslash \mathcal{L}_{k-1}) \subseteq \mathcal{L}_n$, $k \leq n$, by $f_k^*(z) = B_k(z) f_{k*}(z)$ respectively $f_k^{*,n}(z) = B_n(z) f_{k*}(z)$. Note that $\mathcal{L}_n^{*,n} = \{f^{*,n} : f \in \mathcal{L}_n\} = \mathcal{L}_n$.

We denote the Joukowski Transformation $x = \frac{1}{2}(z + z^{-1})$ by $x = J(z)$, mapping the open unit disc $\mathbb{D}$ onto the cut Riemann sphere $\overline{\mathbb{C}}^I$ and the unit circle $\mathbb{T}$ onto the interval I. The inverse mapping is denoted by $z = J^{-1}(x)$ and is chosen so that $z \in \mathbb{D}$ if $x \in \overline{\mathbb{C}}^I$. With the sequence $\mathcal{A} = \{\alpha_0, \alpha_1, \alpha_2, \ldots\} \subset \overline{\mathbb{C}}^I$ we associate a sequence $\mathcal{B} = \{\beta_0, \beta_1, \beta_2, \ldots\} \subset \mathbb{D}$ so that $\beta_k = J^{-1}(\alpha_k)$.

It is proven in [5, Lem. 3.1] that $Z_k(x)$ can be written as a linear combination of $\zeta_k^c(z), \zeta_0$ and $\zeta_{k*}(z)$ iff

$$\alpha_k = J(\beta_k), \tag{2.6}$$

and thus that, if the condition on the placement of poles in (2.6) is satisfied for every point in $\mathcal{A}$ and $\mathcal{B}$, then $b_k(x)$, $k = 0, 1, 2, \ldots$, with $x = J(z), z \in \mathbb{T}$, is a linear combination of

$$B_k^c(z), B_{k-1}^c(z), \ldots, B_1^c(z), B_0^c = B_{0*}, B_{1*}(z), \ldots, B_{(k-1)*}(z), B_{k*}(z). \tag{2.7}$$

In [2], the space $\mathcal{R} = \mathcal{L} + \mathcal{L}_*$ was considered, having poles among $\{\beta_k, 1/\overline{\beta}_k\}_{k=1}^{\infty}$. Here, however, we consider the space $\hat{\mathcal{R}} = \mathcal{L}^c + \mathcal{L}_* = \mathcal{L}^c \cdot \mathcal{L}_*$ with poles among $\{\beta_k, 1/\beta_k\}_{k=1}^{\infty}$. And for m and n nonnegative integers, we define $\hat{\mathcal{R}}_{m,n} = \mathcal{L}_n^c + \mathcal{L}_{m*}$. In the next section, we will be concerned with the construction of two distinct orthogonal bases $\hat{\sigma}_n$ and $\hat{\tau}_n$ for $\hat{\mathcal{R}}$. The steps in the construction will be almost the same as in the case of $\mathcal{R}$ (see [2, p. 168–171]). Note that, when all β_k are real, then $\hat{\mathcal{R}} = \mathcal{R}$ and $\hat{\mathcal{R}}_{m,n} = \mathcal{R}_{m,n} = \mathcal{L}_n + \mathcal{L}_{m*}$. The main difference, however, between $\mathcal{R}$ and $\hat{\mathcal{R}}$ lies in the fact that $\mathcal{R}_* = \mathcal{R}$, while $\hat{\mathcal{R}}_* = \hat{\mathcal{R}}^c \neq \hat{\mathcal{R}}$, unless all β_k are real.

Then, the expressions for $\hat{\sigma}_n$ and $\hat{\tau}_n$ will be used to derive a relation between ORFs on the interval I with respect to a weight function $u(x)$ and ORFs on the unit circle with respect to the weight function $w(\theta) = u(\cos\theta)|\sin\theta|$. Note that $w(\theta)$ is symmetric on $[-\pi, \pi]$ ($w(\theta) = w(-\theta)$). Thus, setting $z = e^{\mathbf{i}\theta}$ and $x = \frac{1}{2}(z + z^{-1}) = \cos\theta$ we have that (see [1, Eq. (3.2)])

$$\int_{-1}^{1} f(x)u(x)dx = \frac{1}{2}\int_{-\pi}^{\pi} f(\cos\theta)w(\theta)d\theta.$$

Hence, the following orthogonality conditions

$$\int_{-1}^{1} \varphi_n(x)\overline{b_k(x)}u(x)dx = 0, \quad k = 0, 1, \ldots, n-1$$

are equivalent with the conditions

$$\int_{-\pi}^{\pi} \varphi_n(\cos\theta)\overline{b_k(\frac{z+z^{-1}}{2})}w(\theta)d\theta = 0, \quad k = 0, 1, \ldots, n-1, \quad z = e^{\mathbf{i}\theta}.$$

Because $b_k(x)$ is a linear combination of (2.7) for $k = 0, 1, 2, \ldots$, the previous conditions hold if the following conditions hold:

$$\int_{-\pi}^{\pi} \varphi_n(\cos\theta)B_{k*}^c(z)w(\theta)d\theta = 0 \quad k = 0, 1, \ldots, n-1, \quad z = e^{\mathbf{i}\theta} \tag{2.8}$$

and

$$\int_{-\pi}^{\pi} \varphi_n(\cos\theta)B_k(z)w(\theta)d\theta = 0 \quad k = 0, 1, \ldots, n-1, \quad z = e^{\mathbf{i}\theta}. \tag{2.9}$$

3 Two distinct ORF sequences for $\hat{\mathcal{R}}$

The ORFs $\hat{\sigma}_n$ and $\hat{\tau}_n$ can be obtained from analytic (in $\mathbb{D}\cup\mathbb{T}$) ORFs associated with auxiliary spaces $\tilde{\mathcal{L}}_n$ that are of the same type as the spaces $\mathcal{L}_n$ but now with respect to an auxiliary set of points $\tilde{\mathcal{B}}$.

We first discuss the sequence of ORFs $\{\hat{\sigma}_n\}_n^\infty$ which corresponds to the nesting

$$\hat{\mathcal{R}}_{0,0} \subset \hat{\mathcal{R}}_{0,1} \subset \hat{\mathcal{R}}_{1,1} \subset \hat{\mathcal{R}}_{1,2} \subset \hat{\mathcal{R}}_{2,2} \subset \dots.$$

Using the notation

$$\hat{\mathcal{R}}_{2n} = \hat{\mathcal{R}}_{n,n} \quad \text{and} \quad \hat{\mathcal{R}}_{2n+1} = \hat{\mathcal{R}}_{n,n+1} \quad \text{for} \quad n = 0, 1, 2, \dots \tag{3.1}$$

we should have $\hat{\sigma}_n \in \hat{\mathcal{R}}_n - \hat{\mathcal{R}}_{n-1}$ and $\hat{\sigma}_n \perp \hat{\mathcal{R}}_{n-1}$.

Let us now consider the sequence of points $\tilde{\mathcal{B}} = \{\tilde{\beta}_0 = 0, \tilde{\beta}_1, \tilde{\beta}_2, \dots\}$ with

$$\tilde{\beta}_{2n} = \beta_n \quad \text{and} \quad \tilde{\beta}_{2n+1} = \overline{\beta}_{n+1} \quad \text{for} \quad n = 0, 1, \dots. \tag{3.2}$$

Let $\tilde{\zeta}_n(z)$ be the Blaschke factors for $\tilde{\mathcal{B}}$ and $\tilde{B}_n(z)$ the corresponding Blaschke products (see (2.4) and (2.5)). Set $\tilde{\mathcal{L}}_n = \operatorname{span}\{1, \tilde{B}_1, \dots, \tilde{B}_n\}$ and $\tilde{\mathcal{L}} = \cup_{n=0}^\infty \tilde{\mathcal{L}}_n$. Also, set $\tilde{\mathcal{L}}_{n*} = \operatorname{span}\{1, \tilde{B}_{1*}, \dots, \tilde{B}_{n*}\}$, $\tilde{\mathcal{L}}_* = \cup_{n=0}^\infty \tilde{\mathcal{L}}_{n*}$ and $\tilde{\mathcal{R}}_{m,n} = \tilde{\mathcal{L}}_{m*} + \tilde{\mathcal{L}}_n$. Note that $\tilde{\mathcal{L}} = \mathcal{L} \cdot \mathcal{L}^c$ and with $\tilde{\mathcal{R}} = \tilde{\mathcal{L}} \cdot \tilde{\mathcal{L}}_*$, it also follows that $\tilde{\mathcal{R}} = (\mathcal{L} \cdot \mathcal{L}^c) \cdot (\mathcal{L}_* \cdot \mathcal{L}_*^c) = (\mathcal{L}_* \cdot \mathcal{L}^c) \cdot (\mathcal{L} \cdot \mathcal{L}_*^c) = \hat{\mathcal{R}} \cdot \hat{\mathcal{R}}_*$. Thus for every $f, g \in \hat{\mathcal{R}}$ there exist $\tilde{f}, \tilde{g} \in \tilde{\mathcal{L}}$ so that $\langle f, g\rangle = \langle \tilde{f}, \tilde{g}\rangle$.

Let $\{\tilde{\phi}_n\}_{n=0}^\infty$ be the ORFs in $\tilde{\mathcal{L}}$ obtained by orthogonalising the basis $\{1, \tilde{B}_1, \tilde{B}_2,$ $\dots\}$ corresponding to the sequence $\tilde{\mathcal{B}}$. Note that by (3.2)

$$\tilde{B}_{2n}(z) = B_n(z) B_n^c(z) \quad \text{and} \quad \tilde{B}_{2n+1}(z) = B_n(z) B_{n+1}^c(z) \quad n \geq 0.$$

Introducing the functions

$$\hat{\sigma}_{2n}(z) = B_{n*}(z)\tilde{\phi}_{2n}^*(z) \quad \text{and} \quad \hat{\sigma}_{2n+1}(z) = B_{n*}(z)\tilde{\phi}_{2n+1}(z) \tag{3.3}$$

results in the following lemma and theorem:

Lemma 3.1. *With $\hat{\sigma}_n$ as in* (3.3) *and $\hat{\mathcal{R}}_n$ as in* (3.1)*, we have that $\hat{\sigma}_n \in \hat{\mathcal{R}}_n$.*

Proof. (The proof is similar to that in [2, Lem. 2.1].) For $n = 2m$, we find that

$$\hat{\sigma}_{2m} = B_{m*}(z)\tilde{\phi}_{2m}^*(z) = \frac{\tilde{B}_{2m}(z)\tilde{\phi}_{2m*}(z)}{B_m(z)} = B_m^c(z)\tilde{\phi}_{2m*}(z).$$

Because the zeros $\{\overline{\beta}_k\}_{k=1}^m$ of the denominator of $\tilde{\phi}_{2m*}$ coincide with the same zeros in the numerator of B_m^c, $\hat{\sigma}_{2m} \in \mathcal{L}_m^c \cdot \mathcal{L}_{m*} = \hat{\mathcal{R}}_{m,m}$.

For $n = 2m + 1$,

$$\hat{\sigma}_{2m+1} = B_{m*}(z)\tilde{\phi}_{2m+1}(z),$$

and hence, because the zeros $\{1/\overline{\beta}_k\}_{k=1}^m$ of the denominator of $\tilde{\phi}_{2m+1}$ coincide with the same zeros in the numerator of B_{m*}, $\hat{\sigma}_{2m+1} \in \mathcal{L}_{m+1}^c \cdot \mathcal{L}_{m*} = \hat{\mathcal{R}}_{m,m+1}$. □

Theorem 3.2. *The sequence $\{\hat{\sigma}_n\}_{n=0}^\infty$ defined by* (3.3) *is an orthogonal basis for $\hat{\mathcal{R}}$ with respect to the inner product $\langle\cdot,\cdot\rangle$, such that $\hat{\sigma}_n \perp \hat{\mathcal{R}}_{n-1}$ for $\hat{\mathcal{R}}_n$ defined by* (3.1) *and $n = 0, 1, 2, \dots$.*

Proof. (The proof is similar to that in [2, Thm. 2.2].) First note that for $f, g \in \tilde{\mathcal{L}}_n$ we have $\langle f, g\rangle = \langle g_*, f_*\rangle = \langle g^{*,n}, f^{*,n}\rangle$. Thus, since $\tilde{\phi}_n \perp \tilde{\mathcal{L}}_{n-1}$, it holds for every $g \in \tilde{\mathcal{L}}_{n-1}$ that

$$0 = \langle g, \tilde{\phi}_n\rangle = \langle \tilde{\phi}_n^*, \tilde{\zeta}_n g^{*,n-1}\rangle,$$

so that $\tilde{\phi}_n^* \perp \tilde{\zeta}_n \tilde{\mathcal{L}}_{n-1}$.

For $n = 2m$ we have to prove that $\hat{\sigma}_{2m} \perp \mathcal{L}_m^c$ and that $\hat{\sigma}_{2m} \perp \mathcal{L}_{(m-1)*}$. Suppose that $f \in \mathcal{L}_m^c$, then $B_{m-1} f \in \mathcal{L}_{m-1} \cdot \mathcal{L}_m^c = \tilde{\mathcal{L}}_{2m-1}$. Thus for every $f \in \mathcal{L}_m^c$

$$\langle \hat{\sigma}_{2m}, f\rangle = \langle \tilde{\phi}_{2m}^* B_{m*}, f\rangle = \langle \tilde{\phi}_{2m}^*, B_m f\rangle = \langle \tilde{\phi}_{2m}^*, \zeta_m B_{m-1} f\rangle = 0,$$

because $\tilde{\phi}_{2m}^* \perp \tilde{\zeta}_{2m} \tilde{\mathcal{L}}_{2m-1} = \zeta_m \tilde{\mathcal{L}}_{2m-1}$. Similarly, for every $f_* \in \mathcal{L}_{(m-1)*}$

$$\langle \hat{\sigma}_{2m}, f_*\rangle = \langle \tilde{\phi}_{2m}^* B_{m*}, f_*\rangle = \langle \tilde{\phi}_{2m}^*, \zeta_m f^{*,m-1}\rangle = 0,$$

because $f^{*,m-1} \in \mathcal{L}_{m-1} \subset \tilde{\mathcal{L}}_{2m-1}$. Thus $\hat{\sigma}_{2m}$ is orthogonal to $\mathcal{L}_m^c$ and to $\mathcal{L}_{(m-1)*}$, hence orthogonal to $\hat{\mathcal{R}}_{m-1,m} = \mathcal{L}_m^c + \mathcal{L}_{(m-1)*}$.

In the same way we prove for $n = 2m+1$ that $\hat{\sigma}_{2m+1} \perp \mathcal{L}_m^c$ and that $\hat{\sigma}_{2m+1} \perp \mathcal{L}_{m*}$. Suppose that $f \in \mathcal{L}_m^c$, then

$$\langle \hat{\sigma}_{2m+1}, f\rangle = \langle \tilde{\phi}_{2m+1} B_{m*}, f\rangle = \langle \tilde{\phi}_{2m+1}, B_m f\rangle = 0,$$

because $B_m f \in \tilde{\mathcal{L}}_{2m} \perp \tilde{\phi}_{2m+1}$. On the other hand, for every $f_* \in \mathcal{L}_{m*}$

$$\langle \hat{\sigma}_{2m+1}, f_*\rangle = \langle \tilde{\phi}_{2m+1} B_{m*}, f_*\rangle = \langle \tilde{\phi}_{2m+1}, f^{*,m}\rangle = 0,$$

because $f^{*,m} \in \mathcal{L}_m \subset \tilde{\mathcal{L}}_{2m} \perp \tilde{\phi}_{2m+1}$. Thus $\hat{\sigma}_{2m+1}$ is orthogonal to $\mathcal{L}_m^c$ and to $\mathcal{L}_{m*}$, hence orthogonal to $\hat{\mathcal{R}}_{m,m}$. □

For the nesting

$$\hat{\mathcal{R}}_{0,0} \subset \hat{\mathcal{R}}_{1,0} \subset \hat{\mathcal{R}}_{1,1} \subset \hat{\mathcal{R}}_{2,1} \subset \hat{\mathcal{R}}_{2,2} \subset \ldots,$$

with the notation

$$\hat{\mathcal{R}}_{2n} = \hat{\mathcal{R}}_{n,n} \quad \text{and} \quad \hat{\mathcal{R}}_{2n+1} = \hat{\mathcal{R}}_{n+1,n} \quad \text{for} \quad n = 0, 1, 2, \ldots, \tag{3.4}$$

we can analogously derive another orthogonal basis for $\hat{\mathcal{R}}$. Let $\{\breve{\phi}_n\}_{n=0}^{\infty}$ be the ORFs in $\tilde{\mathcal{L}}^c$ obtained by orthogonalising the basis $\{1, \tilde{B}_1^c, \tilde{B}_2^c, \ldots\}$. Then with

$$\hat{\tau}_{2n}(z) = B_{n*}(z)\breve{\phi}_{2n}(z) \quad \text{and} \quad \hat{\tau}_{2n+1}(z) = B_{(n+1)*}(z)\breve{\phi}_{2n+1}^*(z) \tag{3.5}$$

we have the following theorem:

Theorem 3.3. *The sequence $\{\hat{\tau}_n\}_{n=0}^{\infty}$ defined by (3.5) is an orthogonal basis for $\hat{\mathcal{R}}$ with respect to the inner product $\langle \cdot, \cdot \rangle$, such that $\hat{\tau}_n \perp \hat{\mathcal{R}}_{n-1}$ for $\hat{\mathcal{R}}_n$ defined by (3.4) and $n = 0, 1, 2, \ldots$.*

4 Relating ORFs on $\mathbb{T}$ and on I

Before we can derive the relation between ORFs on the interval with respect to a weight function $u(x)$ and ORFs on the unit circle with respect to the weight function $w(\theta)$, we need the following lemma:

Lemma 4.1. *Let $h(\theta)$ be a symmetric weight function on $[-\pi,\pi]$ and let $\{\phi_n\}_{n=0}^{\infty}$ be an orthonormal basis for $\mathcal{L}$ on the unit circle with respect to the inner product (2.1) with $d\nu(z) = h(\theta)d\theta$. Then $\{\phi_n^c\}_{n=0}^{\infty}$ is an orthonormal basis for $\mathcal{L}^c$ on the unit circle with respect to the same weight function.*

Proof. For every $f(z) \in \mathcal{L}_{n-1}$ we have that

$$\begin{aligned}
0 &= \langle \phi_n, f\rangle = \int_{-\pi}^{\pi} \phi_n(z) f_*(z) h(\theta) d\theta = \int_{-\pi}^{\pi} \phi_n(e^{\mathbf{i}\theta}) f_*(e^{\mathbf{i}\theta}) h(\theta) d\theta \\
&= -\int_{-\pi}^{\pi} \phi_n(e^{-\mathbf{i}\xi}) f_*(e^{-\mathbf{i}\xi}) h(-\xi) d(-\xi) = \int_{-\pi}^{\pi} \phi_n(\overline{e^{\mathbf{i}\xi}}) f_*(\overline{e^{\mathbf{i}\xi}}) h(\xi) d\xi \\
&= \int_{-\pi}^{\pi} \phi_n(\overline{z}) f_*(\overline{z}) h(\xi) d\xi = \int_{-\pi}^{\pi} \overline{\phi_n(\overline{z}) f_*(\overline{z})} h(\xi) d\xi = \int_{-\pi}^{\pi} \phi_n^c(z) f_*^c(z) h(\xi) d\xi \\
&= \langle \phi_n^c, f^c\rangle.
\end{aligned}$$

Consequently, $\phi_n^c \perp \mathcal{L}_{n-1}^c$. In a similar way we can prove that $1 = \langle \phi_n, \phi_n\rangle = \langle \phi_n^c, \phi_n^c\rangle$. □

Recall that $w(\theta)$ is symmetric on $[-\pi,\pi]$. Thus, it follows from the previous lemma that $\hat{\sigma}_n(z) = \hat{\tau}_{n*}^c(z)$ so that we could continue using only $\hat{\sigma}_n$, disregarding $\hat{\tau}_n$. However, in order not to complicate the notation with $_*$ and ^{c}, we shall continue using both functions.

We are now able to state our main result.

Theorem 4.2. *Suppose the weight functions $u(x)$ and $w(\theta)$, with orthonormal rational functions $\varphi_n(x)$ and $\tilde{\phi}_n(z)$ respectively, are defined as above. Furthermore, assume that the leading coefficient $\tilde{\kappa}_n = \overline{\tilde{\phi}_n^*(\tilde{\beta}_n)}$, i.e. the coefficient of $\tilde{B}_n(z)$ in the expansion of $\tilde{\phi}_n(z)$ with respect to the basis $\{\tilde{B}_0, \tilde{B}_1, \ldots, \tilde{B}_n\}$, is real and positive. Also define the ORFs $\hat{\sigma}_n(z)$ and $\hat{\tau}_n(z)$ as before. If the points in $\mathcal{A} \subset \mathbb{C}^I$ and $\mathcal{B} \subset \mathbb{D}$ satisfy (2.6), then there exist constants $\theta_1, \theta_2 \in [0, 2\pi]$ so that the following relations hold:*

$$\begin{aligned}
\varphi_n(x) &= \{1 + \Re\{\langle \hat{\tau}_{2n}, \hat{\sigma}_{2n}\rangle\}\}^{-\frac{1}{2}} \{\hat{\tau}_{2n}(z) + \hat{\sigma}_{2n}(z)\} e^{\mathbf{i}\theta_1} \\
&= \left\{1 + \frac{\Re\left\{\tilde{\phi}_{2n}^c(\beta_n)\right\}}{\tilde{\kappa}_{2n}}\right\}^{-\frac{1}{2}} \{B_{n*}(z)\tilde{\phi}_{2n}^c(z) + B_n^c(z)\tilde{\phi}_{2n*}(z)\} e^{\mathbf{i}\theta_1} \\
&= \{1 + \Re\{\langle \hat{\sigma}_{2n-1}, \hat{\tau}_{2n-1}\rangle\}\}^{-\frac{1}{2}} \{\hat{\sigma}_{2n-1}(z) + \hat{\tau}_{2n-1}(z)\} e^{\mathbf{i}\theta_2} \\
&= \left\{1 + \Re\left\{\langle \zeta_n \tilde{\phi}_{2n-1}, \tilde{\phi}_{2n-1}^{c*}\rangle\right\}\right\}^{-\frac{1}{2}} \times \\
&\quad \{B_{(n-1)*}(z)\tilde{\phi}_{2n-1}(z) + B_{n-1}^c(z)\tilde{\phi}_{(2n-1)*}^c(z)\} e^{\mathbf{i}\theta_2}.
\end{aligned}$$

Proof. (The proof is similar to that in [9, Thm. 4.1].) The even lines in the equality follow from the definition of $\hat{\sigma}_n(z)$ and $\hat{\tau}_n(z)$. So we only need to prove the odd lines.

To prove the orthogonality of φ_n, we need to verify (2.8) and (2.9). For the first line we find, with $z = e^{\mathbf{i}\theta}$ and $x = \frac{1}{2}(z + z^{-1})$, that for $k = 0, \ldots, n-1$ there is a constant C so that

$$\begin{aligned}\int_{-\pi}^{\pi} \varphi_n(\cos\theta) B_k(z) w(\theta) d\theta &= C \int_{-\pi}^{\pi} \{\hat{\tau}_{2n}(z) + \hat{\sigma}_{2n}(z)\} B_k(z) w(\theta) d\theta \\ &= C[\langle \hat{\tau}_{2n}, B_{k*}\rangle + \langle \hat{\sigma}_{2n}, B_{k*}\rangle] \\ &= 0.\end{aligned}$$

This last equality follows because $k \leq n-1$ and $B_{k*} \in \hat{\mathcal{R}}_{n-1,n-1}$, while $\hat{\sigma}_{2n}(z)$ and $\hat{\tau}_{2n}(z)$ are both orthogonal to that space. Similarly, we have that

$$\begin{aligned}\int_{-\pi}^{\pi} \varphi_n(\cos\theta) B^c_{k*}(z) w(\theta) d\theta &= C \int_{-\pi}^{\pi} \{\hat{\tau}_{2n}(z) + \hat{\sigma}_{2n}(z)\} B^c_{k*}(z) w(\theta) d\theta \\ &= C[\langle \hat{\tau}_{2n}, B^c_k\rangle + \langle \hat{\sigma}_{2n}, B^c_k\rangle] \\ &= 0,\end{aligned}$$

which proves the orthogonality.

Now only the normalisation $\|\varphi\|_2 = \sqrt{\langle \varphi_n, \varphi_n\rangle} = 1$ needs to be checked. Note that

$$\frac{1}{2}\int_{-\pi}^{\pi} |\hat{\tau}_{2n}(z) + \hat{\sigma}_{2n}(z)|^2 w(\theta) d\theta = 1 + \Re\left\{\langle \hat{\tau}_{2n}, \hat{\sigma}_{2n}\rangle\right\}.$$

Rewriting the inner product on the right-hand side gives

$$\begin{aligned}\langle \hat{\tau}_{2n}, \hat{\sigma}_{2n}\rangle &= \int_{-\pi}^{\pi} B_{n*}(z) \tilde{\phi}^c_{2n}(z) B^c_{n*}(z) \tilde{\phi}_{2n}(z) w(\theta) d\theta \\ &= \int_{-\pi}^{\pi} \tilde{\phi}^c_{2n}(z) \tilde{B}_{2n*}(z) \tilde{\phi}_{2n}(z) w(\theta) d\theta \\ &= \langle \tilde{\phi}^c_{2n}, \tilde{\phi}^*_{2n}\rangle = \frac{\tilde{\phi}^c_{2n}(\beta_n)}{\tilde{\kappa}_{2n}}.\end{aligned}$$

The last line follows from [3, Thm. 2.2.3], which states that $\tilde{\kappa}_{2n}\tilde{\phi}^*_{2n} = \tilde{k}_{2n}(z, \tilde{\beta}_{2n})$, where $\tilde{k}_{2n}(z,t)$ denotes the reproducing kernel of $\tilde{\mathcal{L}}_{2n}$, and from the fact that $\tilde{\phi}^c_{2n} \in \tilde{\mathcal{L}}^c_{2n} - \tilde{\mathcal{L}}^c_{2n-1}$ and hence $\tilde{\phi}^c_{2n} \in \tilde{\mathcal{L}}_{2n}$. The multiplication with $e^{\mathbf{i}\theta_1}$ now follows from the fact that the orthonormal rational functions on the interval with respect to the weight function $u(x)$ are unique except for a constant of modulus one.

Proving the orthogonality for the third line can be done in a similar way as before. For the normalisation we now get that

$$\begin{aligned}\langle \hat{\sigma}_{2n-1}, \hat{\tau}_{2n-1}\rangle &= \int_{-\pi}^{\pi} B_{(n-1)*}(z) \tilde{\phi}_{2n-1}(z) B_n(z) \tilde{\phi}^{c*}_{(2n-1)*}(z) w(\theta) d\theta \\ &= \int_{-\pi}^{\pi} \zeta_n \tilde{\phi}_{2n-1}(z) \tilde{\phi}^{c*}_{(2n-1)*}(z) w(\theta) d\theta \\ &= \langle \zeta_n \tilde{\phi}_{2n-1}, \tilde{\phi}^{c*}_{2n-1}\rangle.\end{aligned}$$

□

In [9, Thm. 4.1] four extra relations are given between ORFs on the interval with respect to a varying weight function

$$v_{n,r}(x) = \frac{[(\alpha_n^2-1)(\alpha_{n+1}^2-1)]^{1/2}(1-x^2)}{|(\alpha_n - x)(\alpha_{n+1}-x)|}u(x) \tag{4.1}$$

and ORFs on the unit circle with respect to the weight function $w(\theta)$ for all *real* poles. Similar relations can be found for *complex* poles for $v_{n,c}(x)$ defined by

$$\begin{aligned} v_{n,c}(x) &= \frac{(|\beta_n|^2-1)(|\beta_{n+1}|^2-1)(1-x^2)}{4\beta_n\beta_{n+1}(\alpha_n-x)(\alpha_{n+1}-x)}u(x) \\ &= -\frac{1}{4}(\zeta_n^c(z)-\zeta_{n*}(z))(\zeta_{n+1}^c(z)-\zeta_{(n+1)*}(z))u(x), \end{aligned} \tag{4.2}$$

as will be proven in the next theorem. First note, however, that in general this varying weight function is complex, unless α_n and α_{n+1} are real, for which (4.2) reduces to (4.1), or $\alpha_{n+1} = \overline{\alpha}_n$, for which (4.2) reduces to

$$v_{n,c}(x) = \frac{(|\beta_n|^2-1)^2(1-x^2)}{4|\beta_n|^2|\alpha_n - x|^2}u(x).$$

Theorem 4.3. *Suppose the weight functions $v_{n,c}(x)$ and $w(\theta)$, with orthonormal rational functions $\psi_n(x)$ and $\tilde{\phi}_n(z)$ respectively, are defined as above. Also define the ORFs $\hat{\sigma}_n(z)$ and $\hat{\tau}_n(z)$ as before. If the points in $\mathcal{A} \subset \mathbb{C}^I$ and $\mathcal{B} \subset \mathbb{D}$ satisfy (2.6), then there exist constants C_n and D_n so that the following relations hold:*

$$\begin{aligned} \psi_n(x) &= C_n\frac{\hat{\tau}_{2n+2}(z)-\hat{\sigma}_{2n+2}(z)}{\zeta_{n+1}^c(z)-\zeta_{(n+1)*}(z)} \\ &= C_n\frac{B_{(n+1)*}(z)\tilde{\phi}_{2n+2}^c(z) - B_{n+1}^c(z)\tilde{\phi}_{2n+2*}(z)}{\zeta_{n+1}^c(z)-\zeta_{(n+1)*}(z)} \\ &= D_n\frac{\hat{\sigma}_{2n+1}(z)-\hat{\tau}_{2n+1}(z)}{\zeta_{n+1}^c(z)-\zeta_{(n+1)*}(z)} \\ &= D_n\frac{B_{n*}(z)\tilde{\phi}_{2n+1}(z) - B_n^c(z)\tilde{\phi}_{(2n+1)*}^c(z)}{\zeta_{n+1}^c(z)-\zeta_{(n+1)*}(z)}. \end{aligned}$$

Proof. (The proof is similar to that in [9, Thm. 4.1].) The even lines in the equality follow from the definition of $\hat{\sigma}_n(z)$ and $\hat{\tau}_n(z)$. So we only need to prove the odd lines.

For the third line we have, with $z = e^{\mathbf{i}\theta}$ and $x = \frac{1}{2}(z+z^{-1})$, that

$$\begin{aligned} &\int_{-\pi}^{\pi}(\hat{\tau}_{2n+1}(z)-\hat{\sigma}_{2n+1}(z))(\zeta_n^c(z)-\zeta_{n*}(z))B_k(z)w(\theta)d\theta \\ &\qquad = \langle\hat{\tau}_{2n+1},(\zeta_{n*}^c-\zeta_n)B_{k*}\rangle - \langle\hat{\sigma}_{2n+1},(\zeta_{n*}^c-\zeta_n)B_{k*}\rangle = 0. \end{aligned}$$

This last equality follows because $k \le n-1$ and $(\zeta_{n*}^c-\zeta_n)B_{k*} \in \hat{\mathcal{R}}_{n,n}$, while $\hat{\sigma}_{2n+1}(z)$ and $\hat{\tau}_{2n+1}(z)$ are both orthogonal to that space. Similarly, it holds that

$$\begin{aligned} &\int_{-\pi}^{\pi}(\hat{\tau}_{2n+1}(z)-\hat{\sigma}_{2n+1}(z))(\zeta_n^c(z)-\zeta_{n*}(z))B_{k*}^c(z)w(\theta)d\theta \\ &\qquad = \langle\hat{\tau}_{2n+1},(\zeta_{n*}^c-\zeta_n)B_k^c\rangle - \langle\hat{\sigma}_{2n+1},(\zeta_{n*}^c-\zeta_n)B_k^c\rangle = 0, \end{aligned}$$

which proves the orthogonality of the third line.

Proving the orthogonality for the first line can be done in a similar way as before. □

5 Expressions for the ORFs on the interval and on the unit circle

Let a and b belong to $\{\pm 1/2\}$ and define the weight function $u(x)$ by $u(x) = (1-x)^a(1+x)^b$. Then the related weight functions $w(\theta)$ on the circle are given by Table 1.

For the first case $a = b = -1/2$, the so-called Malmquist basis, which is a basis of ORFs on the unit circle with respect to the Lebesgue measure ($w(\theta) = 1$), is well known and are given by

$$\phi_n(z) = \sqrt{\frac{1-|\beta_n|^2}{2\pi}} \frac{zB_{n-1}(z)}{1-\overline{\beta}_n z}. \tag{5.1}$$

Furthermore, the ORFs on the interval I with respect to the weight functions $u(x)$ are called the Chebyshev ORFs. Note that the case $a = -1/2$ and $b = 1/2$ reduces to the case $a = 1/2$ and $b = -1/2$, replacing x by $-x$, so that there are just three weight functions to be considered on the interval. Based on the Bernstein-Szegő OP with respect to a positive varying weight, expressions were found for the Chebyshev ORFs with all *real* poles for the three weight functions in [8]. Later on these expressions were extended to the case of *complex* poles in [5]. Let i, p and q be given by Table 2 in function of a and b. Then the Chebyshev ORFs $\varphi_n(x)$ on the interval I with $n \geq 1$ are given by

$$\varphi_n(x) = \sqrt{\frac{2^i}{\pi}} \sqrt{1-|\beta_n|^2} \frac{q}{2z^{i-1}+q-3} \left(\frac{z^i B^c_{n-1}(z)}{1-\beta_n z} - \frac{qB_{(n-1)*}(z)}{z-\beta_n} \right). \tag{5.2}$$

If $n = 0$, then φ_0 is given by $\varphi_0 = \sqrt{p/\pi}$.

We shall now derive expressions for the ORFs on the unit circle with respect to the weight functions $w(\theta) = 1 \pm \cos\theta$.

Let $X_n(z)$ be given by

$$X_n(z) = c_n \left(a_n - b_n \frac{z^2 B_{n-1}(z)}{1-\overline{\beta}_n z} + \frac{z^3 B_{n-1}(z)}{1-\overline{\beta}_n z} \right),$$

where c_n is an arbitrary nonzero constant and

$$\begin{aligned} a_n &= \frac{B^2_{n-1}(1)}{(2-\overline{\beta}_n)B_{n-1}(1) + B'_{n-1}(1)(1-\overline{\beta}_n)}, \\ b_n &= 1 + \frac{(1-\overline{\beta}_n)B_{n-1}(1)}{(2-\overline{\beta}_n)B_{n-1}(1) + B'_{n-1}(1)(1-\overline{\beta}_n)}. \end{aligned} \tag{5.3}$$

(a,b)	$u(x)$	$w(\theta)$
$(-\frac{1}{2},-\frac{1}{2})$	$\frac{1}{\sqrt{1-x^2}}$	1
$(\frac{1}{2},-\frac{1}{2})$	$\sqrt{\frac{1-x}{1+x}}$	$1-\cos\theta$
$(-\frac{1}{2},\frac{1}{2})$	$\sqrt{\frac{1+x}{1-x}}$	$1+\cos\theta$
$(\frac{1}{2},\frac{1}{2})$	$\sqrt{1-x^2}$	$\sin^2\theta$

Table 1: The weight functions $u(x)$ and $w(\theta)$ in function of a and b.

(a,b)	i	p	q
$(-\frac{1}{2},-\frac{1}{2})$	1	1	-1
$(\frac{1}{2},-\frac{1}{2})$	2	1	1
$(\frac{1}{2},\frac{1}{2})$	3	2	1

Table 2: Definition of i, p and q in function of a and b.

Then we have that $X_n(1) = X_n^{'}(1) = 0$, and hence $X_n(z)$ has a zero of order two in $z = 1$. So let us now define $\phi_n(z)$ by

$$\phi_n(z) = \begin{cases} c_0, & n = 0 \\ X_n(z)/(z-1)^2, & n \geq 1 \end{cases} . \tag{5.4}$$

Then we have the following theorem:

Theorem 5.1. *The sequence $\{\phi_k\}_{k=0}^n$ defined by (5.4) is an orthogonal basis for $\mathcal{L}_n$ with respect to the inner product (2.1) on the unit circle and the weight function $w(\theta) = 1-\cos\theta$.*

Proof. First note that $\phi_n(z)$ is a linear combination of $\{B_0, \ldots, B_n(z)\}$ and that the pole of order two in $z = 1$ coincides with the same zero of $X_n(z)$. And hence, $\phi_n(z) \in \mathcal{L}_n$. To prove that $\phi_n \perp \mathcal{L}_{n-1}$, it suffices to prove that $\phi_n \perp B_k$ for $k = 0, \ldots, n-1$. We have that

$$\begin{aligned} \langle \phi_n, B_k \rangle &= -\frac{c_n}{2\mathbf{i}} \left(a_n \oint_{\mathbb{T}} \frac{B_{k*}(z)}{z^2} dz + \oint_{\mathbb{T}} \frac{(z-b_n)B_{n-1}(z)}{B_k(z)(1-\overline{\beta}_n z)} dz \right) \\ &= -\frac{c_n}{2\mathbf{i}} \left(-a_n \oint_{\mathbb{T}} \frac{B_{k*}(\overline{z})}{\overline{z}^2} d\overline{z} + 0 \right) \\ &= -\frac{c_n a_n}{2\mathbf{i}} \oint_{\mathbb{T}} B_{k*}^c(z) dz = 0. \end{aligned}$$

So $\phi_n \perp \mathcal{L}_{n-1}$. On the other hand, for $k = n$ it holds that

$$\begin{aligned} \langle \phi_n, B_n \rangle &= -\frac{c_n}{2\mathbf{i}} \left(a_n \oint_{\mathbb{T}} \frac{B_{n*}(z)}{z^2} dz + \oint_{\mathbb{T}} \frac{(z-b_n)B_{n-1}(z)}{B_n(z)(1-\overline{\beta}_n z)} dz \right) \\ &= -\frac{c_n}{2\mathbf{i}} \left(0 + \oint_{\mathbb{T}} \frac{z-b_n}{z-\beta_n} dz \right) \\ &= c_n \pi (b_n - \beta_n), \end{aligned}$$

which is always different from zero, because if $(b_n - \beta_n) = 0$, we would have that

$$\begin{aligned} 0 &= (1-\overline{\beta}_n)B_{n-1}(1) + (1-\beta_n)(2-\overline{\beta}_n)B_{n-1}(1) + B_{n-1}^{'}(1)(1-\beta_n)(1-\overline{\beta}_n) \\ &= |1-\beta_n|^2 \{B_{n-1}(1) + B_{n-1}^{'}(1)\} + 2[1-\Re(\beta_n)]B_{n-1}(1). \end{aligned}$$

Dividing by $|1-\beta_n|^2 B_{n-1}(1) \neq 0$ would then lead to the following contradiction:

$$\begin{aligned} 0 &= 1 + 2\frac{1-\Re(\beta_n)}{|1-\beta_n|^2} + \frac{B_{n-1}^{'}(1)}{B_{n-1}(1)} \\ &= 1 + 2\frac{1-\Re(\beta_n)}{|1-\beta_n|^2} + \sum_{k=1}^{n-1} \frac{1-|\beta_k|^2}{|1-\beta_k|^2} > 0. \end{aligned}$$

The last equality here follows from [8, Lem. 3.3]. □

If the leading coefficient of ϕ_n is assumed to be real and positive, and ϕ_n is orthonormal so that $\langle \phi_n, \phi_n \rangle = \langle \phi_n, \kappa_n B_n \rangle = \kappa_n c_n \pi (b_n - \beta_n) = 1$, then after some computations we find with Q_n given by

$$Q_n = 1 + \sum_{k=1}^{n-1} \frac{1 - |\beta_k|^2}{|1 - \beta_k|^2}, \quad n \geq 1, \tag{5.5}$$

that

$$c_n = \begin{cases} 1/\sqrt{\pi}, & n = 0 \\ \sqrt{\frac{1-|\beta_n|^2}{\pi}} \frac{(1-\overline{\beta}_n)Q_n + 1}{\sqrt{(1+Q_n)(|1-\beta_n|^2 Q_n + 2[1-\Re(\beta_n)])}}, & n \geq 1 \end{cases}, \tag{5.6}$$

and

$$\kappa_n = \sqrt{\frac{1 + Q_n}{\pi(1 - |\beta_n|^2)(|1 - \beta_n|^2 Q_n + 2[1 - \Re(\beta_n)])}}, \quad n \geq 1. \tag{5.7}$$

Corollary 5.2. *The sequence* $\{\phi_k\}_{k=0}^n$ *defined by*

$$\phi_n(z) = \begin{cases} 1/\sqrt{\pi}, & n = 0 \\ X_n(z)/(z+1)^2, & n \geq 1 \end{cases}. \tag{5.8}$$

is an orthonormal basis for $\mathcal{L}_n$ *with respect to the inner product* (2.1) *on the unit circle and the weight function* $w(\theta) = 1 + \cos\theta$*, when in the definition of* $X_n(z)$*,* (5.3) *is replaced by*

$$\begin{aligned} a_n &= -\frac{B_{n-1}^2(-1)}{(2 + \overline{\beta}_n)B_{n-1}(-1) - B'_{n-1}(-1)(1 + \overline{\beta}_n)}, \\ b_n &= -\left(1 + \frac{(1 + \overline{\beta}_n)B_{n-1}(-1)}{(2 + \overline{\beta}_n)B_{n-1}(-1) - B'_{n-1}(-1)(1 + \overline{\beta}_n)}\right). \end{aligned}$$

and (5.5) *and* (5.6) *are replaced by*

$$\begin{aligned} Q_n &= 1 + \sum_{k=1}^{n-1} \frac{1 - |\beta_k|^2}{|1 + \beta_k|^2}, \quad n \geq 1, \\ c_n &= \sqrt{\frac{1 - |\beta_n|^2}{\pi}} \frac{(1 + \overline{\beta}_n)Q_n + 1}{\sqrt{(1 + Q_n)(|1 + \beta_n|^2 Q_n + 2[1 + \Re(\beta_n)])}}, \quad n \geq 1. \end{aligned}$$

Remark 5.3. *If* $\beta_k = 0$ *for every* k*, then* (5.4) *and* (5.8) *reduce to the known expressions for the OPs on the unit circle with respect to the weight function* $w(\theta) = 1 - \cos\theta$ *respectively* $w(\theta) = 1 + \cos\theta$ *(see [1, Table 5.2]).*

6 Illustrating the relation

For the first weight function $a = b = -1/2$, the expression for the Chebyshev ORFs can be determined using the Malmquist basis given by (5.1). This was done in [9, Sec. 5] for all *real* poles, and can be done similarly for *complex* poles, using the relation in Theorem 4.2. Because of this similarity, we shall illustrate the relation for another weight function, namely for the second weight function $a = -b = 1/2$.

With the first or second relation given in Theorem 4.2 and the expression for the orthogonal basis on the unit circle given by (5.4), we get then that

$$
\begin{aligned}
&\{B_{n*}(z)\tilde{\phi}^c_{2n}(z) + B^c_n(z)\tilde{\phi}_{2n*}(z)\} \\
&\quad = \frac{\overline{c}^{\sim}_{2n}}{(z-1)^2}\left(B_{n*}(z)\left\{\overline{a}^{\sim}_{2n} - z^2(\overline{b}^{\sim}_{2n} - z)\frac{B_n(z)B^c_{n-1}(z)}{1-\beta_n z}\right\}\right. \\
&\quad\quad \left. + B^c_n(z)\left\{\overline{a}^{\sim}_{2n}z^2 - (z\overline{b}^{\sim}_{2n} - 1)\frac{B^c_{n*}(z)B_{(n-1)*}(z)}{z-\beta_n}\right\}\right) \\
&\quad = \frac{\overline{c}^{\sim}_{2n}}{(z-1)^2}\left(\frac{z^3B^c_{n-1}(z)}{1-\beta_n z} - z\overline{b}^{\sim}_{2n}\left\{\frac{zB^c_{n-1}(z)}{1-\beta_n z} + \frac{B_{(n-1)*}(z)}{z-\beta_n}\right\} + \frac{B_{(n-1)*}(z)}{z-\beta_n}\right. \\
&\quad\quad \left. + \overline{a}^{\sim}_{2n}\{B_{n*}(z) + z^2B^c_n(z)\}\right) \\
&\quad = \frac{\overline{c}^{\sim}_{2n}}{z-1}\left(\frac{z^2B^c_{n-1}(z)}{1-\beta_n z} - \frac{B_{n-1*}(z)}{z-\beta_n}\right) + \frac{\overline{c}^{\sim}_{2n}}{(z-1)^2}\times \\
&\quad\quad \left(-z(\overline{b}^{\sim}_{2n} - 1)\left\{\frac{zB^c_{n-1}(z)}{1-\beta_n z} + \frac{B_{(n-1)*}(z)}{z-\beta_n}\right\} + \overline{a}^{\sim}_{2n}\{B_{n*}(z) + z^2B^c_n(z)\}\right).
\end{aligned}
$$

Furthermore, we have that

$$
\overline{b}^{\sim}_{2n} - 1 = \frac{\overline{a}^{\sim}_{2n}(1-\beta_n)}{B_n(1)B^c_{n-1}(1)} = \frac{\overline{a}^{\sim}_{2n}(1-\beta_n)}{\zeta_n(1)} = \overline{a}^{\sim}_{2n}(1-\overline{\beta}_n)
$$

so that

$$
\begin{aligned}
&\{B_{n*}(z)\tilde{\phi}^c_{2n}(z) + B^c_n(z)\tilde{\phi}_{2n*}(z)\} \\
&\quad = \frac{\overline{c}^{\sim}_{2n}}{z-1}\left(\frac{z^2B^c_{n-1}(z)}{1-\beta_n z} - \frac{B_{(n-1)*}(z)}{z-\beta_n}\right) + \frac{\overline{c}^{\sim}_{2n}\overline{a}^{\sim}_{2n}}{(z-1)^2}\times \\
&\quad\quad \left(z(\overline{\beta}_n - 1)\left\{\frac{B_{(n-1)*}(z)}{z-\beta_n} + \frac{zB^c_{n-1}(z)}{1-\beta_n z}\right\} + B_{n*}(z) + z^2B^c_n(z)\right) \\
&\quad = \frac{\overline{c}^{\sim}_{2n}(1+\overline{a}^{\sim}_{2n})}{z-1}\left(\frac{z^2B^c_{n-1}(z)}{1-\beta_n z} - \frac{B_{(n-1)*}(z)}{z-\beta_n}\right).
\end{aligned}
$$

If $\tilde{\phi}_{2n}(z)$ is assumed to be orthonormal with a real and positive leading coefficient $\tilde{\kappa}_{2n}$, then $\overline{c}^{\sim}_{2n} = [\tilde{\kappa}_{2n}\pi(\overline{b}^{\sim}_{2n} - \overline{\beta}_n)]^{-1}$, and hence

$$
\overline{c}^{\sim}_{2n}(1+\overline{a}^{\sim}_{2n}) = [\tilde{\kappa}_{2n}\pi(1-\overline{\beta}_n)]^{-1}.
$$

We also have then that

$$
\begin{aligned}
\tilde{\phi}^c_{2n}(\beta_n) &= \frac{\overline{c}^{\sim}_{2n}\overline{a}^{\sim}_{2n}}{(1-\beta_n)^2} = \frac{\overline{a}^{\sim}_{2n}}{\tilde{\kappa}_{2n}\pi(\overline{b}^{\sim}_{2n} - \overline{\beta}_n)(1-\beta_n)^2} \\
&= \frac{\overline{b}^{\sim}_{2n} - 1}{\tilde{\kappa}_{2n}\pi(\overline{b}^{\sim}_{2n} - \overline{\beta}_n)|1-\beta_n|^2(1-\beta_n)} \\
&= \frac{(\overline{b}^{\sim}_{2n} - 1)[(1-\beta_n)\tilde{Q}_{2n} + 1]}{\tilde{\kappa}_{2n}\pi|1-\beta_n|^2(1-\beta_n)(|1-\beta_n|^2\tilde{Q}_{2n} + 2[1-\Re(\beta_n)])} \\
&= \{\tilde{\kappa}_{2n}\pi|1-\beta_n|^2(|1-\beta_n|^2\tilde{Q}_{2n} + 2[1-\Re(\beta_n)])\}^{-1}.
\end{aligned}
$$

Consequently, we get that

$$\overline{c}_{2n}^{\sim}(1+\overline{a}_{2n}^{\sim})\left\{1+\frac{\Re\left\{\tilde{\phi}_{2n}^{c}(\beta_n)\right\}}{\tilde{\kappa}_{2n}}\right\}^{-\frac{1}{2}}$$
$$=\left\{\frac{(1-\beta_n)(|1-\beta_n|^2\tilde{Q}_{2n}+2[1-\Re(\beta_n)])}{\pi(1-\overline{\beta}_n)(\tilde{\kappa}_{2n}^2\pi|1-\beta_n|^2(|1-\beta_n|^2\tilde{Q}_{2n}+2[1-\Re(\beta_n)])+1)}\right\}^{\frac{1}{2}}$$
$$=\sqrt{\frac{1-|\beta_n|^2}{\pi}}e^{-\mathbf{i}\theta_1}K_{2n},$$

where $e^{\mathbf{i}\theta_1}$ is given by

$$e^{\mathbf{i}\theta_1}=\sqrt{\frac{1-\overline{\beta}_n}{1-\beta_n}},$$

and K_{2n} is given by

$$K_{2n}=\left\{\frac{|1-\beta_n|^2\tilde{Q}_{2n}+2[1-\Re(\beta_n)]}{(1-|\beta_n|^2)(\tilde{\kappa}_{2n}^2\pi|1-\beta_n|^2(|1-\beta_n|^2\tilde{Q}_{2n}+2[1-\Re(\beta_n)])+1)}\right\}^{\frac{1}{2}}.$$

From Equation (5.7) it follows that

$$K_{2n}=\sqrt{\frac{|1-\beta_n|^2\tilde{Q}_{2n}+2[1-\Re(\beta_n)]}{(1+\tilde{Q}_{2n})|1-\beta_n|^2+(1-|\beta_n|^2)}}=1.$$

So, with φ_n given by (5.2), and i and q given by Table 2,

$$\{1+\Re\{\langle\hat{\tau}_{2n},\hat{\sigma}_{2n}\rangle\}\}^{-\frac{1}{2}}\{\hat{\tau}_{2n}(z)+\hat{\sigma}_{2n}(z)\}e^{\mathbf{i}\theta_1}=\varphi_n(x).$$

For the third or fourth relation given in Theorem 4.2, we get

$$\begin{aligned}
&\{B_{(n-1)*}(z)\tilde{\phi}_{2n-1}(z)+B_{n-1}^{c}(z)\tilde{\phi}_{(2n-1)*}^{c}(z)\}\\
&= \frac{c_{2n-1}^{\sim}}{(z-1)^2}\left(B_{(n-1)*}(z)\left\{a_{2n-1}^{\sim}-z^2(b_{2n-1}^{\sim}-z)\frac{B_{n-1}(z)B_{n-1}^{c}(z)}{1-\beta_n z}\right\}\right.\\
&\quad \left.+B_{n-1}^{c}(z)\left\{a_{2n-1}^{\sim}z^2-(zb_{2n-1}^{\sim}-1)\frac{B_{(n-1)*}^{c}(z)B_{(n-1)*}(z)}{z-\beta_n}\right\}\right)\\
&= \frac{c_{2n-1}^{\sim}}{(z-1)^2}\left(\frac{z^3B_{n-1}^{c}(z)}{1-\beta_n z}-zb_{2n-1}^{\sim}\left\{\frac{zB_{n-1}^{c}(z)}{1-\beta_n z}+\frac{B_{(n-1)*}(z)}{z-\beta_n}\right\}+\frac{B_{(n-1)*}(z)}{z-\beta_n}\right.\\
&\quad \left.+a_{2n-1}^{\sim}\{B_{n*}(z)+z^2B_{n}^{c}(z)\}\right)\\
&= \frac{c_{2n-1}^{\sim}}{z-1}\left(\frac{z^2B_{n-1}^{c}(z)}{1-\beta_n z}-\frac{B_{(n-1)*}(z)}{z-\beta_n}\right)+\frac{c_{2n-1}^{\sim}}{(z-1)^2}\times\\
&\quad \left(-z(b_{2n-1}^{\sim}-1)\left\{\frac{zB_{n-1}^{c}(z)}{1-\beta_n z}+\frac{B_{(n-1)*}(z)}{z-\beta_n}\right\}+a_{2n-1}^{\sim}\{B_{n*}(z)+z^2B_{n}^{c}(z)\}\right).
\end{aligned}$$

Furthermore,

$$b_{2n-1}^{\sim} - 1 = \frac{a_{2n-1}^{\sim}(1-\overline{\beta}_n)}{B_n^c(1)B_{n-1}(1)} = \frac{a_{2n-1}^{\sim}(1-\overline{\beta}_n)}{\zeta_n^c(1)} = a_{2n-1}^{\sim}(1-\beta_n)$$

so that

$$\begin{aligned}
&\{B_{(n-1)*}(z)\tilde{\phi}_{2n-1}(z) + B_{n-1}^c(z)\tilde{\phi}_{(2n-1)*}^c(z)\} \\
&\quad = \frac{c_{2n-1}^{\sim}}{z-1}\left(\frac{z^2 B_{n-1}^c(z)}{1-\beta_n z} - \frac{B_{(n-1)*}(z)}{z-\beta_n}\right) + \frac{c_{2n-1}^{\sim} a_{2n-1}^{\sim}}{(z-1)^2} \times \\
&\qquad \left(z(\overline{\beta}_n - 1)\left\{\frac{B_{(n-1)*}(z)}{z-\beta_n} + \frac{z B_{n-1}^c(z)}{1-\beta_n z}\right\} + B_{n*}(z) + z^2 B_n^c(z)\right) \\
&\quad = \frac{c_{2n-1}^{\sim}(1+a_{2n-1}^{\sim})}{z-1}\left(\frac{z^2 B_{n-1}^c(z)}{1-\beta_n z} - \frac{B_{(n-1)*}(z)}{z-\beta_n}\right).
\end{aligned}$$

It is clear from this that, with φ_n given by (5.2), and i and q given by Table 2, there exists a constant K so that

$$K\{B_{(n-1)*}(z)\tilde{\phi}_{2n-1}(z) + B_{n-1}^c(z)\tilde{\phi}_{(2n-1)*}^c(z)\} = \varphi_n(x).$$

According to Theorem 4.2, this constant is then given by

$$K = \left\{1 + \Re\left\{\langle \zeta_n \tilde{\phi}_{2n-1}, \tilde{\phi}_{2n-1}^{c*}\rangle\right\}\right\}^{-\frac{1}{2}} e^{\mathbf{i}\theta_2}$$

However, we shall not try to verify this here.

7 Conclusion

We have constructed two distinct orthogonal bases for $\hat{\mathcal{R}}$. Next, we have used these orthogonal bases to extend the relation between orthogonal rational functions (ORFs) on the unit circle and ORFs on the interval $I = [-1, 1]$ for all *real* poles outside I to the general case of arbitrary *complex* poles outside I. Then, we have derived expressions for the ORFs on the unit circle with respect to the weight functions $1 \pm \cos\theta$. Finally, we have used the expression for the ORFs on the unit circle with respect to the weight function $1 - \cos\theta$ together with the relation to derive an expression for the ORFs on the interval with respect to the weight function $\{(1-x)/(1+x)\}^{1/2}$.

Still open for investigation is to find an expression for the ORFs on the unit circle with respect to the weight function $w(\theta) = \sin^2\theta$, which is related to the Chebyshev weight of the second kind, given by $u(x) = \sqrt{1-x^2}$, on the interval.

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