Orthogonal rational functions with real coefficients and semiseparable matrices

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3-5 November 2006

History

- Inverse eigenvalue problem (Boley, Golub, 1987)
- Rutishauser-Gragg-Harrod algorithm (1963, 1984)
- Least squares interpretation (Gragg, Ammar, Reichel, 1990-1991)
- Vector polynomials, rational LS (Van Barel, B., 1992-1994)
- ORF (Van Barel, Fasino, Gemignani, Mastronardi, Van gucht, Delvaux, B., 2004-2005)

ORF scalar case

- Given points $\{z_i\}_{i=1}^N \subset \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.
- Corresponding positive weights $\{w_i^2\}_{i=1}^N$.
- inner product $\langle f, g \rangle = \sum_{k=1}^N w_i^2 f(z_i)\overline{g(z_i)}$.
- Predefined poles $\{\alpha_k\}_{k=1}^N \subset \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.
- Nested spaces $\mathcal{L}_{n-1} \subset \mathcal{L}_n$:
  $\mathcal{L}_n = \{p_n(z)/\prod_{i=1}^n (z - \alpha_i) : p_n \in \Pi_n\}$
- ORF: $\phi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$, $\phi_n \perp \mathcal{L}_{n-1}$
- This defines $\{\phi_n\}_{n=1}^N$

ORF scalar case

Lemma Set

$$\Phi(z) = [\phi_1(z), \phi_2(z), \ldots, \phi_N(z)],$$
$$\mathbf{A} = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_N),$$

Then there exists $\mathbf{H}$ = upper Hessenberg, with positive lower triangular elements such that

$$\Phi(z) = \Phi(z)(z\mathbf{I} - \mathbf{A})\mathbf{H}.$$

ORF scalar case

Theorem Set

$$\begin{aligned}
&\Phi(z) = [\phi_1(z), \phi_2(z), \ldots, \phi_N(z)],\\
&\mathbf{\Phi}_{i,j} = \phi_j(z_i),\\
&\mathbf{A} = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_N),\\
&\mathbf{Z} = \mathrm{diag}(z_1, z_2, \ldots, z_N),\\
&\mathbf{W} = \mathrm{diag}(w_1, w_2, \ldots, w_N),\\
&\mathbf{Q} = \mathbf{W\Phi},
\end{aligned}$$

Then with $\mathbf{H}$ as before

$$\mathbf{Q} = (\mathbf{ZQ} - \mathbf{QA})\mathbf{H}.$$

ORF scalar case

$$\mathbf{Q} = (\mathbf{ZQ} - \mathbf{QA})\mathbf{H} \Rightarrow \mathbf{H}^{-1} = \mathbf{Q}^H\mathbf{ZQ} - \mathbf{A}$$

Hence setting $\mathbf{S} = \mathbf{H}^{-1}$, and fixing $\mathbf{Q}$ completely by letting $\mathbf{n}^H = [w_1\phi_1(z_1), \ldots, w_N\phi_1(z_N)]$, $\mathbf{n}^H\mathbf{Q} = [\|\mathbf{n}\|, 0, \ldots 0] =: \mathbf{n}_1^H$, we get

$$\mathbf{Q}^H[\mathbf{n}|\mathbf{Z}] \left[\begin{array}{c|c} 1 & \\ \hline & \mathbf{Q} \end{array}\right] = [\mathbf{n}_1|\mathbf{S} + \mathbf{A}]$$

an inverse eigenvalue problem.
If all $\alpha_i = 0$

$$\mathbf{Q}^H[\mathbf{n}|\mathbf{Z}] \left[\begin{array}{c|c} 1 & \\ \hline & \mathbf{Q} \end{array}\right] = [\mathbf{n}_1|\mathbf{S}] \quad \text{or} \quad \mathbf{Q}^H[\mathbf{n}|\mathbf{Z}^{-1}] \left[\begin{array}{c|c} 1 & \\ \hline & \mathbf{Q} \end{array}\right] = [\mathbf{n}_1|\mathbf{H}]$$

Semiseparable **S**

Because $\mathbf{H}$ is upper Hessenberg, $\mathbf{S} = \mathbf{H}^{-1}$ is lower semiseparable.
I.e. all submatrices of the lower triangular part of $\mathbf{S}$ (including the diagonal) have rank 1 at most.
Because $\mathbf{S} + \mathbf{A} = \mathbf{Q}^H \mathbf{Z} \mathbf{Q}$ is unitary, $\mathbf{S}$ is also upper semiseparable.

System theoretic applications

Want to find

$$\min \sum_{k=1}^{N} w_k^2 \left| y_k - \frac{p_N(z_k)}{q_N(z_k)} \right|^2$$

first approach:

$$\min \sum_{k=1}^{N} \left[\frac{w_k}{q_N(z_k)} \right]^2 \left| [y_k, -1] \begin{bmatrix} q_N(z_k) \\ p_N(z_k) \end{bmatrix} \right|^2 = \min \sum_{k=1}^{N} \phi_N(z_k)^H W_k \phi_N(z_k)$$

$$\phi_N(z) = \begin{bmatrix} q_N(z) \\ p_N(z) \end{bmatrix}, \quad W_k = \begin{bmatrix} y_k \\ -1 \end{bmatrix} \left[\frac{w_k}{q_N(z_k)} \right]^2 [y_k, -1].$$

$\Rightarrow$ vector orthogonal polynomial ϕ_N

$\Rightarrow$ block version of polynomial IEP

System theoretic applications

Want to find

$$\min \sum_{k=1}^{N} w_k^2 \left| y_k - \frac{p_N(z_k)}{q_N(z_k)} \right|^2$$

second approach: given estimates of α_k

$$\frac{p_N(z)}{q_N(z)} = \sum_{k=1}^{N} a_k \phi_k(z), \quad \phi_k \in \mathcal{L}_k \setminus \mathcal{L}_{k-1}.$$

Linear problem. Needs ORF with $\alpha_k \neq 0$.
For real problems: we have pairs $(z_k, \overline{z}_k)$ and $(\alpha_k, \overline{\alpha}_k)$

Can we then do

$$\mathbf{Q}^H[\mathbf{n}|\mathbf{Z}] \left[\begin{array}{c|c} 1 & \\ \hline & \mathbf{Q} \end{array}\right] = [\mathbf{n}_1|\mathbf{S}+\mathbf{A}]$$

in real arithmetic?

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Real ORF = RORF

Need to introduce complex conjugate pairs simultaneously.
Should result in numerator and denominator having real coefs.
Assume all $\alpha_k \neq 0$ then with $\pi_k(z) = \prod_{i=1}^{k}(z-\alpha_i)(z-\overline{\alpha}_i)$

$$B_k = \left\{ \frac{1}{\pi_k(z)(z-\alpha_{k+1})}, \frac{1}{\pi_{k+1}(z)} \right\} \text{ and } \tilde{B}_k = \left\{ \frac{1}{\pi_{k+1}(z)}, \frac{z}{\pi_{k+1}(z)} \right\}$$

span the same space.

$$\begin{aligned}
\mathcal{L}_N = \mathcal{L}_{2n} &= \operatorname{span}\{B_1, \ldots, B_n\} \\
&= \operatorname{span}\left\{\phi_1, \tilde{\phi}_1, \ldots, \phi_n, \tilde{\phi}_n\right\} \quad (C)ORF \\
&= \operatorname{span}\left\{\tilde{B}_1, \ldots, \tilde{B}_n\right\} \\
&= \operatorname{span}\left\{\psi_1, \tilde{\psi}_1, \ldots, \psi_n, \tilde{\psi}_n\right\} \quad RORF
\end{aligned}$$

ORF $\Rightarrow$ RORF

Because they span the same space, there must be unitary matrices $\mathbf{p}_k$ such that $[\psi_k \ \tilde{\psi}_k] = [\phi_k \ \tilde{\phi}_k]\mathbf{p}_k$.
Set $\mathbf{P} = \mathrm{diag}(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ then the ORF relation $\mathbf{Q}(\mathbf{ZQ} - \mathbf{QA})\mathbf{H}$ can be rewritten as

$$\mathbf{QP} = (\mathbf{ZQ} - \mathbf{QA})\mathbf{HP} = (\mathbf{ZQP} - \mathbf{QPP}^H\mathbf{AP})\mathbf{P}^H\mathbf{HP}$$

$\mathbf{QP}$ consists of blocks $w_j \begin{bmatrix} \psi_i(z_j) & \tilde{\psi}_i(z_j) \\ \psi_i(\overline{z}_j) & \tilde{\psi}_i(\overline{z}_j) \end{bmatrix}$.

RORF but complex values!

This can be made real by $\mathbf{g} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ -\mathbf{i} & \mathbf{i} \end{bmatrix}$ and

$\mathbf{G} = \mathrm{diag}(\mathbf{g}, \ldots, \mathbf{g}) \Rightarrow \mathbf{GQP} \in \mathbb{R}^{N \times N}$

ORF $\Rightarrow$ RORF

Hence

$$\mathbf{GQP} = (\mathbf{GZG}^H\mathbf{GQP} - \mathbf{GQPP}^H\mathbf{AP})\mathbf{P}^H\mathbf{HP}$$

Setting

$$\begin{aligned}
\hat{\mathbf{Q}} &= \mathbf{GQP} \in \mathbb{R}^{N\times N} \\
\hat{\mathbf{Z}} &= \mathbf{GZG}^H \in \mathbb{R}^{N\times N}(\text{block diag}) \\
\tilde{\mathbf{A}} &= \mathbf{P}^H\mathbf{AP} \in \mathbb{C}^{N\times N}(\text{block diag}) \\
\tilde{\mathbf{H}} &= \mathbf{P}^H\mathbf{HP} \in \mathbb{C}^{N\times N}
\end{aligned}$$

we get

$$\hat{\mathbf{Q}} = (\hat{\mathbf{Z}}\hat{\mathbf{Q}} - \hat{\mathbf{Q}}\tilde{\mathbf{A}})\tilde{\mathbf{H}} \quad \text{or} \quad \tilde{\mathbf{S}} + \tilde{\mathbf{A}} = \hat{\mathbf{Q}}^H\hat{\mathbf{Z}}\hat{\mathbf{Q}}.$$

Note that $\tilde{\mathbf{S}}$ is also real except for the block diagonal. Take the complex part out of $\tilde{\mathbf{S}}$ and put it into $\tilde{\mathbf{A}}$ to get $\tilde{\mathbf{S}} + \tilde{\mathbf{A}} = \hat{\mathbf{S}} + \hat{\mathbf{A}}$ with $\hat{\mathbf{S}}$ and $\hat{\mathbf{A}}$ real and $\hat{\mathbf{S}}$ block lower semi-separable and $\hat{\mathbf{A}}$ block diagonal.

The imaginary part of $\tilde{\mathbf{S}}$

What does the imaginary part of $\tilde{\mathbf{S}}$ look like?
Will it be cancelled by the imaginary part of $\tilde{\mathbf{A}}$?
Recall $\tilde{\mathbf{S}} = \tilde{\mathbf{H}}^{-1} = (\mathbf{P}^H \mathbf{H} \mathbf{P})^{-1} = \mathbf{P}^H \mathbf{S} \mathbf{P}$.

$$\mathbf{S} = \begin{bmatrix} u_1 v_1^* & x_1 \tilde{y}_1^* & \cdots & x_1 y_n^* & x_1 \tilde{y}_n^* \\ \tilde{u}_1 v_1^* & \tilde{u}_1 \tilde{v}_1^* & \cdots & \tilde{x}_1 y_n^* & \tilde{x}_1 \tilde{y}_n^* \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ u_n v_1^* & u_n \tilde{v}_1^* & \cdots & u_n v_n^* & x_1 \tilde{y}_n^* \\ \tilde{u}_n v_1^* & \tilde{u}_n \tilde{v}_1^* & \cdots & \tilde{u}_n v_n^* & \tilde{u}_n \tilde{v}_n^* \end{bmatrix}$$

Transformation with $\mathbf{P}$ keeps the rank 1 structure, except on the diagonal blocks.
Making it block semiseparable will also make it real!

The imaginary part of $\tilde{\mathbf{S}}$

Consider a block:

$$\mathbf{S}_{ii} = \begin{bmatrix} u_i v_i^* & x_i \tilde{y}_i^* \\ \tilde{u}_i v_i^* & \tilde{u}_i \tilde{v}_i^* \end{bmatrix} = \begin{bmatrix} u_i \\ \tilde{u}_i \end{bmatrix} [v_1 \;\; \tilde{v}_i] + \begin{bmatrix} 0 & \Delta_i \\ 0 & 0 \end{bmatrix}, \;\; \Delta_i = x_i \tilde{y}_i^* - u_i \tilde{v}_i^*.$$

The transformation with $\mathbf{P}$ will keep the rank 1 structure off the block diagonal of $\mathbf{S}$ and for the first term of the block.
Taking out

$$\mathbf{d}_i = \mathbf{p}_i^H \begin{bmatrix} 0 & \Delta_i \\ 0 & 0 \end{bmatrix} \mathbf{p}_i$$

will make the remaing block semiseparable matrix real.
Hence $\mathrm{diag}(\mathbf{d}_1, \ldots, \mathbf{d}_n)$ must be the complex part of $\tilde{\mathbf{S}}$.

The imaginary part of $\tilde{\mathbf{A}}$

This proves existence of the real decomposition, but not a way of computing it.
We compute the imaginary part of $\tilde{\mathbf{A}}$ and require that $\Im(\tilde{\mathbf{S}} + \tilde{\mathbf{A}}) = 0$, which will give equations to find the $\mathbf{p}_i$.

$$\tilde{\mathbf{a}} = \mathbf{p}^H \begin{bmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{bmatrix} \mathbf{p} = \begin{bmatrix} e & f \\ -\overline{f} & \overline{e} \end{bmatrix}, \quad \begin{aligned} e &= \Re(\alpha) + \mathbf{i}\Im(\alpha)\cos(2t) \\ f &= \sin(2t)\Im(\alpha)(\sin(\gamma) - \mathbf{i}\cos(\gamma)) \end{aligned}$$

$t, \gamma = \nu - \mu \in \mathbb{R}$.
The blocks of $\mathbf{d}$ of $\Im(\tilde{\mathbf{S}})$ are of the form

$$\Delta \begin{bmatrix} \sin(t)\cos(t)e^{\mathbf{i}\mu} & \cos^2(t)e^{\mathbf{i}\nu} \\ -\sin^2(t)e^{\mathbf{i}(2\mu-\nu)} & -\sin(t)\cos(t)e^{\mathbf{i}\mu} \end{bmatrix}$$

$\mu, \nu, t \in \mathbb{R}$.

The solution

Setting $\Im(\tilde{\mathbf{S}} + \tilde{\mathbf{A}}) = 0$ leads to the equations

$$\begin{aligned}
\frac{d}{2}\sin(2t)\cos(\rho) &= \Im(\alpha)\cot(\gamma) \\
d\cos^2(t)\cos(\rho+\gamma) + \Im(\alpha)\sin(2t)\sin(\gamma) &= \Im(\alpha)\frac{\cot(t)}{\sin(\gamma)} \\
d\sin^2(t)\cos(\rho-\gamma) + \Im(\alpha)\sin(2t)\sin(\gamma) &= \Im(\alpha)\frac{\tan(t)}{\sin(\gamma)}
\end{aligned}$$

$\rho = \delta + \mu$, $\Delta = de^{i\delta}$.
The blocks of $\hat{\mathbf{A}}$ are of the form

$$\hat{\mathbf{a}} = \Re(\alpha)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \Im(\alpha)\begin{bmatrix} \cot(\gamma) & \frac{\cot(t)}{\sin(\gamma)} \\ -\frac{\tan(t)}{\sin(\gamma)} & -\cot(\gamma) \end{bmatrix}$$

note $\lambda(\hat{\mathbf{a}}) = \{\alpha, \overline{\alpha}\}$.

The algorithm

$$\left[\begin{array}{c|c} \mathbf{I} & \\ \hline & \hat{\mathbf{Q}}_{k-1}^T \end{array}\right] \left[\begin{array}{c|c} \begin{matrix}\mathbf{n}_k\\ \mathbf{n}_{k-1}\\ \vdots\\ \mathbf{n}_1\end{matrix} & \begin{matrix}\hat{\mathbf{z}}_k\\ \\ \\ \\ \end{matrix} \\ \hline & \hat{\mathbf{Z}}_{k-1} \end{array}\right] \left[\begin{array}{c|c|c} \mathbf{I} & & \\ \hline & \mathbf{I} & \\ \hline & & \hat{\mathbf{Q}}_{k-1} \end{array}\right] =$$

$$\left[\begin{array}{c|c|c} \begin{matrix}\mathbf{n}_k\\ \omega_{k-1}\\ 0\\ 0\end{matrix} & \begin{matrix}\hat{\mathbf{z}}_k\\ \\ \\ \\ \end{matrix} & \\ \hline & & \hat{\mathbf{S}}_{k-1}+\hat{\mathbf{A}}_{k-1} \end{array}\right] \rightarrow \left[\begin{array}{c|c} \begin{matrix}\omega_k\\ 0\\ 0\\ 0\end{matrix} & \\ \hline & \hat{\mathbf{S}}_k+\hat{\mathbf{A}}_k \end{array}\right]$$

$$\mathbf{n}_k = w_k \begin{bmatrix} \Re(\eta_1(z_k)) & \Re(\tilde{\eta}_1(z_k)) \\ \Im(\eta_1(z_k)) & \Im(\tilde{\eta}_1(z_k)) \end{bmatrix}, \quad [\eta_1(z), \tilde{\eta}_1(z)] = \frac{[1, z]}{\tilde{\pi}_1(z)}$$

The algorithm: step k

$$\left[\begin{array}{c|c||c|c} \mathbf{n}_k & \hat{\mathbf{z}}_k & 0 & \cdots \\ \hline \omega_{k-1} & 0 & U_1 V_1^T + \hat{\mathbf{a}}_1 & \cdots \\ 0 & \vdots & \vdots & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}\right]$$

Q_1 defined by

$$Q_1^T \begin{bmatrix} \mathbf{n}_k \\ \omega_{k-1} \end{bmatrix} = \begin{bmatrix} \omega_k \\ 0 \end{bmatrix}.$$

Implies a similarity transformation involving first 2 block rows and columns

$$Q_1^T \begin{bmatrix} \hat{\mathbf{z}}_k & 0 \\ 0 & U_1 V_1^T + \hat{\mathbf{a}}_1 \end{bmatrix} Q_1 = \begin{bmatrix} U_1' V_1'^T + \hat{\mathbf{a}}_1' & X_1' Y_2''^T \\ U_2'' V_1'^T & U_2'' V_2''^T + \hat{\mathbf{a}}_2'' \end{bmatrix}$$

The algorithm: step $k.1$

Transform block rows/columns 1-2

$$Q_1^T \begin{bmatrix} \hat{\mathbf{z}}_k & 0 \\ 0 & U_1 V_1^T + \hat{\mathbf{a}}_1 \end{bmatrix} Q_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} U_1' V_1'^T + \hat{\mathbf{a}}_1' & X_1' Y_2''^T \\ U_2'' V_1'^T & U_2'' V_2''^T + \hat{\mathbf{a}}_2'' \end{bmatrix}$$

Set

$$\begin{bmatrix} V_1' & X_1' \\ V_2'' & X_2'' \end{bmatrix} = \begin{bmatrix} A_1^T \\ B_1^T \end{bmatrix} [V_1 \;\; X_1], \quad Q_1 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$$

$a \Rightarrow U_1', \hat{\mathbf{a}}_1', \qquad b \Rightarrow Y_2'',$
$c \Rightarrow U_2'', \qquad d \Rightarrow \hat{\mathbf{a}}_2''$
Next transform block rows/columns 2-3:

The algorithm: step $k.2$

Transform block rows/columns 2-3

$$Q_2^T \begin{bmatrix} U_2'' V_2''^T + \hat{\mathbf{a}}_2'' & X_2'' Y_3^T \\ U_3 V_2''^T & U_3 V_3^T + \hat{\mathbf{a}}_2 \end{bmatrix} Q_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} U_2' V_2'^T + \hat{\mathbf{a}}_2' & X_2' Y_3''^T \\ U_3'' V_2'^T & U_3'' V_3''^T + \hat{\mathbf{a}}_3'' \end{bmatrix}$$

Once we know Q_2, the set

$$\begin{bmatrix} U_2' & V_2' & X_2' & Y_2' \\ U_3'' & V_3'' & X_2'' & Y_3'' \end{bmatrix} = Q_2^T \begin{bmatrix} U_2'' & V_2'' & X_2'' & Y_2'' \\ U_3 & V_3 & X_3 & Y_3 \end{bmatrix}$$

The algorithm: step $k.2$

Transform block rows/columns 2-3

$$Q_2 = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}$$

Then

$$\begin{bmatrix} \hat{\mathbf{a}}_2'' & X_2'' Y_3^T - U_2'' V_3^T \\ 0 & \hat{\mathbf{a}}_3 \end{bmatrix} \begin{bmatrix} A_2 \\ C_2 \end{bmatrix} = \begin{bmatrix} A_2 \\ C_2 \end{bmatrix} \hat{\mathbf{a}}_2'$$

$\Rightarrow A_2, C_2$ can be obtained from the eigenvectors of the 4×4 matrix corresponding to α_2 and $\overline{\alpha}_2$.

$$\begin{bmatrix} A_2 \\ C_2 \end{bmatrix} = c\mathbf{E}\mathbf{g}^H \begin{bmatrix} \frac{1}{\cot(t_2')} & 0 \\ -\frac{\cot(\gamma_2')}{\cot(t_2')} & -\frac{1}{\sin(\gamma_2')} \end{bmatrix}$$

B_2, D_2 similarly (not important, only needed for intermediate results).

The algorithm: step $k.(k-1)$

After transform of block rows/columns $(k-1, k)$

$$\begin{bmatrix} U'_1 {V'_1}^T + \hat{\mathbf{a}}_1 & \cdots & X'_1 {Y'_{k-1}}^T & X'_1 {Y''_k}^T \\ \vdots & \ddots & \cdots & \vdots \\ U'_{k-1} {V'_1}^T & \cdots & U'_{k-1} {V'_{k-1}}^T + \hat{\mathbf{a}}'_{k-1} & X'_{k-1} {Y''_k}^T \\ U''_k {V'_1}^T & \cdots & U''_k {V'_{k-1}}^T & U''_k {V''_k}^T + \hat{\mathbf{a}}''_k \end{bmatrix}$$

Just set $[U'_k \quad Y'_k] = [U''_k \quad Y''_k]$ and solve

$$U'_k {V''_k}^T + \hat{\mathbf{a}}''_k = U'_k {V'_k}^T + \hat{\mathbf{a}}'_k$$

for $V'_k, \gamma'_k, \cot(t'_k)$.