Rational wavelets on the unit circle and the real line

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abstract

Suppose $\{\phi_k\}_{k=0}^n$ is an orhonornal basis for the function space $\mathcal{L}_n$ of polynomials or rational functions of degree n. Set $\mathcal{V}_n = \mathcal{L}_{2^n}$. Then $k_n(z,w) = \sum_{k=0}^{2^n} \phi_k(z)\overline{\phi_k(w)}$, is a reproducing kernel for $\mathcal{V}_n$. For fixed w, it is known that the solution of the problem $\min\{\|f\| : f(w) = 1, f \in \mathcal{V}_n\}$ is given by $k_n(z,w)/k_n(w,w)$. Thus, such reproducing kernels are functions localized in the neighbourhood of $z = w$. Moreover, by an appropriate choice of the parameters $\{\xi_{nk}\}_{k=0}^n$, the functions $\{\varphi_{n,k}(z) = k_n(z,\xi_{nk})\}_{k=0}^{2^n}$ will be an orthogonal basis for $\mathcal{V}_n$. The orthogonal complement $\mathcal{W}_n = \mathcal{V}_{n+1} \ominus \mathcal{V}_n$ is spanned by the functions $\{\psi_{n,k}(z) = l_n(z,\eta_{nk})\}_{k=0}^{2^n-1}$ for an appropriate choice of the parameters $\{\eta_{nk}\}_{k=0}^{2^n-1}$ where $l_n = k_{n+1} - k_n$ is the reproducing kernel for $\mathcal{W}_n$. These observations form the basic ingredients for a wavelet type of analysis for orthogonal rational functions on the unit circle or the real line with respect to an arbitrary probability measure.

We investigate the case where orthogonality is considered on the real line and on the unit circle of the complex plane. In the case of the real line, the inner product is given by $\langle f,g\rangle = \int f(x)g(x)d\mathring{\mu}(x)$ where $d\mathring{\mu}$ is of the form $d\mathring{\mu}(x) = (1+x^2)^{-1}d\mu(x)$ with $\int d\mathring{\mu}(x) = 1$ defined on the extended real line $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. In the case of the unit circle, the inner product is given by $\langle f,g\rangle = \int f(t)\overline{g(t)}d\mu(t)$ where the integral is over the complex unit circle and $\int d\mu(t) = 1$.

This work is a continuation of similar ideas used for polynomials orthogonal on an interval by Fischer and Prestin [2] and for the trigonometric wavelets of Chui and Mhaskar [1].

References

[1] C.K. Chui and H.N. Mhaskar. On trigonometric wavelets. *Constr. Approx.*, 9:167–190, 1993.

[2] B. Fischer and J. Prestin. Wavelets based on orthogonal polynomials. *Math. Comp.*, 66:1593–1618, 1997.