SCHUR-NEVANLINNA SEQUENCES OF RATIONAL FUNCTIONS

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Abstract. We study certain sequences of rational functions with poles outside the unit circle. Such kind of sequences are recursively constructed based on sequences of complex numbers with norm less than one. In fact, such sequences are closely related to the Schur-Nevanlinna algorithm for Schur functions on the one hand and on the other hand to orthogonal rational functions on the unit circle. We shall see that rational functions belonging to a Schur-Nevanlinna sequence can be used to parameterize the set of all solutions of an interpolation problem of Nevanlinna-Pick type for Schur functions.

1. Introduction

A function g which maps the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}$ into the complex plane $\mathbb{C}$ is a *Schur function* or belongs to the *Schur class* $\mathcal{S}$ (in the open unit disk $\mathbb{D}$) if g is holomorphic in $\mathbb{D}$ and if its values $g(z)$ are bounded by 1 for $z \in \mathbb{D}$, i.e. g is a holomorphic function such that the kernel

$$S_g(z,\zeta) := \frac{1 - g(z)\overline{g(\zeta)}}{1 - z\overline{\zeta}}, \quad z, \zeta \in \mathbb{D},$$

is non-negative Hermitian. More explicitly this kernel condition means that for every choice of $m \in \mathbb{N} := \{1, 2, 3, \ldots\}$ and of the m points $z_1, z_2, \ldots, z_m \in \mathbb{D}$ the complex $(m \times m)$-matrix $\big(S_g(z_j, z_k)\big)_{j,k=1}^{m}$ is non-negative Hermitian. The equivalence of these conditions follows from the considerations on the classical Nevanlinna-Pick interpolation problem (see [20], [18]).

In the present work, we study a multiple point Nevanlinna-Pick interpolation problem for Schur functions. A multiple point interpolation problem is a problem where not only values for the function itself, but also for its derivatives up to a certain order are prescribed. Here we consider the following problem:

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(MNP) Given $n \in \mathbb{N}$, mutually distinct points $z_1, z_2, \ldots, z_n \in \mathbb{D}$, $l_1, l_2, \ldots, l_n \in \mathbb{N}$, and $w_{js} \in \mathbb{C}$, $s = 0, 1, \ldots, l_j - 1$, $j = 1, 2, \ldots, n$. Find necessary and sufficient conditions for the existence of a $g \in \mathcal{S}$ such that

$$\frac{1}{s!} g^{(s)}(z_j) = w_{js}, \quad s = 0, 1, \ldots, l_j - 1, \ j = 1, 2, \ldots, n. \tag{1.1}$$

Moreover, describe the set of solutions $\mathcal{S}_\Delta$ of all $g \in \mathcal{S}$ fulfilling (1.1).

Note that Problem (MNP) can be conceived as a generalization of the Schur coefficient problem (see [23], [1]) on the one hand and on the other hand of the classical Nevanlinna-Pick problem (see [20], [18]).

As is well-known, it is frequently the case that a finite interpolation problem of Nevanlinna-Pick type can be reduced, in a suitable way, to the study of a truncated trigonometric moment problem. Moreover, there exists several approaches to the solution of such a problem and several generalizations of the problem, too (see, e.g., [11], [7], [16], [10], [2], [12], [24], [3], [25], [6]). In particular, it is well-known that there is a $g \in \mathcal{S}$ satisfying (1.1) if and only if the generalized Schwarz-Pick matrix $\mathbf{P}_\Delta$ which can be computed from the data given in Problem (MNP) is non-negative Hermitian. In Section 2, we introduce briefly $\mathbf{P}_\Delta$. Moreover, we recall some basic facts on the Schur-Nevanlinna algorithm for Schur functions which was introduced by Nevanlinna [18] as an extension of the classical algorithm of Schur [23]. In fact, we shall deduce that, starting from a $g \in \mathcal{S}_\Delta$, the feasibility of this algorithm for g is closely related to the case that $\mathbf{P}_\Delta$ is even positive Hermitian. Furthermore, we present some basics on linear fractional transformations and on rational functions.

As the main result of this note, in Section 6 we will see that, for the non-uniqueness case, i.e. if $\mathbf{P}_\Delta$ is a positive Hermitian matrix, the set of solutions $\mathcal{S}_\Delta$ can be characterized by a linear fractional transformation which is determined by certain rational functions, where $\mathcal{S}$ is the set of parameters. In fact, we obtain that a function g belongs to $\mathcal{S}_\Delta$ if and only if there exists a Schur function h fulfilling

$$g(z) = \frac{\delta_m^{[m]}(z) + b_{\alpha_m}(z)\gamma_m(z)h(z)}{\gamma_m^{[m]}(z) + b_{\alpha_m}(z)\delta_m(z)h(z)}, \quad z \in \mathbb{D}, \tag{1.2}$$

where b_{α_m} is a Blaschke factor, the rational functions δ_m, γ_m are elements of a Schur-Nevanlinna sequence, and $\delta_m^{[m]}$, $\gamma_m^{[m]}$ denote their adjoint rational functions.

Some basic facts on Schur-Nevanlinna sequences of rational functions are explained in Section 3. In fact, such kind of sequences are connected with the Schur-Nevanlinna algorithm for Schur functions on the one hand and on the other hand to orthogonal rational functions on the unit circle. Similar as in the case of orthogonal functions, we will see that the validity of some Christoffel-Darboux formulae is an important property of Schur-Nevanlinna sequences of rational functions (see Theorem 4.2 and the inverse question discussed in Section 5).

Note that already in [24] the set of solutions $\mathcal{S}_\Delta$ is characterized by a linear fractional transformation which is determined by some polynomials (without calculating these functions precise). We will also mention that in [19] a modified Schur-Nevanlinna algorithm is discussed with respect to the real line case and a multiple point interpolation problem for Nevanlinna functions. The essential new feature of this paper is that the functions δ_m and γ_m which appear in (1.2) are closely related to the orthogonal rational functions on the unit circle which were introduced by M.M. Djrbashian [8] (see also [4] and other papers cited there). But the explicit interplay between Schur-Nevanlinna sequences and orthogonal rational functions will be done in a forthcoming work.

2. Preliminaries

For Problem (MNP), we assume that the following data are given: $n \in \mathbb{N}$, mutually different points $z_1, z_2, \dots, z_n \in \mathbb{D}$, numbers $l_1, l_2, \dots, l_n \in \mathbb{N}$ and $w_{js} \in \mathbb{C}$, $s = 0, 1, \dots, l_j - 1$, $j = 1, 2, \dots, n$. We denote this data set by Δ:

$$\Delta := \left\{ \left(z_j, l_j, (w_{js})_{s=0}^{l_j-1} \right)_{j=1}^{n} \right\} \tag{2.1}$$

and put

$$m := \sum_{j=1}^{n} l_j - 1. \tag{2.2}$$

For a given function $g \in \mathcal{S}$ we define similarly

$$\Delta_g := \left\{ \left(z_j, l_j, \left(\frac{1}{s!} g^{(s)}(z_j) \right)_{s=0}^{l_j-1} \right)_{j=1}^{n} \right\}.$$

In particular, $g \in \mathcal{S}_\Delta$ if and only if $\Delta_g = \Delta$.

Furthermore, the *generalized Schwarz-Pick matrix* (with respect to the data Δ) of size $(m+1) \times (m+1)$ is defined as

$$\mathbf{P}_\Delta := \left(\mathbf{P}_{jk} \right)_{j,k=1}^{n},$$

where the complex $(l_j \times l_k)$-matrices

$$\mathbf{P}_{jk} := \left(p_{jk}^{s,t} \right)_{\substack{s=0,1,\dots,l_j-1 \\ t=0,1,\dots,l_k-1}}, \quad j, k = 1, 2, \dots, n,$$

are determined by the entries

$$\begin{aligned} p_{jk}^{s,t} \;:=\; & \sum_{r=0}^{\min\{s,t\}} \frac{(s+t-r)!}{(s-r)!r!(t-r)!} \frac{z_j^{t-r} \overline{z_k}^{s-r}}{(1 - z_j \overline{z_k})^{s+t-r+1}} \\ & - \sum_{\ell=0}^{s} \sum_{h=0}^{t} \sum_{r=0}^{\min\{\ell,h\}} \frac{(h+\ell-r)!}{(\ell-r)!r!(h-r)!} \frac{z_j^{h-r} \overline{z_k}^{h-r}}{(1 - z_j \overline{z_k})^{h+\ell-r+1}} w_{j,s-\ell} \overline{w_{k,t-h}}, \\ & s = 0, 1, \dots, l_j - 1,\; t = 0, 1, \dots, l_k - 1. \end{aligned}$$

In the sequel, 0 stands also for the zero matrix of appropriate size, and if $\mathbf{A}$, $\mathbf{B}$ are Hermitian matrices of the same size, then $\mathbf{A} \geq \mathbf{B}$ (resp., $\mathbf{A} > \mathbf{B}$) means that $\mathbf{A} - \mathbf{B}$ is a non-negative (resp., positive) Hermitian matrix.

Now we recall a well-known criterion for the solvability of Problem (MNP); see, e.g., [11], [12], [3].

Theorem 2.1. *For a given data set Δ as in* (2.1), Problem (MNP) *has a solution if and only if* $\mathbf{P}_\Delta \geq 0$. *The solution is unique if and only if* $\mathbf{P}_\Delta \geq 0$ *with* $\det \mathbf{P}_\Delta = 0$.

Since the main goal of this paper is to obtain the description of $\mathcal{S}_\Delta$ via (1.2) for the non-uniqueness case we will always assume in the further course

$$\mathbf{P}_\Delta > 0.$$

The next considerations are aimed at showing that $\mathbf{P}_\Delta > 0$ is closely related to the feasibility of the Schur-Nevanlinna algorithm at least $m+1$ times for a $g \in \mathcal{S}_\Delta$.

The algorithm presented below goes back to Nevanlinna [18] and is based on the following version of Schwarz's lemma (see, e.g., [4, Theorem 1.2.3] for a proof). Here b_z denotes the elementary Blaschke factor corresponding to $z \in \mathbb{D}$, i.e.

$$b_z(v) := \begin{cases} \dfrac{\overline{z}}{|z|} \dfrac{z - v}{1 - \overline{z}v} & \text{if } z \neq 0, \\ v & \text{if } z = 0. \end{cases} \tag{2.3}$$

Remark 2.2. If $g \in \mathcal{S}$ such that $g(z) = 0$ then also $h := \dfrac{g}{b_z} \in \mathcal{S}$.

Now we recall the *Schur-Nevanlinna algorithm*: Given a function $g \in \mathcal{S}$ and some points $\alpha_0, \alpha_1, \alpha_2, \ldots \in \mathbb{D}$. We set $g_0 := g$ and consider $s_0 := g_0(\alpha_0)$. If $s_0 \in \mathbb{D}$ we can define

$$g_1(z) := \frac{1}{b_{\alpha_0}(z)} \frac{g_0(z) - s_0}{1 - \overline{s_0} g_0(z)}$$

and, if the function g_k, $k = 0, 1, 2, \ldots$, is already defined and

$$s_k := g_k(\alpha_k) \tag{2.4}$$

belongs to $\mathbb{D}$, then

$$g_{k+1}(z) := \frac{1}{b_{\alpha_k}(z)} \frac{g_k(z) - s_k}{1 - \overline{s_k} g_k(z)}, \quad k = 0, 1, 2, \ldots . \tag{2.5}$$

If $g \in \mathcal{S}$ and $\alpha_0, \alpha_1, \alpha_2, \ldots \in \mathbb{D}$ such that the Schur-Nevanlinna algorithm can be carried out at least r times (that is after obtaining g_r and s_r), then $(s_k)_{k=0}^r$ from (2.4) is called the sequence of *Schur parameters* associated with the pair $[g, (\alpha_k)_{k=0}^r]$.

The algorithm (cf. (2.5)) defines Schur functions $g_0, g_1, g_2, \ldots$. It breaks down after the k-th step (that is after obtaining g_k and s_k) if and only if $s_k \in \mathbb{T}$. In particular, the Schur-Nevanlinna algorithm breaks down after the k-th step if and only if g is a Blaschke product of degree k (cf. [18]). Therefore, using some basic facts on generalized Schwarz-Pick matrices one can conclude the following statement. (cf. [14, Section 5], [17, Corollary 3.6]).

Theorem 2.3. *If $g \in \mathcal{S}$ then the Schur-Nevanlinna algorithm can be carried out at least $m+1$ times for g (and any points $\alpha_0, \alpha_1, \ldots, \alpha_{m+1} \in \mathbb{D}$) if and only if $\mathbf{P}_{\Delta_g} > 0$.*

In the sequel, we will also apply some well-known results on linear fractional transformations (see, e.g., [22], [10], [9]). If

$$\mathbf{\Theta} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a complex (2×2)-matrix and w is a complex number such that $cw + d \neq 0$, then we set

$$T_{\mathbf{\Theta}}(w) := \frac{aw + b}{cw + d}. \tag{2.6}$$

Note that the relation

$$T_{\mathbf{\Theta}_1}\big(T_{\mathbf{\Theta}_2}(w)\big) = T_{\mathbf{\Theta}_1 \mathbf{\Theta}_2}(w) \tag{2.7}$$

is fulfilled, and that in the case $\det \mathbf{\Theta} \neq 0$ the inverse mapping $T_{\mathbf{\Theta}}^{-1}$ is given by

$$T_{\mathbf{\Theta}}^{-1}(w) = T_{\mathbf{\Theta}^{-1}}(w) = \frac{dw - b}{-cw + a}. \tag{2.8}$$

Note that, in view of (2.6)-(2.8), the relation (2.5) can also be written as

$$g_k(z) = \frac{b_{\alpha_k}(z)g_{k+1}(z) + s_k}{\overline{s_k}b_{\alpha_k}(z)g_{k+1}(z) + 1} = T_{\widehat{\Xi}_k(z)}\big(g_{k+1}(z)\big)$$

with

$$\widehat{\Xi}_k(z) := \begin{pmatrix} b_{\alpha_k}(z) & s_k \\ \overline{s_k}b_{\alpha_k}(z) & 1 \end{pmatrix} = \begin{pmatrix} 1 & s_k \\ \overline{s_k} & 1 \end{pmatrix} \begin{pmatrix} b_{\alpha_k}(z) & 0 \\ 0 & 1 \end{pmatrix} \tag{2.9}$$

and hence

$$\begin{aligned} g(z) \equiv g_0(z) &= T_{\widehat{\Xi}_0(z)}\Big(T_{\widehat{\Xi}_1(z)}\Big(\cdots\Big(T_{\widehat{\Xi}_k(z)}\big(g_{k+1}(z)\big)\cdots\Big)\Big)\Big) \\ &= T_{\widehat{\Xi}_0(z)\widehat{\Xi}_1(z)\cdots\widehat{\Xi}_k(z)}\big(g_{k+1}(z)\big). \end{aligned} \tag{2.10}$$

In the next section, we shall treat those sequences of rational functions which are closely related to the Schur-Nevanlinna algorithm on the one hand and on the other hand to the orthogonal rational functions on the unit circle introduced by M.M. Djrbashian [8] (see also [4]). Here, for fixed points $\alpha_0, \alpha_1, \alpha_2, \ldots \in \mathbb{D}$ the notation $\mathfrak{H}_k$, $k = 0, 1, 2, \ldots$, stands for the space of rational functions x which admit for some complex polynomial p of degree not greater than k the representation

$$x = \frac{p}{q_k},$$

where the complex polynomial q_k of degree not greater than $k+1$ is given by

$$q_k(v) = \prod_{j=0}^{k}(1 - \overline{\alpha_j}v).$$

As suggested in [4], the following transform of a rational function into another plays a key role. For $x \in \mathfrak{H}_k$, $k = 0, 1, 2, \ldots$, by the *adjoint rational function* $x^{[k]}$ of x (adjoint with respect to $\alpha_0, \alpha_1, \ldots, \alpha_k$) we mean the rational function which is uniquely determined via the formula

$$x^{[k]}(v) = \frac{1}{v}B_k(v)\overline{x\Big(\frac{1}{\overline{v}}\Big)}, \tag{2.11}$$

where B_k stands for the *Blaschke product* (of degree $k+1$) with respect to the points $\alpha_0, \alpha_1, \ldots, \alpha_k$, i.e.

$$B_k(v) := \prod_{j=0}^{k} b_{\alpha_j}(v).$$

Some information on the calculation of $x^{[k]}$, $k = 0, 1, 2, \ldots$, can be found in [4, Section 2.2]. Note that the results on adjoint rational function in [4] are explained relating to the special case $\alpha_0 = 0$. But it is not hard to restate these with their proofs to the present situation. For instance, if $x \in \mathfrak{H}_k$, $k = 0, 1, 2, \ldots$, then also $x^{[k]} \in \mathfrak{H}_k$ and $(x^{[k]})^{[k]} = x$ in that case.

3. Some basics on Schur-Nevanlinna sequences of rational functions

In this section, as a rational extension of the classical considerations of Schur [23] and Nevanlinna [18] (see also [13, Section 3] for an extension to the case of matrix-valued polynomials) we study some sequences of rational functions formed by given sequences of points and parameters belonging to $\mathbb{D}$.

If $\tau = 0$ or $\tau \in \mathbb{N}$ or $\tau = \infty$, if $\mathbb{I} := \{0, 1, 2, \ldots, \tau\}$, and if $(\alpha_k)_{k\in\mathbb{I}}$ and $(\kappa_k)_{k\in\mathbb{I}}$ are sequences of points belonging to $\mathbb{D}$, then we define sequences of rational functions $(\gamma_k)_{k\in\mathbb{I}}$ and $(\delta_k)_{k\in\mathbb{I}}$ by the relations

$$\gamma_0(v) := \sqrt{\frac{1-|\alpha_0|^2}{1-|\kappa_0|^2}}\frac{1}{1-\overline{\alpha_0}v}, \quad \delta_0(v) := \overline{\kappa_0}\sqrt{\frac{1-|\alpha_0|^2}{1-|\kappa_0|^2}}\frac{1}{1-\overline{\alpha_0}v} \tag{3.1}$$

and, for $k \in \mathbb{I} \setminus \{0\}$, recursively

$$\gamma_k(v) := \sqrt{\frac{1-|\alpha_k|^2}{(1-|\alpha_{k-1}|^2)(1-|\kappa_k|^2)}}\frac{1-\overline{\alpha_{k-1}}v}{1-\overline{\alpha_k}v}\left(b_{\alpha_{k-1}}(v)\gamma_{k-1}(v) + \overline{\kappa_k}\delta_{k-1}^{[k-1]}(v)\right),$$

$$\delta_k(v) := \sqrt{\frac{1-|\alpha_k|^2}{(1-|\alpha_{k-1}|^2)(1-|\kappa_k|^2)}}\frac{1-\overline{\alpha_{k-1}}v}{1-\overline{\alpha_k}v}\left(b_{\alpha_{k-1}}(v)\delta_{k-1}(v) + \overline{\kappa_k}\gamma_{k-1}^{[k-1]}(v)\right),$$

where $\delta_{k-1}^{[k-1]}$ and $\gamma_{k-1}^{[k-1]}$ stands for the adjoint rational function of δ_{k-1} and γ_{k-1} (with respect to the points $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}$; see (2.11)). We call $[(\gamma_k)_{k\in\mathbb{I}}, (\delta_k)_{k\in\mathbb{I}}]$ the *Schur-Nevanlinna pair of rational functions corresponding to* $(\alpha_k, \kappa_k)_{k\in\mathbb{I}}$.

With the matrix function

$$\mathbf{\Theta}_k := \begin{pmatrix} b_{\alpha_k}\gamma_k & \delta_k^{[k]} \\ b_{\alpha_k}\delta_k & \gamma_k^{[k]} \end{pmatrix}, \quad k \in \mathbb{I}, \tag{3.2}$$

the recurrence formulae above can be written, for $k \in \mathbb{I} \setminus \{0\}$, in matricial form as

$$\mathbf{\Theta}_k(v) = \sqrt{\frac{1-|\alpha_k|^2}{1-|\alpha_{k-1}|^2}}\frac{1-\overline{\alpha_{k-1}}v}{1-\overline{\alpha_k}v}\mathbf{\Theta}_{k-1}(v)\widehat{\mathbf{\Theta}}_k(v), \tag{3.3}$$

where

$$\widehat{\mathbf{\Theta}}_k(v) := \frac{1}{\sqrt{1-|\kappa_k|^2}}\begin{pmatrix} 1 & \kappa_k \\ \overline{\kappa_k} & 1 \end{pmatrix}\begin{pmatrix} b_{\alpha_k}(v) & 0 \\ 0 & \eta_k\overline{\eta_{k-1}} \end{pmatrix}, \quad k \in \mathbb{I} \setminus \{0\}, \tag{3.4}$$

and

$$\eta_k := \begin{cases} -1 & \text{if } \alpha_k = 0, \\ \dfrac{\overline{\alpha_k}}{|\alpha_k|} & \text{if } \alpha_k \neq 0, \end{cases} \qquad k \in \mathbb{I}. \tag{3.5}$$

Proposition 3.1. *Let $\tau \in \mathbb{N}$ or $\tau = \infty$, let $\mathbb{I} := \{0, 1, 2, \ldots, \tau\}$, and let $(\alpha_k)_{k\in\mathbb{I}}$ and $(\kappa_k)_{k\in\mathbb{I}}$ be sequences of points belonging to $\mathbb{D}$. Further, let $(\gamma_k)_{k\in\mathbb{I}}$ and $(\delta_k)_{k\in\mathbb{I}}$ be sequences of rational functions such that γ_0, δ_0 are defined as in (3.1) and γ_k, δ_k belong to $\mathfrak{H}_k$ for $k \in \mathbb{I} \setminus \{0\}$. Then $[(\gamma_k)_{k\in\mathbb{I}}, (\delta_k)_{k\in\mathbb{I}}]$ is the Schur-Nevanlinna pair of rational functions corresponding to $(\alpha_k, \kappa_k)_{k\in\mathbb{I}}$ if and only if, for each $k \in \mathbb{I} \setminus \{0\}$, the following backward recurrence relations hold:*

$$\eta_k\overline{\eta_{k-1}}\gamma_k(v) - \overline{\kappa_k}\delta_k^{[k]}(v) = \frac{(1-\overline{\alpha_k}\alpha_{k-1})\sqrt{1-|\kappa_k|^2}}{\sqrt{(1-|\alpha_k|^2)(1-|\alpha_{k-1}|^2)}}\big(b_{\alpha_k}(v) - b_{\alpha_k}(\alpha_{k-1})\big)\gamma_{k-1}(v),$$

$$\eta_k\overline{\eta_{k-1}}\delta_k(v) - \overline{\kappa_k}\gamma_k^{[k]}(v) = \frac{(1-\overline{\alpha_k}\alpha_{k-1})\sqrt{1-|\kappa_k|^2}}{\sqrt{(1-|\alpha_k|^2)(1-|\alpha_{k-1}|^2)}}\big(b_{\alpha_k}(v) - b_{\alpha_k}(\alpha_{k-1})\big)\delta_{k-1}(v).$$

Proof. Let $k \in \mathbb{I} \setminus \{0\}$. Evidently, the identity

$$\begin{pmatrix} 1 & \kappa_k \\ \overline{\kappa_k} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\kappa_k \\ -\overline{\kappa_k} & 1 \end{pmatrix} = \begin{pmatrix} 1-|\kappa_k|^2 & 0 \\ 0 & 1-|\kappa_k|^2 \end{pmatrix}$$

is satisfied. Therefore, (3.3) is equivalent to the relation

$$\begin{pmatrix} \gamma_k(v) & \overline{\eta_k}\eta_{k-1}\delta_k^{[k]}(v) \\ \delta_k(v) & \overline{\eta_k}\eta_{k-1}\gamma_k^{[k]}(v) \end{pmatrix} \begin{pmatrix} 1 & -\kappa_k \\ -\overline{\kappa_k} & 1 \end{pmatrix} = \sqrt{\frac{(1-|\alpha_k|^2)(1-|\kappa_k|^2)}{1-|\alpha_{k-1}|^2}} \frac{1-\overline{\alpha_{k-1}}v}{1-\overline{\alpha_k}v} \mathbf{\Theta}_{k-1}(v).$$

Hence, by considering the first column of $\mathbf{\Theta}_{k-1}(v)$ and using

$$\overline{\eta_k}\eta_{k-1}\frac{1-\overline{\alpha_k}\alpha_{k-1}}{1-|\alpha_k|^2}\big(b_{\alpha_k}(v)-b_{\alpha_k}(\alpha_{k-1})\big) = \frac{1-\overline{\alpha_{k-1}}v}{1-\overline{\alpha_k}v} b_{\alpha_{k-1}}(v), \tag{3.6}$$

one can finally conclude the assertion. □

In the further course of this section, $[(\gamma_k)_{k\in\mathbb{I}}, (\delta_k)_{k\in\mathbb{I}}]$ stands always for the Schur-Nevanlinna pair of rational functions corresponding to $(\alpha_k, \kappa_k)_{k\in\mathbb{I}}$, where $(\alpha_k)_{k\in\mathbb{I}}$ and $(\kappa_k)_{k\in\mathbb{I}}$ are some sequences of points belonging to $\mathbb{D}$.

By combination of the forward recursions defining the Schur-Nevanlinna pair of rational functions $[(\gamma_k)_{k\in\mathbb{I}}, (\delta_k)_{k\in\mathbb{I}}]$, the backward recursions stated in Proposition 3.1, and (3.6) one can see that the pair $[(\gamma_k)_{k\in\mathbb{I}}, (\delta_k)_{k\in\mathbb{I}}]$ fulfills also the following three-term recurrence relations.

Corollary 3.2. *For each* $k \in \mathbb{I} \setminus \{0,1\}$*,*

$$\begin{aligned} \overline{\kappa_{k-1}}\gamma_k(v) &= \sqrt{\tfrac{1-|\alpha_k|^2}{(1-|\alpha_{k-1}|^2)(1-|\kappa_k|^2)}} \tfrac{1-\overline{\alpha_{k-1}}v}{1-\overline{\alpha_k}v} \Big(\overline{\kappa_{k-1}} b_{\alpha_{k-1}}(v) + \overline{\kappa_k}\eta_{k-1}\overline{\eta_{k-2}}\Big)\gamma_{k-1}(v) \\ &\quad - \overline{\kappa_k}\eta_{k-1}\overline{\eta_{k-2}}\sqrt{\tfrac{(1-|\alpha_k|^2)(1-|\kappa_{k-1}|^2)}{(1-|\alpha_{k-2}|^2)(1-|\kappa_k|^2)}} \tfrac{1-\overline{\alpha_{k-2}}v}{1-\overline{\alpha_k}v} b_{\alpha_{k-2}}(v)\gamma_{k-2}(v) \end{aligned}$$

and

$$\begin{aligned} \overline{\kappa_{k-1}}\delta_k(v) &= \sqrt{\tfrac{1-|\alpha_k|^2}{(1-|\alpha_{k-1}|^2)(1-|\kappa_k|^2)}} \tfrac{1-\overline{\alpha_{k-1}}v}{1-\overline{\alpha_k}v} \Big(\overline{\kappa_{k-1}} b_{\alpha_{k-1}}(v) + \overline{\kappa_k}\eta_{k-1}\overline{\eta_{k-2}}\Big)\delta_{k-1}(v) \\ &\quad - \overline{\kappa_k}\eta_{k-1}\overline{\eta_{k-2}}\sqrt{\tfrac{(1-|\alpha_k|^2)(1-|\kappa_{k-1}|^2)}{(1-|\alpha_{k-2}|^2)(1-|\kappa_k|^2)}} \tfrac{1-\overline{\alpha_{k-2}}v}{1-\overline{\alpha_k}v} b_{\alpha_{k-2}}(v)\delta_{k-2}(v), \end{aligned}$$

where $\gamma_0(v)$*,* $\delta_0(v)$ *are given as in* (3.1) *and*

$$\gamma_1(v) = \sqrt{\tfrac{1-|\alpha_1|^2}{(1-|\kappa_0|^2)(1-|\kappa_1|^2)}} \tfrac{1}{1-\overline{\alpha_1}v}\Big(b_{\alpha_0}(v) - \eta_0\overline{\kappa_1}\kappa_0\Big),$$

$$\delta_1(v) = \sqrt{\tfrac{1-|\alpha_1|^2}{(1-|\kappa_0|^2)(1-|\kappa_1|^2)}} \tfrac{1}{1-\overline{\alpha_1}v}\Big(\overline{\kappa_0} b_{\alpha_0}(v) - \eta_0\overline{\kappa_1}\Big).$$

A key tool in the proof of the description (1.2) for the set $\mathcal{S}_\Delta$ is an application of some well-known results on Potapov's $\mathbf{J}$-theory (see, e.g., [21, 22]). Hereby, the special choice of the (2×2)-signature matrix

$$\mathbf{J} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

will be essential in the considerations below. Recall that a complex (2×2)-matrix $\mathbf{\Theta}(v)$ is said to be $\mathbf{J}$-*contractive* (resp., $\mathbf{J}$-*unitary*), if

$$\mathbf{J} \geq \mathbf{\Theta}(v)^*\mathbf{J}\mathbf{\Theta}(v) \quad \Big(\text{resp., } \mathbf{J} = \mathbf{\Theta}(v)^*\mathbf{J}\mathbf{\Theta}(v)\Big),$$

where $\mathbf{\Theta}(v)^*$ denotes the adjoint matrix of $\mathbf{\Theta}(v)$.

Theorem 3.3. *For each $k \in \mathbb{I}$,*

$$\mathbf{\Theta}_k(v) = \frac{\sqrt{1-|\alpha_k|^2}}{1-\overline{\alpha_k}v}\widehat{\mathbf{\Theta}}_0(v)\widehat{\mathbf{\Theta}}_1(v)\cdots\widehat{\mathbf{\Theta}}_k(v), \tag{3.7}$$

where $\mathbf{\Theta}_k(v)$ and $\widehat{\mathbf{\Theta}}_\ell(v)$, $\ell = 1, 2, \ldots, k$, is given by (3.2) and (3.4) as well as

$$\widehat{\mathbf{\Theta}}_0(v) := \frac{1}{\sqrt{1-|\kappa_0|^2}}\begin{pmatrix} 1 & \kappa_0 \\ \overline{\kappa_0} & 1 \end{pmatrix}\begin{pmatrix} b_{\alpha_0}(v) & 0 \\ 0 & -\eta_0 \end{pmatrix}.$$

*In particular, if $v \in \mathbb{D}$ (resp., $v \in \mathbb{T}$) then the matrix $\frac{1-\overline{\alpha_k}v}{\sqrt{1-|\alpha_k|^2}}\mathbf{\Theta}_k(v)$ is **J**-contractive (resp., **J**-unitary).*

Proof. Let $k \in \mathbb{I}$. The alleged representation (3.7) of $\mathbf{\Theta}_k(v)$ is an easy consequence of (3.3) and the choice of γ_0 and δ_0. It remains to prove that if $v \in \mathbb{D}$ (resp., $v \in \mathbb{T}$) then the matrix $\frac{1-\overline{\alpha_k}v}{\sqrt{1-|\alpha_k|^2}}\mathbf{\Theta}_k(v)$ is **J**-contractive (resp., **J**-unitary). But this follows immediately from (3.7) and the fact that, for all $\ell \in \{0, 1, \ldots, k\}$,

$$\mathbf{J} = \frac{1}{1-|\kappa_\ell|^2}\begin{pmatrix} 1 & \kappa_\ell \\ \overline{\kappa_\ell} & 1 \end{pmatrix}^* \mathbf{J}\begin{pmatrix} 1 & \kappa_\ell \\ \overline{\kappa_\ell} & 1 \end{pmatrix} \tag{3.8}$$

as well as that if $v \in \mathbb{D}$ (resp., $v \in \mathbb{T}$) then

$$\begin{gathered}\mathbf{J} \geq \begin{pmatrix} b_{\alpha_\ell}(v) & 0 \\ 0 & u \end{pmatrix}^* \mathbf{J}\begin{pmatrix} b_{\alpha_\ell}(v) & 0 \\ 0 & u \end{pmatrix} \\ \left(\text{resp., } \mathbf{J} = \begin{pmatrix} b_{\alpha_\ell}(v) & 0 \\ 0 & u \end{pmatrix}^* \mathbf{J}\begin{pmatrix} b_{\alpha_\ell}(v) & 0 \\ 0 & u \end{pmatrix}\right)\end{gathered} \tag{3.9}$$

for a $u \in \mathbb{T}$, i.e. the fact that $\widehat{\mathbf{\Theta}}_\ell(v)$ is **J**-contractive (resp., **J**-unitary). □

By forming the determinants of both sides in (3.7), it follows that:

Corollary 3.4. *For each $k \in \mathbb{I}$,*

$$b_{\alpha_k}(v)\left(\gamma_k(v)\gamma_k^{[k]}(v) - \delta_k(v)\delta_k^{[k]}(v)\right) = -\eta_k\frac{1-|\alpha_k|^2}{(1-\overline{\alpha_k}v)^2}B_k(v).$$

Because of $B_k(\alpha_\ell) = 0$, $\ell = 0, 1, \ldots, k-1$ (cf. (2.3)), the equality in Corollary 3.4 implies particularly:

Corollary 3.5. *For each $k \in \mathbb{I} \setminus \{0\}$ and each $\ell \in \{0, 1, \ldots, k-1\}$,*

$$\gamma_k(\alpha_\ell)\gamma_k^{[k]}(\alpha_\ell) = \delta_k(\alpha_\ell)\delta_k^{[k]}(\alpha_\ell).$$

In view of some well-known results on **J**-contractive matrices (see, e.g., [9, Theorem 1.6.1]), Theorem 3.3 yields also that:

Corollary 3.6. *For each $k \in \mathbb{I}$ and each $v \in \mathbb{D} \cup \mathbb{T}$, the relations*

$$\gamma_k^{[k]}(v) \neq 0, \quad \frac{1}{|\gamma_k^{[k]}(v)|} \leq \frac{|1-\overline{\alpha_k}v|}{\sqrt{1-|\alpha_k|^2}}, \quad \left|\frac{\delta_k^{[k]}(v)}{\gamma_k^{[k]}(v)}\right| < 1, \quad \text{and} \quad \left|\frac{b_{\alpha_k}(v)\delta_k(v)}{\gamma_k^{[k]}(v)}\right| < 1$$

are satisfied.

Since Corollary 3.6 includes a localization of the zeros of $\gamma_k^{[k]}$, $k \in \mathbb{I} \setminus \{0\}$, the next Corollary is an easy conclusion of Proposition 3.1 and (3.3) with $v = \alpha_{k-1}$.

Corollary 3.7. *For each $k \in \mathbb{I} \setminus \{0\}$, we have $\gamma_k^{[k]}(\alpha_{k-1}) \neq 0$,*

$$\overline{\kappa_k} = \eta_k \overline{\eta_{k-1}} \frac{\delta_k(\alpha_{k-1})}{\gamma_k^{[k]}(\alpha_{k-1})},$$

$$\sqrt{1-|\kappa_k|^2} = \eta_k \overline{\eta_{k-1}} \frac{\sqrt{(1-|\alpha_k|^2)(1-|\alpha_{k-1}|^2)}}{1-\overline{\alpha_k}\alpha_{k-1}} \frac{\gamma_{k-1}^{[k-1]}(\alpha_{k-1})}{\gamma_k^{[k]}(\alpha_{k-1})},$$

and, in particular, $\kappa_k = 0 \iff \delta_k(\alpha_{k-1}) = 0$.

Note that, if we put $\alpha_{-1} := 0$, $\eta_{-1} := -1$, and $\gamma_{-1}^{[-1]}(\alpha_{-1}) := 1$ then the relations in Corollary 3.7 hold also in the case $k = 0$.

In spite of the symmetry of the recurrence relations, one can not easily interchange the roles of γ_k and δ_k in Corollary 3.7. In fact, the condition $\delta_k^{[k]}(\alpha_{k-1}) \neq 0$ is not true in general.

Remark 3.8. For each $k \in \mathbb{I} \setminus \{0\}$, if $\delta_k^{[k]}(\alpha_{k-1}) \neq 0$ then Proposition 3.1 and (3.3) yield

$$\overline{\kappa_k} = \eta_k \overline{\eta_{k-1}} \frac{\gamma_k(\alpha_{k-1})}{\delta_k^{[k]}(\alpha_{k-1})},$$

$$\sqrt{1-|\kappa_k|^2} = \eta_k \overline{\eta_{k-1}} \frac{\sqrt{(1-|\alpha_k|^2)(1-|\alpha_{k-1}|^2)}}{1-\overline{\alpha_k}\alpha_{k-1}} \frac{\delta_{k-1}^{[k-1]}(\alpha_{k-1})}{\delta_k^{[k]}(\alpha_{k-1})},$$

and, in particular, $\kappa_k = 0 \iff \gamma_k(\alpha_{k-1}) = 0$ on the one hand. On the other hand from (3.3), $\gamma_k^{[k]}(\alpha_{k-1}) \neq 0$, and Corollary 3.5, one can see

$$\delta_{k-1}^{[k-1]}(\alpha_{k-1}) = 0 \iff \delta_k^{[k]}(\alpha_{k-1}) = 0 \implies \gamma_k(\alpha_{k-1}) = 0, \quad k \in \mathbb{I} \setminus \{0\},$$

but $\kappa_k = 0$ implies also $\gamma_k(\alpha_{k-1}) = 0$ and the condition $\gamma_k(\alpha_{k-1}) = 0$ supplies conversely $\delta_k^{[k]}(\alpha_{k-1}) = 0$ or $\kappa_k = 0$.

Remark 3.9. If $\kappa_0 = 0$ then obviously

$$\delta_0^{[0]}(\alpha_0) = 0$$

and, by an application of $\gamma_k^{[k-1]}(\alpha_0) \neq 0$ and Corollary 3.5, one can inductively derive from the recurrence relations of a Schur-Nevanlinna pair of rational functions (use, e.g., (3.3)) that

$$\delta_k^{[k]}(\alpha_0) = 0, \quad \gamma_k(\alpha_0) = 0, \quad k \in \mathbb{I} \setminus \{0\}.$$

4. Christoffel-Darboux formulae

In the present section, we shall show that, similar as in the case of orthogonal rational functions (cf. [4, Theorem 3.1.3]), arbitrary Schur-Nevanlinna pairs of rational functions $[(\gamma_k)_{k\in\mathbb{I}}, (\delta_k)_{k\in\mathbb{I}}]$ fulfill also some Christoffel-Darboux formulae. To prove these, we remark at first the following identities for $[(\gamma_k)_{k\in\mathbb{I}}, (\delta_k)_{k\in\mathbb{I}}]$. Here and in the sequel $[(\gamma_k)_{k\in\mathbb{I}}, (\delta_k)_{k\in\mathbb{I}}]$ stands again for the Schur-Nevanlinna pair of rational functions corresponding to $(\alpha_k, \kappa_k)_{k\in\mathbb{I}}$, where $(\alpha_k)_{k\in\mathbb{I}}$ and $(\kappa_k)_{k\in\mathbb{I}}$ are some sequences of points belonging to $\mathbb{D}$.

Lemma 4.1. *For each $k \in \mathbb{I} \setminus \{0\}$, the following relations hold:*

$$\begin{aligned}&\big(1-b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(w)}\,\big)\Big(\delta_k^{[k]}(v)\overline{\delta_k^{[k]}(w)}-\gamma_k(v)\overline{\gamma_k(w)}\Big)\\&\quad=\big(1-b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}\,\big)\Big(\delta_{k-1}^{[k-1]}(v)\overline{\delta_{k-1}^{[k-1]}(w)}-b_{\alpha_{k-1}}(v)\gamma_{k-1}(v)\overline{b_{\alpha_{k-1}}(w)\gamma_{k-1}(w)}\Big),\end{aligned}$$

$$\begin{aligned}&\big(1-b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(w)}\,\big)\Big(\gamma_k^{[k]}(v)\overline{\gamma_k^{[k]}(w)}-\delta_k(v)\overline{\delta_k(w)}\Big)\\&\quad=\big(1-b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}\,\big)\Big(\gamma_{k-1}^{[k-1]}(v)\overline{\gamma_{k-1}^{[k-1]}(w)}-b_{\alpha_{k-1}}(v)\delta_{k-1}(v)\overline{b_{\alpha_{k-1}}(w)\delta_{k-1}(w)}\Big),\end{aligned}$$

and

$$\begin{aligned}&\big(1-b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(w)}\,\big)\Big(\delta_k^{[k]}(v)\overline{\gamma_k^{[k]}(w)}-\gamma_k(v)\overline{\delta_k(w)}\Big)\\&\quad=\big(1-b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}\,\big)\Big(\delta_{k-1}^{[k-1]}(v)\overline{\gamma_{k-1}^{[k-1]}(w)}-b_{\alpha_{k-1}}(v)\gamma_{k-1}(v)\overline{b_{\alpha_{k-1}}(w)\delta_{k-1}(w)}\Big).\end{aligned}$$

Proof. Note that, in view of (2.3), for $z \in \mathbb{D}$ and complex numbers v, w ($\neq \frac{1}{\overline{z}}$) it follows that

$$1-b_z(v)\overline{b_z(w)}=\frac{(1-|z|^2)(1-v\overline{w})}{(1-v\overline{z})(1-z\overline{w})}. \tag{4.1}$$

Let $k \in \mathbb{I} \setminus \{0\}$. Considering the first row of the matrix function $\mathbf{\Theta}_k$, from (3.3), (3.8), and (3.9) one can see

$$\begin{aligned}&\delta_k^{[k]}(v)\overline{\delta_k^{[k]}(w)}-\gamma_k(v)\overline{\gamma_k(w)}\\&\quad=-\Big(\gamma_k(v)\quad\overline{\eta_k}\eta_{k-1}\delta_k^{[k]}(v)\Big)\,\mathbf{J}\,\Big(\gamma_k(w)\quad\overline{\eta_k}\eta_{k-1}\delta_k^{[k]}(w)\Big)^*\\&\quad=-\left(\sqrt{\tfrac{1-|\alpha_k|^2}{1-|\alpha_{k-1}|^2}}\,\tfrac{1-\overline{\alpha_{k-1}}v}{1-\overline{\alpha_k}v}\Big(b_{\alpha_{k-1}}(v)\gamma_{k-1}(v)\quad\delta_{k-1}^{[k-1]}(v)\Big)\tfrac{1}{\sqrt{1-|\kappa_k|^2}}\begin{pmatrix}1&\kappa_k\\\overline{\kappa_k}&1\end{pmatrix}\right)\mathbf{J}\\&\qquad\left(\sqrt{\tfrac{1-|\alpha_k|^2}{1-|\alpha_{k-1}|^2}}\,\tfrac{1-\overline{\alpha_{k-1}}w}{1-\overline{\alpha_k}w}\Big(b_{\alpha_{k-1}}(w)\gamma_{k-1}(w)\quad\delta_{k-1}^{[k-1]}(w)\Big)\tfrac{1}{\sqrt{1-|\kappa_k|^2}}\begin{pmatrix}1&\kappa_k\\\overline{\kappa_k}&1\end{pmatrix}\right)^*\\&\quad=-\tfrac{1-|\alpha_k|^2}{1-|\alpha_{k-1}|^2}\,\tfrac{1-\overline{\alpha_{k-1}}v}{1-\overline{\alpha_k}v}\,\tfrac{1-\alpha_{k-1}\overline{w}}{1-\alpha_k\overline{w}}\Big(b_{\alpha_{k-1}}(v)\gamma_{k-1}(v)\quad\delta_{k-1}^{[k-1]}(v)\Big)\begin{pmatrix}\overline{b_{\alpha_{k-1}}(w)\gamma_{k-1}(w)}\\-\overline{\delta_{k-1}^{[k-1]}(w)}\end{pmatrix}\\&\quad=\tfrac{1-|\alpha_k|^2}{1-|\alpha_{k-1}|^2}\,\tfrac{1-\overline{\alpha_{k-1}}v}{1-\overline{\alpha_k}v}\,\tfrac{1-\alpha_{k-1}\overline{w}}{1-\alpha_k\overline{w}}\big(\delta_{k-1}^{[k-1]}(v)\overline{\delta_{k-1}^{[k-1]}(w)}-b_{\alpha_{k-1}}(v)\gamma_{k-1}(v)\overline{b_{\alpha_{k-1}}(w)\gamma_{k-1}(w)}\,\big)\end{aligned}$$

and, hence, an application of (4.1) yields the first identity. Similarly, observing the second row (resp., the first and the second row) of $\mathbf{\Theta}_k$, in view of (3.3), (3.8), (3.9), and (4.1), one can obtain the second (resp., the third) identity. □

Theorem 4.2. *For $k \in \mathbb{I}$, the following Christoffel-Darboux formulae hold:*

$$\begin{aligned}&\big(1-b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}\,\big)\sum_{\ell=0}^{k}\gamma_\ell(v)\overline{\gamma_\ell(w)}\\&\quad=\delta_k^{[k]}(v)\overline{\delta_k^{[k]}(w)}-b_{\alpha_k}(v)\gamma_k(v)\overline{b_{\alpha_k}(w)\gamma_k(w)}+\frac{1-|\alpha_k|^2}{(1-\overline{\alpha_k}v)(1-\alpha_k\overline{w})},\end{aligned}$$

$$\begin{aligned}&\big(1-b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}\,\big)\sum_{\ell=0}^{k}\delta_\ell(v)\overline{\delta_\ell(w)}\\&=\gamma_k^{[k]}(v)\overline{\gamma_k^{[k]}(w)}-b_{\alpha_k}(v)\delta_k(v)\overline{b_{\alpha_k}(w)\delta_k(w)}-\frac{1-|\alpha_k|^2}{(1-\overline{\alpha_k}v)(1-\alpha_k\overline{w})},\end{aligned}$$

and

$$\big(1-b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}\,\big)\sum_{\ell=0}^{k}\gamma_\ell(v)\overline{\delta_\ell(w)}=\delta_k^{[k]}(v)\overline{\gamma_k^{[k]}(w)}-b_{\alpha_k}(v)\gamma_k(v)\overline{b_{\alpha_k}(w)\delta_k(w)}.$$

Proof. According to the definition, we have

$$\gamma_0(v)=\sqrt{\frac{1-|\alpha_0|^2}{1-|\kappa_0|^2}}\frac{1}{1-\overline{\alpha_0}v},\quad \delta_0(v)=\overline{\kappa_0}\sqrt{\frac{1-|\alpha_0|^2}{1-|\kappa_0|^2}}\frac{1}{1-\overline{\alpha_0}v}.$$

Hence, in view of (4.1),

$$\begin{aligned}&(1-|\alpha_0|^2)(1-v\overline{w})\\&=1-\overline{\alpha_0}v-\alpha_0\overline{w}+|\alpha_0|^2v\overline{w}-(|\alpha_0|^2-\overline{\alpha_0}v-\alpha_0\overline{w}+v\overline{w})\\&=(1-\overline{\alpha_0}v)(1-\alpha_0\overline{w})-(\alpha_0-v)(\overline{\alpha_0}-\overline{w}),\end{aligned}$$

(2.11), and (2.3), we get

$$\begin{aligned}&\big(1-b_{\alpha_0}(v)\overline{b_{\alpha_0}(w)}\,\big)\sum_{\ell=0}^{0}\gamma_\ell(v)\overline{\gamma_\ell(w)}\\&=\frac{(1-|\alpha_0|^2)(1-v\overline{w})}{(1-\overline{\alpha_0}v)(1-\alpha_0\overline{w})}\frac{1-|\alpha_0|^2}{1-|\kappa_0|^2}\frac{1}{(1-\overline{\alpha_0}v)(1-\alpha_0\overline{w})}\\&=\frac{(1-|\alpha_0|^2)(1-\overline{\alpha_0}v)(1-\alpha_0\overline{w})}{(1-|\kappa_0|^2)(1-\overline{\alpha_0}v)^2(1-\alpha_0\overline{w})^2}-\frac{(1-|\alpha_0|^2)(\alpha_0-v)(\overline{\alpha_0}-\overline{w})}{(1-|\kappa_0|^2)(1-\overline{\alpha_0}v)^2(1-\alpha_0\overline{w})^2}\qquad(4.2)\\&=\frac{(|\kappa_0|^2-|\kappa_0|^2+1)(1-|\alpha_0|^2)}{(1-|\kappa_0|^2)(1-\overline{\alpha_0}v)(1-\alpha_0\overline{w})}-\frac{(1-|\alpha_0|^2)(\alpha_0-v)(\overline{\alpha_0}-\overline{w})}{(1-|\kappa_0|^2)(1-\overline{\alpha_0}v)^2(1-\alpha_0\overline{w})^2}\\&=\delta_0^{[0]}(v)\overline{\delta_0^{[0]}(w)}-b_{\alpha_0}(v)\gamma_0(v)\overline{b_{\alpha_0}(w)\gamma_0(w)}+\frac{1-|\alpha_0|^2}{(1-\overline{\alpha_0}v)(1-\alpha_0\overline{w})}.\end{aligned}$$

Thus, for the case $k=0$ the first identity is verified. Now we assume that, for $k\in\mathbb{I}\setminus\{0\}$, the formula

$$\begin{aligned}&\big(1-b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(w)}\,\big)\sum_{\ell=0}^{k-1}\gamma_\ell(v)\overline{\gamma_\ell(w)}\\&=\delta_{k-1}^{[k-1]}(v)\overline{\delta_{k-1}^{[k-1]}(w)}-b_{\alpha_{k-1}}(v)\gamma_{k-1}(v)\overline{b_{\alpha_{k-1}}(w)\gamma_{k-1}(w)}+\frac{1-|\alpha_{k-1}|^2}{(1-\overline{\alpha_{k-1}}v)(1-\alpha_{k-1}\overline{w})}\end{aligned}$$

is already proved. Therefore, an application of the first equality in Lemma 4.1 and (4.1) implies

$$\begin{aligned}
&\big(1-b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}\,\big)\sum_{\ell=0}^{k}\gamma_\ell(v)\overline{\gamma_\ell(w)}\\
&=\frac{1-b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}}{1-b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(w)}}\big(1-b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(w)}\,\big)\sum_{\ell=0}^{k-1}\gamma_\ell(v)\overline{\gamma_\ell(w)}\\
&\qquad+\gamma_k(v)\overline{\gamma_k(w)}-b_{\alpha_k}(v)\gamma_k(v)\overline{b_{\alpha_k}(w)\gamma_k(w)}\\
&=\frac{1-b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}}{1-b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(w)}}\Big(\delta_{k-1}^{[k-1]}(v)\overline{\delta_{k-1}^{[k-1]}(w)}-b_{\alpha_{k-1}}(v)\gamma_{k-1}(v)\overline{b_{\alpha_{k-1}}(w)\gamma_{k-1}(w)}\Big)\\
&\qquad+\frac{\big(1-b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}\,\big)\big(1-|\alpha_{k-1}|^2\big)}{\big(1-b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(w)}\,\big)\big(1-\overline{\alpha_{k-1}}v\big)\big(1-\alpha_{k-1}\overline{w}\big)}\\
&\qquad+\gamma_k(v)\overline{\gamma_k(w)}-b_{\alpha_k}(v)\gamma_k(v)\overline{b_{\alpha_k}(w)\gamma_k(w)}\\
&=\delta_k^{[k]}(v)\overline{\delta_k^{[k]}(w)}-\gamma_k(v)\overline{\gamma_k(w)}+\frac{1-|\alpha_k|^2}{(1-\overline{\alpha_k}v)(1-\alpha_k\overline{w})}\\
&\qquad+\gamma_k(v)\overline{\gamma_k(w)}-b_{\alpha_k}(v)\gamma_k(v)\overline{b_{\alpha_k}(w)\gamma_k(w)}\\
&=\delta_k^{[k]}(v)\overline{\delta_k^{[k]}(w)}-b_{\alpha_k}(v)\gamma_k(v)\overline{b_{\alpha_k}(w)\gamma_k(w)}+\frac{1-|\alpha_k|^2}{(1-\overline{\alpha_k}v)(1-\alpha_k\overline{w})}.
\end{aligned}$$

Consequently, for each $k\in\mathbb{I}$, the first formula is inductively shown. Similarly, the second and third formulae can be verified by using Lemma 4.1 and (4.1). □

Obviously, the formulae in Theorem 4.2 can be restated as follows:

Corollary 4.3. *For $k\in\mathbb{I}\setminus\{0\}$, the following Christoffel-Darboux formulae hold:*

$$\big(1-b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}\,\big)\sum_{\ell=0}^{k-1}\gamma_\ell(v)\overline{\gamma_\ell(w)}=\delta_k^{[k]}(v)\overline{\delta_k^{[k]}(w)}-\gamma_k(v)\overline{\gamma_k(w)}+\frac{1-|\alpha_k|^2}{(1-\overline{\alpha_k}v)(1-\alpha_k\overline{w})},$$

$$\big(1-b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}\,\big)\sum_{\ell=0}^{k-1}\delta_\ell(v)\overline{\delta_\ell(w)}=\gamma_k^{[k]}(v)\overline{\gamma_k^{[k]}(w)}-\delta_k(v)\overline{\delta_k(w)}-\frac{1-|\alpha_k|^2}{(1-\overline{\alpha_k}v)(1-\alpha_k\overline{w})},$$

and

$$\big(1-b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}\,\big)\sum_{\ell=0}^{k-1}\gamma_\ell(v)\overline{\delta_\ell(w)}=\delta_k^{[k]}(v)\overline{\gamma_k^{[k]}(w)}-\gamma_k(v)\overline{\delta_k(w)}.$$

Using the same strategy as in the case of orthogonal rational functions (cf. [4, Corollary 3.1.4]), from the (first and second) Christoffel-Darboux formulae with $v=w$ one can conclude the following statement (cf. Corollary 3.6).

Corollary 4.4. *For $k\in\mathbb{I}\setminus\{0\}$, if $w\in\mathbb{C}$ with $|\kappa_0|>|b_{\alpha_0}(w)|$ then $\delta_k^{[k]}(w)\neq 0$ and*

$$\left|\frac{\gamma_k(w)}{\delta_k^{[k]}(w)}\right|<1$$

as well as for each $v\in\mathbb{D}\cup\mathbb{T}$ we have $\gamma_k^{[k]}(v)\neq 0$ and

$$\left|\frac{\delta_k(v)}{\gamma_k^{[k]}(v)}\right|<1.$$

Remark 4.5. Let $k \in \mathbb{I} \setminus \{0\}$. Since $x^{[k]}(\alpha_k) = 0 \iff x \in \mathfrak{H}_{k-1}$ for each $x \in \mathfrak{H}_k$, Corollary 4.4 includes particularly that $\gamma_k \in \mathfrak{H}_k \setminus \mathfrak{H}_{k-1}$ and that if $|\kappa_0| > |b_{\alpha_0}(\alpha_k)|$ then also $\delta_k \in \mathfrak{H}_k \setminus \mathfrak{H}_{k-1}$. Furthermore, the case $\delta_k \in \mathfrak{H}_{k-1}$ is possible in general (cf. Remark 3.8, Remark 3.9) and Theorem 4.2 implies

$$\begin{aligned}\delta_k \in \mathfrak{H}_{k-1} \quad &\iff \quad \sum_{\ell=0}^{k} \gamma_\ell(v)\overline{\gamma_\ell(\alpha_k)} = \frac{1}{1-\overline{\alpha_k}v} \quad \iff \quad \sum_{\ell=0}^{k} |\gamma_\ell(\alpha_k)|^2 = \frac{1}{1-|\alpha_k|^2} \\ &\iff \quad \sum_{\ell=0}^{k} \gamma_\ell(v)\overline{\delta_\ell(\alpha_k)} = 0 \quad \iff \quad \sum_{\ell=0}^{k} \gamma_\ell(\alpha_k)\overline{\delta_\ell(\alpha_k)} = 0.\end{aligned}$$

5. A characterization of Schur-Nevanlinna sequences

In the previous section (see, e.g., Theorem 4.2), we have explained that Schur-Nevanlinna sequences of rational functions fulfill some Christoffel-Darboux formulae. Referring to this, in the present section we study now an inverse problem. Roughly speaking, we shall see that (similar as in the case of orthogonal rational functions) the realization of Christoffel-Darboux formulae is in a way also a sufficient condition for rational functions to be a Schur-Nevanlinna pair.

Remark 5.1. Let $\tau \in \mathbb{N}$ or $\tau = \infty$, let $\mathbb{I} := \{0, 1, 2, \dots, \tau\}$, and let $(\alpha_k)_{k \in \mathbb{I}}$ be a sequence of points belonging to $\mathbb{D}$. Further, let $k \in \mathbb{I} \setminus \{0\}$ and let $\gamma_\ell, \delta_\ell \in \mathfrak{H}_\ell$, $\ell = 0, 1, \dots, k$. Clearly, the following statements are equivalent:

(i) The first (resp., second or third) identity of Theorem 4.2 is satisfied.
(ii) The first (resp., second or third) identity of Corollary 4.3 is satisfied.

Lemma 5.2. *Let $\tau \in \mathbb{N}$ or $\tau = \infty$, let $\mathbb{I} := \{0, 1, 2, \dots, \tau\}$, and let $(\alpha_k)_{k \in \mathbb{I}}$ be a sequence of points belonging to $\mathbb{D}$. Further, let $k \in \mathbb{I} \setminus \{0\}$ and let $\gamma_k, \delta_k \in \mathfrak{H}_k$ as well as $\gamma_{k-1}, \delta_{k-1} \in \mathfrak{H}_{k-1}$. The following statements are equivalent:*

(i) *The first identity of* Lemma 4.1 *is satisfied.*
(ii) *The second identity of* Lemma 4.1 *is satisfied.*

Proof. If we fix the complex number w then, in view of (2.11) and forming the adjoint with respect to the $k+2$ points $\alpha_0, \alpha_1, \dots, \alpha_k, \alpha_{k-1}$, the first identity of Lemma 4.1 is equal to

$$\begin{aligned}&\big(b_{\alpha_{k-1}}(v) - b_{\alpha_{k-1}}(w)\big)\Big(\delta_k(v)\delta_k^{[k]}(w) - \gamma_k^{[k]}(v)\gamma_k(w)\Big) \\ &\quad = \big(b_{\alpha_k}(v) - b_{\alpha_k}(w)\big)\Big(b_{\alpha_{k-1}}(v)\delta_{k-1}(v)\delta_{k-1}^{[k-1]}(w) - \gamma_{k-1}^{[k-1]}(v)b_{\alpha_{k-1}}(w)\gamma_{k-1}(w)\Big).\end{aligned}$$

Since, by fixing now the point v and adjoining, this relation is equal to

$$\begin{aligned}&\big(\overline{b_{\alpha_{k-1}}(v)}b_{\alpha_{k-1}}(w) - 1\big)\Big(\overline{\delta_k(v)}\delta_k(w) - \overline{\gamma_k^{[k]}(v)}\gamma_k^{[k]}(w)\Big) \\ &\quad = \big(\overline{b_{\alpha_k}(v)}b_{\alpha_k}(w) - 1\big)\Big(\overline{b_{\alpha_{k-1}}(v)\delta_{k-1}(v)}b_{\alpha_{k-1}}(w)\delta_{k-1}(w) - \overline{\gamma_{k-1}^{[k-1]}(v)}\gamma_{k-1}^{[k-1]}(w)\Big),\end{aligned}$$

we obtain finally the equivalence of (i) and (ii). □

Lemma 5.3. *Let $\tau \in \mathbb{N}$ or $\tau = \infty$, let $\mathbb{I} := \{0, 1, 2, \dots, \tau\}$, and let $(\alpha_k)_{k \in \mathbb{I}}$ be a sequence of points belonging to $\mathbb{D}$. Further, let $(\gamma_k)_{k \in \mathbb{I}}$ and $(\delta_k)_{k \in \mathbb{I}}$ be sequences of rational functions such that γ_0, δ_0 are given as in (3.1) for some $\kappa_0 \in \mathbb{D}$ and that γ_k, δ_k belong to $\mathfrak{H}_k$, $k \in \mathbb{I} \setminus \{0\}$. The following statements are equivalent:*

(i) *For each* $k \in \mathbb{I}\setminus\{0\}$*, the first* (*resp., second or third*) *identity of* Lemma 4.1 *is fulfilled.*

(ii) *For each* $k \in \mathbb{I}\setminus\{0\}$*, the first* (*resp., second or third*) *identity of* Theorem 4.2 *is fulfilled.*

Proof. By using the same arguments as in the proof of Theorem 4.2, one can inductively show that (i) implies (ii). It remains to verify that (ii) implicates also (i). For each $k \in \mathbb{I}\setminus\{0\}$, from (4.1), Remark 5.1, the first identity of Corollary 4.3, (4.2), and the first identity of Theorem 4.2 it follows

$$\begin{aligned}
&\big(1 - b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(w)}\,\big)\Big(\delta_k^{[k]}(v)\overline{\delta_k^{[k]}(w)} - \gamma_k(v)\overline{\gamma_k(w)}\Big)\\
&= \big(1 - b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(w)}\,\big)\Big(\delta_k^{[k]}(v)\overline{\delta_k^{[k]}(w)} - \gamma_k(v)\overline{\gamma_k(w)} + \frac{1-|\alpha_k|^2}{(1-\overline{\alpha_k}v)(1-\alpha_k\overline{w})}\Big)\\
&\quad - \frac{(1-|\alpha_k|^2)(1-|\alpha_{k-1}|^2)(1-v\overline{w})}{(1-\overline{\alpha_k}v)(1-\alpha_k\overline{w})(1-\overline{\alpha_{k-1}}v)(1-\alpha_{k-1}\overline{w})}\\
&= \big(1 - b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(w)}\,\big)\big(1 - b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}\,\big)\sum_{\ell=0}^{k-1}\gamma_\ell(v)\overline{\gamma_\ell(w)}\\
&\quad - \frac{(1-|\alpha_k|^2)(1-|\alpha_{k-1}|^2)(1-v\overline{w})}{(1-\overline{\alpha_k}v)(1-\alpha_k\overline{w})(1-\overline{\alpha_{k-1}}v)(1-\alpha_{k-1}\overline{w})}\\
&= \big(1 - b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}\,\big)\Big(\delta_{k-1}^{[k-1]}(v)\overline{\delta_{k-1}^{[k-1]}(w)} - b_{\alpha_{k-1}}(v)\gamma_{k-1}(v)\overline{b_{\alpha_{k-1}}(w)\gamma_{k-1}(w)}\Big).
\end{aligned}$$

Consequently, with respect to the first kind of identities it is shown that (ii) yields (i). Similarly, one can prove by a straightforward calculation that the implication referring to the second (resp., third) kind of identities is fulfilled. □

Theorem 5.4. *Let* $\tau = 0$ *or* $\tau \in \mathbb{N}$ *or* $\tau = \infty$*, let* $\mathbb{I} := \{0, 1, 2, \ldots, \tau\}$*, and let* $(\alpha_k)_{k\in\mathbb{I}}$ *be a sequence of points belonging to* $\mathbb{D}$*. Further, for each* $k \in \mathbb{I}$*, let* γ_k*,* δ_k *be rational functions belonging to* $\mathfrak{H}_k$ *such that the following three conditions are fulfilled:*

(I) *The first* (*resp., second*) *identity of* Theorem 4.2 *is fulfilled.*

(II) *The third identity of* Theorem 4.2 *is fulfilled.*

(III) $\arg\Big[\gamma_k^{[k]}(\alpha_{k-1})\Big] = \arg\Big[\frac{\eta_k\overline{\eta_{k-1}}}{1-\overline{\alpha_k}\alpha_{k-1}}\gamma_{k-1}^{[k-1]}(\alpha_{k-1})\Big]$*, where* $\alpha_{-1} := 0$*,* $\eta_{-1} := -1$*, and* $\gamma_{-1}^{[-1]}(\alpha_{-1}) := 1$.

The relation $\gamma_k^{[k]}(\alpha_{k-1}) \neq 0$ *holds and if we put*

$$\kappa_k := \overline{\eta_k}\eta_{k-1}\frac{\overline{\delta_k(\alpha_{k-1})}}{\overline{\gamma_k^{[k]}(\alpha_{k-1})}}, \quad k \in \mathbb{I},$$

then $(\kappa_k)_{k\in\mathbb{I}}$ *is a sequence of points belonging to* $\mathbb{D}$ *and* $[(\gamma_k)_{k\in\mathbb{I}}, (\delta_k)_{k\in\mathbb{I}}]$ *is the Schur-Nevanlinna pair of rational functions corresponding to* $(\alpha_k, \kappa_k)_{k\in\mathbb{I}}$.

Proof. First, we consider the case $k = 0$. From (I) and (II) we get

$$\gamma_0(v)\overline{\gamma_0(w)} = \delta_0^{[0]}(v)\overline{\delta_0^{[0]}(w)} + \frac{1-|\alpha_0|^2}{(1-\overline{\alpha_0}v)(1-\alpha_0\overline{w})} \tag{5.1}$$

$$\Big(\text{resp.,}\ \delta_0(v)\overline{\delta_0(w)} = \gamma_0^{[0]}(v)\overline{\gamma_0^{[0]}(w)} - \frac{1-|\alpha_0|^2}{(1-\overline{\alpha_0}v)(1-\alpha_0\overline{w})}\Big) \tag{5.2}$$

and

$$\gamma_0(v)\overline{\delta_0(w)} = \delta_0^{[0]}(v)\overline{\gamma_0^{[0]}(w)}. \tag{5.3}$$

Since (5.1) (resp., (5.2)) implies $|\gamma_0(v)|^2 \neq 0$ (resp., $|\gamma_0^{[0]}(v)|^2 \neq 0$), in view of (2.11) it follows at any rate

$$\gamma_0^{[0]}(0) \neq 0$$

and, hence, (5.3) and the definition of κ_0 yield

$$\delta_0^{[0]}(v) = -\eta_0\kappa_0\gamma_0(v).$$

Thus, from (5.1) (resp., (5.2)), $\gamma_0 \in \mathfrak{H}_0$, (2.11), and (III), i.e. $-\overline{\eta_0}\gamma_0^{[0]}(0) \in [0,\infty)$, one can finally conclude that κ_0 belongs to $\mathbb{D}$ and that (3.1) is fulfilled. Particularly for the case $\tau = 0$ it is shown that $[(\gamma_k)_{k\in\mathbb{I}}, (\delta_k)_{k\in\mathbb{I}}]$ is the Schur-Nevanlinna pair of rational functions corresponding to $(\alpha_k, \kappa_k)_{k\in\mathbb{I}}$.

Now let $\tau \in \mathbb{N}$ or $\tau = \infty$ and let $k \in \mathbb{I} \setminus \{0\}$. Because of (I), Lemma 5.3, and Lemma 5.2 we obtain (cf. Corollary 4.4) that

$$\gamma_k^{[k]}(v) \neq 0, \quad v \in \mathbb{D} \cup \mathbb{T},$$

and, by using $b_{\alpha_{k-1}}(\alpha_{k-1}) = 0$, that

$$\begin{aligned}
&\gamma_k^{[k]}(v)\overline{\gamma_k^{[k]}(\alpha_{k-1})} - \delta_k(v)\overline{\delta_k(\alpha_{k-1})} \\
&= \left(1 - b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(\alpha_{k-1})}\right)\left(\gamma_k^{[k]}(v)\overline{\gamma_k^{[k]}(\alpha_{k-1})} - \delta_k(v)\overline{\delta_k(\alpha_{k-1})}\right) \\
&= \left(1 - b_{\alpha_k}(v)\overline{b_{\alpha_k}(\alpha_{k-1})}\right)\gamma_{k-1}^{[k-1]}(v)\overline{\gamma_{k-1}^{[k-1]}(\alpha_{k-1})}.
\end{aligned} \tag{5.4}$$

In particular, we have

$$\gamma_k^{[k]}(\alpha_{k-1}) \neq 0$$

as well as, by the choice of κ_k, (4.1), and (III), therefore $\kappa_k \in \mathbb{D}$ and

$$\begin{aligned}
\sqrt{1-|\kappa_k|^2} &= \sqrt{1 - \left|\frac{\delta_k(\alpha_{k-1})}{\gamma_k^{[k]}(\alpha_{k-1})}\right|^2} \\
&= \frac{\sqrt{(1-|\alpha_k|^2)(1-|\alpha_{k-1}|^2)}}{|1-\overline{\alpha_k}\alpha_{k-1}|}\left|\frac{\gamma_{k-1}^{[k-1]}(\alpha_{k-1})}{\gamma_k^{[k]}(\alpha_{k-1})}\right| \\
&= \eta_k\overline{\eta_{k-1}}\frac{\sqrt{(1-|\alpha_k|^2)(1-|\alpha_{k-1}|^2)}}{1-\overline{\alpha_k}\alpha_{k-1}}\frac{\gamma_{k-1}^{[k-1]}(\alpha_{k-1})}{\gamma_k^{[k]}(\alpha_{k-1})}.
\end{aligned} \tag{5.5}$$

In view of (2.11), the relation (5.4) implies

$$\gamma_k(v)\gamma_k^{[k]}(\alpha_{k-1}) - \delta_k^{[k]}(v)\delta_k(\alpha_{k-1}) = \left(b_{\alpha_k}(v) - b_{\alpha_k}(\alpha_{k-1})\right)\gamma_{k-1}(v)\gamma_{k-1}^{[k-1]}(\alpha_{k-1}).$$

Consequently, an application of (5.5) yields

$$\begin{aligned}
\eta_k\overline{\eta_{k-1}}\gamma_k(v) - \overline{\kappa_k}\delta_k^{[k]}(v) &= \eta_k\overline{\eta_{k-1}}\gamma_k(v) - \eta_k\overline{\eta_{k-1}}\frac{\delta_k(\alpha_{k-1})}{\gamma_k^{[k]}(\alpha_{k-1})}\delta_k^{[k]}(v) \\
&= \eta_k\overline{\eta_{k-1}}\frac{\gamma_{k-1}^{[k-1]}(\alpha_{k-1})}{\gamma_k^{[k]}(\alpha_{k-1})}\left(b_{\alpha_k}(v) - b_{\alpha_k}(\alpha_{k-1})\right)\gamma_{k-1}(v) \\
&= \frac{(1-\overline{\alpha_k}\alpha_{k-1})\sqrt{1-|\kappa_k|^2}}{\sqrt{(1-|\alpha_k|^2)(1-|\alpha_{k-1}|^2)}}\left(b_{\alpha_k}(v) - b_{\alpha_k}(\alpha_{k-1})\right)\gamma_{k-1}(v).
\end{aligned} \tag{5.6}$$

Furthermore, from (II), Lemma 5.3, and $b_{\alpha_{k-1}}(\alpha_{k-1}) = 0$ it follows

$$\begin{aligned}
&\delta_k^{[k]}(v)\overline{\gamma_k^{[k]}(\alpha_{k-1})} - \gamma_k(v)\overline{\delta_k(\alpha_{k-1})} \\
&\quad = \big(1 - b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(\alpha_{k-1})}\big)\Big(\delta_k^{[k]}(v)\overline{\gamma_k^{[k]}(\alpha_{k-1})} - \gamma_k(v)\overline{\delta_k(\alpha_{k-1})}\Big) \\
&\quad = \big(1 - b_{\alpha_k}(v)\overline{b_{\alpha_k}(\alpha_{k-1})}\big)\delta_{k-1}^{[k-1]}(v)\overline{\gamma_{k-1}^{[k-1]}(\alpha_{k-1})}.
\end{aligned}$$

By forming the adjoint rational functions, we get

$$\delta_k(v)\gamma_k^{[k]}(\alpha_{k-1}) - \gamma_k^{[k]}(v)\delta_k(\alpha_{k-1}) = \big(b_{\alpha_k}(v) - b_{\alpha_k}(\alpha_{k-1})\big)\delta_{k-1}(v)\gamma_{k-1}^{[k-1]}(\alpha_{k-1}).$$

Accordingly, the equality (5.5) provides

$$\begin{aligned}
\eta_k\overline{\eta_{k-1}}\delta_k(v) - \overline{\kappa_k}\gamma_k^{[k]}(v) &= \eta_k\overline{\eta_{k-1}}\delta_k(v) - \eta_k\overline{\eta_{k-1}}\frac{\delta_k(\alpha_{k-1})}{\gamma_k^{[k]}(\alpha_{k-1})}\gamma_k^{[k]}(v) \\
&= \eta_k\overline{\eta_{k-1}}\frac{\gamma_{k-1}^{[k-1]}(\alpha_{k-1})}{\gamma_k^{[k]}(\alpha_{k-1})}\big(b_{\alpha_k}(v) - b_{\alpha_k}(\alpha_{k-1})\big)\delta_{k-1}(v) \qquad (5.7)\\
&= \frac{(1-\overline{\alpha_k}\alpha_{k-1})\sqrt{1-|\kappa_k|^2}}{\sqrt{(1-|\alpha_k|^2)(1-|\alpha_{k-1}|^2)}}\big(b_{\alpha_k}(v) - b_{\alpha_k}(\alpha_{k-1})\big)\delta_{k-1}(v).
\end{aligned}$$

In the end, by virtue of (5.6), (5.7), and Proposition 3.1 one can conclude that $[(\gamma_k)_{k\in\mathbb{I}}, (\delta_k)_{k\in\mathbb{I}}]$ is the Schur-Nevanlinna pair of rational functions corresponding to $(\alpha_k, \kappa_k)_{k\in\mathbb{I}}$. □

6. Solution of Problem (MNP) in the non-uniqueness case

In this section, we shall show that the set $\mathcal{S}_\Delta$ of solutions of Problem (MNP) can be parameterized by the linear fractional transformation (1.2), where γ_m and δ_m are some elements of a Schur-Nevanlinna pair of rational functions with m defined as in (2.2), if a data set Δ (as in (2.1)) is given such that $\mathbf{P}_\Delta > 0$.

With the points $z_1, z_2, \dots, z_n$ in Δ we form here a sequence $(\alpha_k)_{k=0}^m$ in which z_j appears according to its multiplicity l_j. For instance, we can choose $\alpha_k := \beta_k$ with

$$\beta_k := z_j \quad \text{if} \quad \sum_{r=1}^{j-1} l_r \le k < \sum_{r=1}^{j} l_r, \quad j = 1, 2, \dots, n.$$

But in fact, the order that equal points are successors is not essential in the following, i.e. for an arbitrary bijective mapping p of $\{0, 1, \dots, m\}$ onto itself we can put

$$\alpha_k := \beta_{p(k)}, \quad k = 0, 1, \dots, m. \tag{6.1}$$

In the sequel, then $[(\gamma_k)_{k\in\mathbb{I}}, (\delta_k)_{k\in\mathbb{I}}]$ stands for the Schur-Nevanlinna pair of rational functions corresponding to $(\alpha_k, \kappa_k)_{k\in\mathbb{I}}$, where $\mathbb{I} := \{0, 1, \dots, m\}$ and $(\kappa_k)_{k\in\mathbb{I}}$ is a certain sequence of points belonging to $\mathbb{D}$.

Note that, Corollary 3.6 implies, for all $h \in \mathcal{S}$,

$$\gamma_m^{[m]}(z) + b_{\alpha_m}(z)\delta_m(z)h(z) \neq 0, \quad z \in \mathbb{D}, \tag{6.2}$$

and, moreover, that the function g_0:

$$g_0(z) := \frac{\delta_m^{[m]}(z)}{\gamma_m^{[m]}(z)}, \quad z \in \mathbb{D}, \tag{6.3}$$

belongs to $\mathcal{S}$.

Lemma 6.1. *For each $h \in \mathcal{S}$, the function g:*

$$g(z) := \frac{\delta_m^{[m]}(z) + b_{\alpha_m}(z)\gamma_m(z)h(z)}{\gamma_m^{[m]}(z) + b_{\alpha_m}(z)\delta_m(z)h(z)}, \quad z \in \mathbb{D}, \tag{6.4}$$

belongs to $\mathcal{S}$ and $\Delta_g = \Delta_{g_0}$.

Proof. Let $z \in \mathbb{D}$. In view of (2.6), with

$$\mathbf{\Theta}(z) := \frac{1-\overline{\alpha_m}z}{\sqrt{1-|\alpha_m|^2}} \begin{pmatrix} b_{\alpha_m}(z)\gamma_m(z) & \delta_m^{[m]}(z) \\ b_{\alpha_m}(z)\delta_m(z) & \gamma_m^{[m]}(z) \end{pmatrix}$$

the relation (6.4) can be written as

$$g(z) = T_{\mathbf{\Theta}(z)}\big(h(z)\big).$$

From Theorem 3.3 we see that the matrix $\mathbf{\Theta}(z)$ is $\mathbf{J}$-contractive. Therefore, $\mathbf{\Theta}$ is a $\mathbf{J}$-contractive holomorphic matrix function and, as a well-known result on linear fractional transformations (cf. [9, Theorem 1.6.1]), $T_{\mathbf{\Theta}}$ maps the class $\mathcal{S}$ into the class $\mathcal{S}$. Hence, it follows that according to (6.4) we have $g \in \mathcal{S}$ if $h \in \mathcal{S}$.
Now let the function g_0 be defined as in (6.3). Consequently, by virtue of (6.2), (6.4), and Corollary 3.4 we obtain that

$$\begin{aligned} g(z) - g_0(z) &= \frac{\delta_m^{[m]}(z) + b_{\alpha_m}(z)\gamma_m(z)h(z)}{\gamma_m^{[m]}(z) + b_{\alpha_m}(z)\delta_m(z)h(z)} - \frac{\delta_m^{[m]}(z)}{\gamma_m^{[m]}(z)} \\ &= \frac{b_{\alpha_m}(z)h(z)\big(\gamma_m(z)\gamma_m^{[m]}(z) - \delta_m(z)\delta_m^{[m]}(z)\big)}{\big(\gamma_m^{[m]}(z) + b_{\alpha_m}(z)\delta_m(z)h(z)\big)\gamma_m^{[m]}(z)} \\ &= \frac{-\eta_m(1-|\alpha_m|^2)h(z)}{(1-\overline{\alpha_m}z)^2\big(\gamma_m^{[m]}(z) + b_{\alpha_m}(z)\delta_m(z)h(z)\big)\gamma_m^{[m]}(z)} B_m(z). \end{aligned} \tag{6.5}$$

Since the Blaschke product B_m has a zero of order l_j at the point z_j, $j = 1, 2, \dots, n$, one can finally conclude

$$g^{(s)}(z_j) = g_0^{(s)}(z_j), \quad s = 0, 1, \dots, l_j - 1,\ j = 1, 2, \dots, n,$$

i.e. $\Delta_g = \Delta_{g_0}$. □

Lemma 6.2. *If $[(\widetilde{\gamma}_k)_{k\in\mathbb{I}}, (\widetilde{\delta}_k)_{k\in\mathbb{I}}]$ is a Schur-Nevanlinna pair of rational functions corresponding to $(\alpha_k, \widetilde{\kappa}_k)_{k\in\mathbb{I}}$ with some $\widetilde{\kappa}_k \in \mathbb{D}$, $k \in \mathbb{I}$, such that $\Delta_{\widetilde{g}_0} = \Delta_{g_0}$, where g_0 is defined as in (6.3) and the function $\widetilde{g}_0$ similarly by*

$$\widetilde{g}_0(z) := \frac{\widetilde{\delta}_m^{[m]}(z)}{\widetilde{\gamma}_m^{[m]}(z)}, \quad z \in \mathbb{D}, \tag{6.6}$$

then for each $k \in \mathbb{I}$ the equality $\widetilde{\kappa}_k = \kappa_k$ holds, i.e. $[(\widetilde{\gamma}_k)_{k\in\mathbb{I}}, (\widetilde{\delta}_k)_{k\in\mathbb{I}}]$ is the Schur-Nevanlinna pair of rational functions corresponding to $(\alpha_k, \kappa_k)_{k\in\mathbb{I}}$.

Proof. Since (3.1) and (2.11) imply

$$\widetilde{\kappa}_0 = \frac{\widetilde{\delta}_0^{[0]}(z)}{\widetilde{\gamma}_0^{[0]}(z)}, \quad \kappa_0 = \frac{\delta_0^{[0]}(z)}{\gamma_0^{[0]}(z)}, \tag{6.7}$$

in the case $m = 0$ it follows evidently $\widetilde{\kappa}_0 = \kappa_0$, i.e. that $[(\widetilde{\gamma}_k)_{k\in\mathbb{I}}, (\widetilde{\delta}_k)_{k\in\mathbb{I}}]$ is the Schur-Nevanlinna pair of rational functions corresponding to $(\alpha_k, \kappa_k)_{k\in\mathbb{I}}$.
Now let $m > 0$. In view of the recursions defining a Schur-Nevanlinna pair of rational functions, (2.11), and $\Delta_{\widetilde{g}_0} = \Delta_{g_0}$ we obtain that the values of the functions

$$\frac{\widetilde{\delta}^{[m]}_{m-1}(z) + b_{\alpha_{m-1}}(z)\widetilde{\gamma}_{m-1}(z)\widetilde{\kappa}_m}{\widetilde{\gamma}^{[m]}_{m-1}(z) + b_{\alpha_{m-1}}(z)\widetilde{\delta}_{m-1}(z)\widetilde{\kappa}_m}, \quad \frac{\delta^{[m]}_{m-1}(z) + b_{\alpha_{m-1}}(z)\gamma_{m-1}(z)\kappa_m}{\gamma^{[m]}_{m-1}(z) + b_{\alpha_{m-1}}(z)\delta_{m-1}(z)\kappa_m}, \quad z \in \mathbb{D},$$

and their derivatives up to the order $l_j - 1$ at the points z_j, $j = 1, 2, \dots, n$ coincide. Because of Lemma 6.1, a successive continuation of this procedure yields that, for each $k \in \mathbb{I} \setminus \{0\}$, the values of the functions

$$\frac{\widetilde{\delta}^{[k-1]}_{k-1}(z) + b_{\alpha_{k-1}}(z)\widetilde{\gamma}_{k-1}(z)\widetilde{\kappa}_k}{\widetilde{\gamma}^{[k-1]}_{k-1}(z) + b_{\alpha_{k-1}}(z)\widetilde{\delta}_{k-1}(z)\widetilde{\kappa}_k}, \quad \frac{\delta^{[k-1]}_{k-1}(z) + b_{\alpha_{k-1}}(z)\gamma_{k-1}(z)\kappa_k}{\gamma^{[k-1]}_{k-1}(z) + b_{\alpha_{k-1}}(z)\delta_{k-1}(z)\kappa_k}, \quad z \in \mathbb{D}, \tag{6.8}$$

and their derivatives up to the order $r_j - 1$ at the points z_j contained in the sequence $(\alpha_\ell)_{\ell=0}^k$ (where r_j stands for the number how many times) coincide and, in particular, that

$$\frac{\widetilde{\delta}^{[0]}_0(\alpha_0)}{\widetilde{\gamma}^{[0]}_0(\alpha_0)} = \frac{\delta^{[0]}_0(\alpha_0)}{\gamma^{[0]}_0(\alpha_0)}. \tag{6.9}$$

In the following, by induction on k, we verify that $\widetilde{\kappa}_k = \kappa_k$, $k \in \mathbb{I}$. For $k = 0$, the equalities (6.7) and (6.9) supply immediately

$$\widetilde{\kappa}_0 = \kappa_0.$$

Now let $k \in \mathbb{I}\setminus\{0\}$ and we assume that $\widetilde{\kappa}_\ell = \kappa_\ell$, $\ell = 0, 1, \dots, k-1$. In view of (6.8) and the induction assumption we obtain that the values of the functions

$$\frac{\delta^{[k-1]}_{k-1}(z) + b_{\alpha_{k-1}}(z)\gamma_{k-1}(z)\widetilde{\kappa}_k}{\gamma^{[k-1]}_{k-1}(z) + b_{\alpha_{k-1}}(z)\delta_{k-1}(z)\widetilde{\kappa}_k}, \quad \frac{\delta^{[k-1]}_{k-1}(z) + b_{\alpha_{k-1}}(z)\gamma_{k-1}(z)\kappa_k}{\gamma^{[k-1]}_{k-1}(z) + b_{\alpha_{k-1}}(z)\delta_{k-1}(z)\kappa_k}, \quad z \in \mathbb{D},$$

and their derivatives up to the order $r_j - 1$ at the points z_j contained in the sequence $(\alpha_\ell)_{\ell=0}^k$ (where r_j stands again for the number how many times) coincide on the one hand and on the other hand Corollary 3.4 provides

$$\begin{aligned}
&\frac{\delta^{[k-1]}_{k-1}(z) + b_{\alpha_{k-1}}(z)\gamma_{k-1}(z)\widetilde{\kappa}_k}{\gamma^{[k-1]}_{k-1}(z) + b_{\alpha_{k-1}}(z)\delta_{k-1}(z)\widetilde{\kappa}_k} - \frac{\delta^{[k-1]}_{k-1}(z) + b_{\alpha_{k-1}}(z)\gamma_{k-1}(z)\kappa_k}{\gamma^{[k-1]}_{k-1}(z) + b_{\alpha_{k-1}}(z)\delta_{k-1}(z)\kappa_k} \\
&\quad = \frac{\big(\widetilde{\kappa}_k - \kappa_k\big) b_{\alpha_{k-1}}(z)\big(\gamma_{k-1}(z)\gamma^{[k-1]}_{k-1}(z) - \delta_{k-1}(z)\delta^{[k-1]}_{k-1}(z)\big)}{\big(\gamma^{[k-1]}_{k-1}(z) + b_{\alpha_{k-1}}(z)\delta_{k-1}(z)\widetilde{\kappa}_k\big)\big(\gamma^{[k-1]}_{k-1}(z) + b_{\alpha_{k-1}}(z)\delta_{k-1}(z)\kappa_k\big)} \\
&\quad = \frac{-\big(\widetilde{\kappa}_k - \kappa_k\big)\eta_{k-1}\big(1 - |\alpha_{k-1}|^2\big)B_{k-1}(z)}{\big(1 - \overline{\alpha_{k-1}}z\big)^2\big(\gamma^{[k-1]}_{k-1}(z) + b_{\alpha_{k-1}}(z)\delta_{k-1}(z)\widetilde{\kappa}_k\big)\big(\gamma^{[k-1]}_{k-1}(z) + b_{\alpha_{k-1}}(z)\delta_{k-1}(z)\kappa_k\big)}.
\end{aligned}$$

Since $\eta_{k-1}(1 - |\alpha_{k-1}|^2) \neq 0$ and since the Blaschke product B_{k-1} has only k zeros (at the points $\alpha_0, \alpha_1, \dots, \alpha_{k-1}$), one can finally conclude $\widetilde{\kappa}_k = \kappa_k$. Thus, for each $k \in \mathbb{I}$ the identity $\widetilde{\kappa}_k = \kappa_k$ is satisfied, i.e. $[(\widetilde{\gamma}_k)_{k\in\mathbb{I}}, (\widetilde{\delta}_k)_{k\in\mathbb{I}}]$ is the Schur-Nevanlinna pair of rational functions corresponding to $(\alpha_k, \kappa_k)_{k\in\mathbb{I}}$. □

Now we are able to prove the main result of this paper, i.e. that in the non-uniqueness case the set $\mathcal{S}_\Delta$ of all solutions of Problem (MNP) is given by a linear fractional transformation of the form stated in (1.2).

Theorem 6.3. *Let Δ be a data set as in* (2.1) *whereby $\mathbf{P}_\Delta > 0$. Further, let $g_\bullet \in \mathcal{S}_\Delta$, let $(\alpha_k)_{k=0}^m$ be given as in* (6.1), *and let $(s_k)_{k=0}^m$ be the sequence of Schur parameters associated with $[g_\bullet, (\alpha_k)_{k=0}^m]$. If $g \in \mathcal{S}$ then the following statements are equivalent:*

(i) *$g \in \mathcal{S}_\Delta$.*
(ii) *$(s_k)_{k=0}^m$ is the sequence of Schur parameters associated with $[g, (\alpha_k)_{k=0}^m]$.*

Moreover, if we put $\mathbb{I} := \{0, 1, \dots, m\}$, $\kappa_0 := s_0$, and $\kappa_k := -s_k \eta_{k-1}$, $k = 1, 2, \dots, m$, and if $[(\gamma_k)_{k\in\mathbb{I}}, (\delta_k)_{k\in\mathbb{I}}]$ stands for the Schur-Nevanlinna pair of rational functions corresponding to $(\alpha_k, \kappa_k)_{k\in\mathbb{I}}$ then the relation

$$g(z) = \frac{\delta_m^{[m]}(z) + b_{\alpha_m}(z)\gamma_m(z)h(z)}{\gamma_m^{[m]}(z) + b_{\alpha_m}(z)\delta_m(z)h(z)}, \quad z \in \mathbb{D},$$

establishes a bijective correspondence between the set $\mathcal{S}_\Delta$ of all solutions g of Problem (MNP) *and the class $\mathcal{S}$ of all Schur functions h.*

Proof. Let $z \in \mathbb{D}$. Note that, in view of $\mathbf{P}_\Delta > 0$ and Theorem 2.3, for any solution of Problem (MNP) the Schur-Nevanlinna algorithm can be carried out (at least) $m + 1$ times. Consequently, we can always suppose in the following a given Schur function g for which the Schur-Nevanlinna algorithm can be carried out (at least) $m + 1$ times. In particular (cf. (2.4) and (2.10)), we find Schur parameters $\widetilde{s}_0, \widetilde{s}_1, \dots, \widetilde{s}_m \in \mathbb{D}$ associated with the pair $[g, (\alpha_k)_{k=0}^m]$ and a unique Schur function h_{m+1} such that the relation

$$g(z) = T_{\mathbf{\Xi}(z)}\big(h_{m+1}(z)\big) \tag{6.10}$$

is satisfied, where

$$\mathbf{\Xi}(z) := \overset{\curvearrowright}{\prod_{k=0}^{m}} \widetilde{\mathbf{\Xi}}_k(z) \quad \Big(:= \widetilde{\mathbf{\Xi}}_0(z)\widetilde{\mathbf{\Xi}}_1(z)\cdots\widetilde{\mathbf{\Xi}}_m(z)\Big)$$

and where

$$\widetilde{\mathbf{\Xi}}_k(z) := \begin{pmatrix} b_{\alpha_k}(z) & \widetilde{s}_k \\ \overline{\widetilde{s}_k} b_{\alpha_k}(z) & 1 \end{pmatrix}, \quad k \in \mathbb{I}.$$

By virtue of (2.9) and (3.5), with $\eta_{-1} := -1$ one can also write

$$\mathbf{\Xi}(z) = \left(\overset{\curvearrowright}{\prod_{k=0}^{m}} \begin{pmatrix} 1 & -\widetilde{s}_k\eta_{k-1} \\ -\overline{\widetilde{s}_k\eta_{k-1}} & 1 \end{pmatrix} \begin{pmatrix} b_{\alpha_k}(z) & 0 \\ 0 & \eta_k\overline{\eta_{k-1}} \end{pmatrix} \right) \begin{pmatrix} 1 & 0 \\ 0 & -\overline{\eta_m} \end{pmatrix}. \tag{6.11}$$

According to Section 3, we define now $[(\widetilde{\gamma}_k)_{k\in\mathbb{I}}, (\widetilde{\delta}_k)_{k\in\mathbb{I}}]$ as the Schur-Nevanlinna pair of rational functions corresponding to $(\alpha_k, \widetilde{\kappa}_k)_{k\in\mathbb{I}}$ with $\widetilde{\kappa}_k := -\widetilde{s}_k\eta_{k-1}$, $k \in \mathbb{I}$. Thus, setting

$$\widetilde{\mathbf{\Theta}}_k(v) := \frac{1}{\sqrt{1 - |\widetilde{\kappa}_k|^2}} \begin{pmatrix} 1 & \widetilde{\kappa}_k \\ \overline{\widetilde{\kappa}_k} & 1 \end{pmatrix} \begin{pmatrix} b_{\alpha_k}(v) & 0 \\ 0 & \eta_k\overline{\eta_{k-1}} \end{pmatrix}, \quad k \in \mathbb{I},$$

Theorem 3.3 yields the identity

$$\begin{pmatrix} b_{\alpha_m}(v)\widetilde{\gamma}_m(v) & \widetilde{\delta}_m^{[m]}(v) \\ b_{\alpha_m}(v)\widetilde{\delta}_m(v) & \widetilde{\gamma}_m^{[m]}(v) \end{pmatrix} = \frac{\sqrt{1-|\alpha_m|^2}}{1-\overline{\alpha_m}v} \widetilde{\mathbf{\Theta}}_0(v)\widetilde{\mathbf{\Theta}}_1(v)\cdots\widetilde{\mathbf{\Theta}}_m(v)$$

on the one hand and on the other hand from (6.11) follows

$$\Xi(z) = \left(\prod_{k=0}^{\overset{\curvearrowright}{m}} \sqrt{1-|\widetilde{\kappa}_k|^2}\,\widetilde{\Theta}_k(z) \right) \begin{pmatrix} 1 & 0 \\ 0 & -\overline{\eta_m} \end{pmatrix}.$$

Hence, by (6.10), (6.2), (2.6), and (2.7), we see that

$$g(z) = \frac{\widetilde{\delta}_m^{[m]}(z) + b_{\alpha_m}(z)\widetilde{\gamma}_m(z)\big(-\eta_m h_{m+1}(z)\big)}{\widetilde{\gamma}_m^{[m]}(z) + b_{\alpha_m}(z)\widetilde{\delta}_m(z)\big(-\eta_m h_{m+1}(z)\big)}. \tag{6.12}$$

Moreover, via the construction of the rational functions $\widetilde{\gamma}_m$ and $\widetilde{\delta}_m$, Lemma 6.1 implies that the Schur function $\widetilde{g}_0$ given as in (6.6) fulfills $\Delta_g = \Delta_{\widetilde{g}_0}$. In particular, since $g_\bullet \in \mathcal{S}_\Delta$, the considerations above supply that $g_\bullet$ admits a representation

$$g_\bullet(z) = \frac{\delta_m^{[m]}(z) + b_{\alpha_m}(z)\gamma_m(z)\big(-\eta_m h_\bullet(z)\big)}{\gamma_m^{[m]}(z) + b_{\alpha_m}(z)\delta_m(z)\big(-\eta_m h_\bullet(z)\big)}$$

for a unique $h_\bullet \in \mathcal{S}$ and that the Schur function g_0 given as in (6.3) fulfills the identity $\Delta_{g_\bullet} = \Delta_{g_0}$. Consequently, if $g \in \mathcal{S}_\Delta$ then

$$\Delta_{\widetilde{g}_0} = \Delta_g = \Delta_{g_\bullet} = \Delta_{g_0}$$

and, hence, Lemma 6.2 yields for each $k \in \mathbb{I}$ the identity $\widetilde{\kappa}_k = \kappa_k$ (i.e. $\widetilde{s}_k = s_k$). Therefore, (i) implicates (ii) and, in addition, (6.12) leads to (1.2). Conversely, if g admits the representation (1.2) for some $h \in \mathcal{S}$ then from Lemma 6.1 one can get

$$\Delta_g = \Delta_{g_0} = \Delta_{g_\bullet},$$

i.e. $g \in \mathcal{S}_\Delta$. Finally, if (ii) is fulfilled then (6.12) implies that g admits the representation (1.2) with $h(z) := -\eta_m h_{m+1}(z)$, $z \in \mathbb{D}$. □

Observe that the equivalence of (i) and (ii) in Theorem 6.3 is closely related to Schur's result that, for each $l \in \mathbb{N}$, there is a one-to-one correspondence between the first l Taylor coefficients of a Schur function at the point $z = 0$ and the first l corresponding Schur parameters. Clearly, applying appropriate conformal mappings of the open unit disk $\mathbb{D}$ onto itself, one can obtain a similar result with respect to arbitrarily points $z_1, z_2, \ldots, z_n \in \mathbb{D}$. Nevertheless, it seems to be really hard and unwieldy to derive directly from this classical result the equivalence of (i) and (ii), since the underlying sequence $(\alpha_k)_{k=0}^m$ has only to fulfill (6.1) and hence the points $\alpha_0, \alpha_1, \ldots, \alpha_m$ are not strictly in the order as in the example before (6.1).

If in (1.2) (i.e. in the description of $\mathcal{S}_\Delta$ by the linear fractional transformation according to Theorem 6.3) the point $z \in \mathbb{D}$ is fixed then the set

$$\mathfrak{K}_\Delta(z) := \{g(z) : \ g \in \mathcal{S}_\Delta\} \tag{6.13}$$

is a closed disk in the unit disk $\mathbb{D}$, the boundary of which is sometimes called *Weyl circle*. Using some well-known properties of linear fractional transformations (cf. [24, Proposition 2]), it can easily be shown that the center c_z and the radius r_z of this Weyl circle are given by

$$c_z = \frac{\delta_m^{[m]}(z)\overline{\gamma_m^{[m]}(z)} - b_{\alpha_m}(z)\gamma_m(z)\overline{b_{\alpha_m}(z)\delta_m(z)}}{|\gamma_m^{[m]}(z)|^2 - |b_{\alpha_m}(z)\delta_m(z)|^2},$$

$$r_z = \frac{|b_{\alpha_m}(z)|\,|\gamma_m(z)\gamma_m^{[m]}(z) - \delta_m(z)\delta_m^{[m]}(z)|}{|\gamma_m^{[m]}(z)|^2 - |b_{\alpha_m}(z)\delta_m(z)|^2}.$$

Consequently, in view of Theorem 4.2, (4.1), and Corollary 3.4 the parameters of this Weyl circle can also be computed by the formulae

$$c_z = \frac{(1-|z|^2)\sum_{\ell=0}^{m}\gamma_\ell(z)\overline{\delta_\ell(z)}}{(1-|z|^2)\sum_{\ell=0}^{m}|\delta_\ell(z)|^2+1}, \quad r_z = \frac{|B_m(z)|}{(1-|z|^2)\sum_{\ell=0}^{m}|\delta_\ell(z)|^2+1},$$

where B_m denotes the Blaschke product (of degree $m+1$) with respect to the points $\alpha_0, \alpha_1, \dots, \alpha_m$ given via (6.1). Furthermore, (3.1) and Corollary 4.4 imply

$$|\gamma_m^{[m]}(z)|^2 - |b_{\alpha_m}(z)\delta_m(z)|^2 > \left(1-|b_{\alpha_m}(z)|^2\right)|\gamma_m^{[m]}(z)|^2 > 0$$

as well as Corollary 3.6 and (4.1) yield

$$\frac{1}{(1-|z|^2)|\gamma_m^{[m]}(z)|^2} \le \frac{1}{1-|b_{\alpha_m}(z)|^2}.$$

Summing up, we have proved the following.

Corollary 6.4. *Under the assumptions of* Theorem 6.3, *if $z \in \mathbb{D}$ is fixed then the set $\mathfrak{K}_\Delta(z)$ in* (6.13) *can be described by*

$$\mathfrak{K}_\Delta(z) = \{w : \; |w - c_z| \le r_z\}$$

where the parameters c_z and r_z are given by the relations above. In particular, if $z \in \mathbb{D} \setminus \{z_1, z_2, \dots, z_n\}$ is fixed, then for each $g \in \mathcal{S}_\Delta$ the following estimate holds

$$|g(z) - c_z| < \frac{|B_m(z)|}{(1-|z|^2)|\gamma_m^{[m]}(z)|^2} \le \frac{|B_m(z)|}{1-|b_{\alpha_m}(z)|^2}.$$

Following the geometrical considerations, one can also see that the Weyl circle with center c_z and radius r_z can be described as an Apollonius circle (cf. [15], [5]).

Corollary 6.5. *Under the assumptions of* Theorem 6.3, *if $z \in \mathbb{D}$ is fixed then the set $\mathfrak{K}_\Delta(z)$ in* (6.13) *can be described by*

$$\mathfrak{K}_\Delta(z) = \left\{ v : \; \left| \frac{v - a_{1,z}}{v - a_{2,z}} \right| \le |b_{\alpha_m}(z)| d_z \right\},$$

where

$$a_{1,z} := \frac{\delta_m^{[m]}(z)}{\gamma_m^{[m]}(z)}, \quad a_{2,z} := \frac{\gamma_m(z)}{\delta_m(z)}, \quad d_z := \left| \frac{\delta_m(z)}{\gamma_m^{[m]}(z)} \right|.$$

Remark 6.6. Clearly, $\mathfrak{K}_\Delta(z_j)$, $j = 1, 2, \dots, n$, contains only the value w_{j0}. But following the idea of [24, Section 6], if we consider instead the set

$$\mathfrak{K}'_\Delta(z_j) := \left\{ \frac{1}{l_j!} g^{(l_j)}(z_j) : \; g \in \mathcal{S}_\Delta \right\}, \quad j = 1, 2, \dots, n,$$

and if we choose just in (6.1) a sequence $(\alpha_k)_{k\in\mathbb{I}}$ so that $\alpha_m = z_j$ then by a straightforward calculation from (6.5) it follows

$$\mathfrak{K}'_\Delta(z_j) = \{w : \; |w - c'_{z_j}| \le r'_{z_j}\},$$

where the Schur function g_0 is defined as in (6.3) and

$$c'_{z_j} = \frac{1}{l_j!} g_0^{(l_j)}(z_j), \quad r'_{z_j} = \frac{\prod\limits_{\substack{k=1\\k\neq j}}^{n} |b_{z_k}(z_j)|^{l_k}}{|\gamma_m^{[m]}(z_j)|^2(1-|z_j|^2)^{l_j+1}} \le \frac{1}{(1-|z_j|^2)^{l_j}} \prod_{\substack{k=1\\k\neq j}}^{n} |b_{z_k}(z_j)|^{l_k}.$$

Finally, we point out that the rational functions γ_m and δ_m, which occur in the linear fractional transformation of Theorem 6.3, can be constructed from the interpolation data Δ, but indirectly. One needs to determine the corresponding Schur parameters first, which is not easy to do in general. A way out is the following. Since the definition of Schur-Nevanlinna sequences of rational functions according to Section 3 is done with a view to orthogonal rational functions on the unit circle and their recurrence relations presented in [4, Chapter 4], one can also use the theory of orthogonal rational functions to compute the corresponding Schur parameters or the functions γ_m and δ_m. This will be explained in detail in a forthcoming work.

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