Schur-Nevanlinna-Potapov sequences of rational matrix functions

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Abstract
We study particular sequences of rational matrix functions with poles outside the unit circle. These Schur-Nevanlinna-Potapov sequences are recursively constructed based on some complex numbers with norm less than one and some strictly contractive matrices. The main theme of this paper is a thorough analysis of the matrix functions belonging to the sequences in question. Essentially, such sequences are closely related to the theory of orthogonal rational matrix functions on the unit circle. As a further crosslink, we explain that the functions belonging to Schur-Nevanlinna-Potapov sequences can be used to describe the solution set of an interpolation problem of Nevanlinna-Pick type for matricial Schur functions.

Key words: orthogonal rational matrix functions, Christoffel-Darboux formulae, Schur algorithm, Nevanlinna-Pick problem

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1. Introduction

Since the fundamental work of Szegö in the 1920s, orthogonal polynomials on the unit circle have been a basic tool in the analysis of miscellaneous problems in mathematics and engineering. This topic were studied by several authors. For instance, the important role of Szegö’s orthogonal polynomials for moment problems, numerical quadrature, approximation, and interpolation as well as the application of these methods in engineering are today well elaborated (see, e.g., the books [32, 20, 1, 30] and for discussing the case of matrix polynomials [8, 18, 11, 31]).

One of the recent and far-ranging generalizations of orthogonal polynomials that has emerged is the concept of orthogonal rational functions (see, e.g., the books [5, 21]). The
probably first work on this score was already done by Djrbashian in the 1960s (see [10]). Some steps towards extending the theory of orthogonal rational functions on the unit circle to the matrix case come across in [16, 17, 6, 24]. The present paper is closely related to the line of investigations stated there.

The main objective here is in carefully studying certain sequences of rational matrix functions which are recursively defined by a sequence of points belonging to the open unit disk (to arrange the poles) and a sequence of strictly contractive matrices (as arbitrary parameters). The considerations below on these so-called Schur-Nevanlinna-Potapov (SNP for short) sequences are actually motivated by the investigations on dual Szegő pairs of orthogonal rational matrix functions stated in [24]. Moreover, the conception of SNP sequences is leaned on those in [14, Section 2] (see also [11, Section 3.7]), where the special case of matrix polynomials is handled.

Roughly speaking, the dual Szegő pairs of rational functions analyzed in [24] are square matrix-valued functions, left and right orthogonal with respect to a non-negative Hermitian matrix Borel measure on the unit circle. They satisfy some recurrence relations which are very similar to the recurrence relations that are studied in this paper. Both kind of recursions are given on the basis of a sequence of points in the open unit disk and an associated sequence of matrix-valued parameters. In the case of Szegő pairs these (Szegő) parameters are square contractive matrices. In the present paper though, the parameters are not square in general. Therefore, the SNP sequences considered below are rational matrix-valued functions that need not be square in general. Moreover, the initial conditions of the recurrence relations here are unlike those in [24].

Since the recursions are very similar, one might expect properties for SNP sequences that are very similar to the properties of dual Szegő pairs. As an example we shall prove Christoffel-Darboux type relations and we shall see that the SNP sequences can be used to parameterize the solution set of an interpolation problem of Nevanlinna-Pick type, just like the dual Szegő pairs could. However, we are here in a modified situation and the details of the properties are somewhat different. For instance, the Christoffel-Darboux formulae lead to a characterization of SNP sequences which is much more coupled than in the case of the similar result for dual Szegő pairs, where the two associated types of Szegő pairs can be separately characterized by such formulae in a sense. In addition, there is also some kind of orthogonality behind the sequences studied here, but not that straightforward as for dual Szegő pairs. It is our intention to explore this fact in a forthcoming paper.

A brief synopsis is as follows. In Section 2 we introduce the central notations of this paper and explain some characteristics about these. In doing so, we will reveal a multitude on interrelations between the rational matrix functions in question. Like in the case of orthogonal functions (cf. [16, 24]), in Section 3 we will see that the Christoffel-Darboux formulae are important tools for studying SNP sequences of rational matrix functions. Roughly speaking, we get that these formulae are necessary and sufficient for SNP sequences. Finally, in Section 4 we shall show that SNP sequences of rational matrix functions can be used to describe the solution set of interpolation problems of Nevanlinna-Pick type for matrix-valued Schur functions in the non-degenerate case. In particular, based on SNP sequences, we point out the well-known fact that the solution set of such kind of interpolation problems can be
parameterized by linear fractional matrix transformations.

Of course, the parameterization of the solution set by linear fractional matrix transformations does not come as a surprise. In the literature much more general settings of interpolation problems can be found which lead particularly to descriptions of this set (see, e.g., [13, 3, 4, 27, 2]). Our primary point of concern lies in the approach and the thorough analysis of the rational matrix functions which leads to the descriptions presented here. Thereby, the essential feature is the close relationship with the theory of orthogonal rational matrix functions on the unit circle. In particular, the present paper is a contribution to clarify that also in the matrix case this theory is related to interpolation problems of Nevanlinna-Pick type like the classical Szegő theory of orthogonal polynomials is related to certain Taylor coefficient problems (see, e.g., [19]).

2. SNP pairs of rational matrix functions

Throughout this paper, let $p$ and $q$ be positive integers. Let $\mathbb{D} := \{ w \in \mathbb{C} : |w| < 1 \}$ and $\mathbb{T} := \{ z \in \mathbb{C} : |z| = 1 \}$ be the unit disk and the unit circle of the complex plane $\mathbb{C}$.

In this section, we give some fundamental information on SNP sequences of rational matrix functions. In fact, we introduce and begin to study particular sequences of rational matrix-valued functions defined by a sequence of points belonging to $\mathbb{D}$ and a sequence of strictly contractive $p \times q$ matrices. Recall that a complex $p \times q$ matrix $A$ is said to be contractive (resp., strictly contractive) if $I_q - A^*A$ is a non-negative (resp., positive) Hermitian matrix, where $I_q$ stands for the identity matrix of size $q \times q$.

The definition of SNP sequences is motivated by the studies in [24] on dual pairs of orthogonal rational matrix functions on the unit circle. The results of the present section can also be seen as a rational extension of the matrix polynomial case treated in [14, Section 2] and as a matrix extension of the scalar rational case discussed in [7, Section 3].

Henceforth, let $\mathbb{N}_0$ and $\mathbb{N}$ denote the set of all non-negative integers and the set of all positive integers, respectively. Moreover, let $\tau \in \mathbb{N}_0$ or $\tau = \infty$. If $j \in \mathbb{N}_0$ then $\mathbb{N}_{j, \tau}$ stands for the set of all integers $k$ satisfying $j \leq k \leq \tau$.

For a fixed sequence $(\alpha_j)_{j=0}^{\tau}$ of points belonging to $\mathbb{D}$ and some $k \in \mathbb{N}_{0, \tau}$ the notation $\tilde{\mathcal{R}}_{\alpha,k}$ stands for the space of rational functions $x$ that admit for some complex polynomial $\pi_k$ of degree not greater than $k$ the representation

$$x = \frac{\pi_k}{\pi_{\alpha,k+1}},$$

where $\pi_{\alpha,k+1}$ is the polynomial of degree not greater than $k + 1$ given by

$$\pi_{\alpha,k+1}(v) = \prod_{j=0}^{k} (1 - \alpha_j v), \quad v \in \mathbb{C}.$$

Let $k \in \mathbb{N}_{0, \tau}$. Then $\tilde{\mathcal{R}}_{\alpha,k}^{p \times q}$ denotes the set of all complex $p \times q$ matrix functions whose entries belong to $\tilde{\mathcal{R}}_{\alpha,k}$. Further, we use (with $\frac{1}{0} := \infty$) the settings
\[ \mathbb{P}_{a,k} := \bigcup_{j=0}^{k} \left\{ \frac{1}{\alpha_j} \right\} \quad \text{and} \quad \mathbb{Z}_{a,k} := \bigcup_{j=0}^{k} \{\alpha_j\}. \]

Note that \( \mathbb{Z}_{a,k} \subset \mathbb{D} \) so that \( \mathbb{D} \cup \mathbb{T} \subset \mathbb{C} \setminus \mathbb{P}_{a,k} \).

The following transform of a rational matrix-valued function into another plays a key role in the sequel (cf. \([5, 7, 16, 24]\)). For each \( X \in \mathcal{R}^{p \times q}_{\mathbb{a},k} \), the notation \( X^{[a,k]} \) (with respect to \( X \) and \( \alpha_0, \alpha_1, \ldots, \alpha_k \)) stands for the rational matrix-valued function which is uniquely determined via the formula

\[ X^{[a,k]}(v) = \frac{1}{v} B_{a,k}(v) \left( X \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right)^{*}, \quad v \in \mathbb{C} \setminus (\mathbb{P}_{a,k} \cup \mathbb{Z}_{a,k} \cup \{0\}), \tag{1} \]

where \( B_{a,k} \) denotes the Blaschke product (of degree \( k + 1 \)) in reference to the underlying points \( \alpha_0, \alpha_1, \ldots, \alpha_k \), i.e.

\[ B_{a,k}(v) := \prod_{j=0}^{k} b_{\alpha_j}(v) \tag{2} \]

with

\[ b_{\alpha_j}(v) := \begin{cases} v & \text{if } \alpha_j = 0, \\ \frac{\alpha_j - v}{|\alpha_j|} & \text{if } \alpha_j \neq 0. \end{cases} \tag{3} \]

In doing so, \( A^{*} \) stands for the adjoint matrix of a complex matrix \( A \). We will also use the notation \( A^{\frac{1}{2}} \) for the non-negative Hermitian square root of a non-negative Hermitian matrix \( A \) and \( A^{-\frac{1}{2}} = (A^{-1})^{\frac{1}{2}} = (A^{\frac{1}{2}})^{-1} \) for its inverse.

Let \( (\alpha_k)_{k=0}^{r} \) be a sequence of points belonging to \( \mathbb{D} \) and let \( (F_k)_{k=0}^{r} \) be a sequence of strictly contractive \( p \times q \) matrices. Then we define sequences of rational matrix-valued functions \( (O_k)_{k=0}^{r} \) and \( (Q_k)_{k=0}^{r} \) by

\[ O_0(v) := \sqrt{1 - |\alpha_0|^2} \left( I_q - F_0^{*}F_0 \right)^{-\frac{1}{2}} F_0^{*}, \quad Q_0(v) := \sqrt{1 - |\alpha_0|^2} \left( I_p - F_0^{*}F_0 \right)^{-\frac{1}{2}} \tag{4} \]

for each \( v \in \mathbb{C} \setminus \mathbb{P}_{a,0} \) and recursively via

\[ O_k(v) := d_k \frac{1 - \alpha_{k-1}^{*} v}{1 - \alpha_{k} v} \left( I_q - F_k^{*}F_k \right)^{-\frac{1}{2}} \left( b_{\alpha_{k-1}}(v)O_{k-1}(v) + F_k^{*}Q_k^{[a,k-1]}(v) \right), \]

\[ Q_k(v) := d_k \frac{1 - \alpha_{k-1}^{*} v}{1 - \alpha_{k} v} \left( b_{\alpha_{k-1}}(v)Q_{k-1}(v) + O_{k-1}^{[a,k-1]}(v)F_k \right) \left( I_p - F_k^{*}F_k \right)^{-\frac{1}{2}} \]

for each \( k \in \mathbb{N}_{1,r} \) and each \( v \in \mathbb{C} \setminus \mathbb{P}_{a,k} \), where

\[ d_k := \sqrt{\frac{1 - |\alpha_k|^2}{1 - |\alpha_{k-1}|^2}}. \tag{5} \]

We call \( [(O_k)_{k=0}^{r}, (Q_k)_{k=0}^{r}] \) the first SNP pair of rational matrix functions corresponding to \( (\alpha_k, F_k)_{k=0}^{r} \). Similarly, we bring in \( (P_k)_{k=0}^{r} \) and \( (R_k)_{k=0}^{r} \) by

\[ P_0(v) := \sqrt{1 - |\alpha_0|^2} \left( I_q - F_0^{*}F_0 \right)^{-\frac{1}{2}}, \quad R_0(v) := \sqrt{1 - |\alpha_0|^2} \left( I_p - F_0^{*}F_0 \right)^{-\frac{1}{2}} \tag{6} \]
for each \( v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,0} \) and recursively via

\[
P_k(v) := d_k \frac{1 - \alpha_k - v}{1 - \alpha_k v} \left( I_q - F_k^v F_k \right)^{-\frac{1}{2}} \left( b_{\alpha_k - 1}(v) P_{k-1}(v) + F_k^v P_{\alpha,k-1}^{[\alpha,k-1]}(v) \right),
\]

\[
R_k(v) := d_k \frac{1 - \alpha_k - v}{1 - \alpha_k v} \left( b_{\alpha_k - 1}(v) R_{k-1}(v) + P_{\alpha,k-1}^{[\alpha,k-1]}(v) F_k^v \right) \left( I_p - F_k F_k^v \right)^{-\frac{1}{2}}
\]

for each \( k \in \mathbb{N}_{1,\tau} \) and each \( v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k} \). We call \( [(P_k)_{k=0}^\tau, (R_k)_{k=0}^\tau] \) the second SNP pair of rational matrix functions corresponding to \((\alpha_k, F_k)_{k=0}^\tau\).

We remark that the recurrence relations of SNP pairs of rational matrix functions are closely related to those of dual pairs of orthogonal functions discussed in [24]. More precisely, they have a similar shape, but the initial conditions are different and the functions here are not square matrices in general. In fact, we have \( O_k \in \mathcal{R}_\alpha^{q \times p} \), \( Q_k \in \mathcal{R}_\alpha^{r \times q} \), \( P_k \in \mathcal{R}_\alpha^{r \times r} \), and \( R_k \in \mathcal{R}_\alpha^{q \times p} \) for each \( k \in \mathbb{N}_{0,\tau} \). Moreover, in the scalar situation \( p = q = 1 \) the identities \( O_k = R_k \) and \( P_k = Q_k \) hold for each \( k \in \mathbb{N}_{0,\tau} \). These scalar rational functions were studied in [7], whereas the particular case of polynomials (i.e. the special choice \( \alpha_k = 0 \) for every \( k \in \mathbb{N}_{0,\tau} \)) already occurred in the classical papers [28, 29] of Schur.

For a strictly contractive \( p \times q \) matrix \( S \), we use from now on the notation

\[
H_S := \begin{pmatrix} (I_p - SS^*)^{-\frac{1}{2}} & S(I_q - S^*S)^{-\frac{1}{2}} \\ S^*(I_p - SS^*)^{-\frac{1}{2}} & (I_q - S^*S)^{-\frac{1}{2}} \end{pmatrix}. \tag{7}
\]

In view of (7) and [11, Lemma 1.1.12], by setting

\[
\Theta_k(v) := \begin{pmatrix} b_{\alpha_k}(v) Q_k(v) & O_k^{[\alpha,k]}(v) \\ b_{\alpha_k}(v) R_k(v) & P_k^{[\alpha,k]}(v) \end{pmatrix} \tag{8}
\]

\( \text{resp., } \Xi_k(v) := \begin{pmatrix} b_{\alpha_k}(v) P_k(v) & b_{\alpha_k}(v) O_k(v) \\ R_k^{[\alpha,k]}(v) & Q_k^{[\alpha,k]}(v) \end{pmatrix} \)

and (for technical reasons)

\[
\eta_k := \begin{cases} -1 & \text{if } \alpha_k = 0, \\ \frac{\alpha_k}{|\alpha_k|} & \text{if } \alpha_k \neq 0, \end{cases} \tag{9}
\]

for each \( k \in \mathbb{N}_{0,\tau} \), the recursion formulae above can be paraphrased as

\[
\Theta_k(v) = \sqrt{\frac{1 - |\alpha_k|^2}{1 - |\alpha_{k-1}|^2} - \frac{1 - \alpha_k - v}{1 - \alpha_k v} \Theta_{k-1}(v) \Theta_k(v)}
\]

\( \text{resp., } \Xi_k(v) = \sqrt{\frac{1 - |\alpha_k|^2}{1 - |\alpha_{k-1}|^2} - \frac{1 - \alpha_k - v}{1 - \alpha_k v} \Xi_{k}(v) \Xi_{k-1}(v)} \) \tag{10}
for each $k \in \mathbb{N}_{1,r}$ and each $v \in \mathbb{C} \setminus \mathbb{P}_{a,k}$, where

$$
\Theta_k(v) := H_{F_k} \begin{pmatrix} b_{a_k}(v)I_p & 0 \\ 0 & \eta_k \eta_{k-1} I_q \end{pmatrix} 
$$

(11)

(resp., $\widehat{\Theta}_k(v) := \begin{pmatrix} b_{a_k}(v)I_q & 0 \\ 0 & \eta_k \eta_{k-1} I_p \end{pmatrix} H_{F_k}$)

with the short notation 0 for the zero matrix of appropriate size. For later reference we use (11) also for $k = 0$, setting by definition $\eta_{-1} := -1$.

**Proposition 2.1.** Let $\tau \in \mathbb{N}$ or $\tau = \infty$, let $(F_k)_{k=0}^\tau$ be a sequence of strictly contractive $p \times q$ matrices, let $(\alpha_k)_{k=0}^\tau$ be a sequence of points belonging to $\mathbb{D}$, and let

$$
c_k := \frac{1 - \overline{\alpha_k} \alpha_{k-1}}{(1 - |\alpha_k|^2)(1 - |\alpha_{k-1}|^2)}, \quad k \in \mathbb{N}_{1,r}.
$$

Furthermore, let $(O_k)_{k=0}^\tau$ and $(Q_k)_{k=0}^\tau$ be sequences of functions such that $O_0, Q_0$ are defined as in (4) and that $O_k \in \mathcal{R}^{q \times p}_{a,k}$ and $Q_k \in \mathcal{R}^{q \times p}_{a,k}$ for all $k \in \mathbb{N}_{1,r}$. Then $[(O_k)_{k=0}^\tau, (Q_k)_{k=0}^\tau]$ is the first SNP pair of rational matrix functions corresponding to $(\alpha_k, F_k)_{k=0}^\tau$ if and only if, for each $k \in \mathbb{N}_{1,r}$ and each $v \in \mathbb{C} \setminus \mathbb{P}_{a,k}$, the following backward recurrence relations hold:

$$
\eta_k \eta_{k-1} O_k(v) - F_k^* Q_k^{[a,k]}(v) = c_k (b_{a_k}(v) - b_{\alpha_k}(\alpha_{k-1}))(I_q - F_k^* F_k)^{1/2} O_{k-1}(v), \\
\eta_k \eta_{k-1} Q_k(v) - O_k^{[a,k]}(v) F_k^* = c_k (b_{a_k}(v) - b_{\alpha_k}(\alpha_{k-1})) Q_{k-1}(v)(I_p - F_k^* F_k)^{1/2}.
$$

Moreover, if $(P_k)_{k=0}^\tau$ and $(R_k)_{k=0}^\tau$ are sequences so that $P_0, R_0$ are given by (6) and that $P_k \in \mathcal{R}^{q \times q}_{a,k}$ and $R_k \in \mathcal{R}^{q \times q}_{a,k}$ for all $k \in \mathbb{N}_{1,r}$. Then $[(P_k)_{k=0}^\tau, (R_k)_{k=0}^\tau]$ is the second SNP pair of rational matrix functions corresponding to $(\alpha_k, F_k)_{k=0}^\tau$ if and only if, for each $k \in \mathbb{N}_{1,r}$ and each $v \in \mathbb{C} \setminus \mathbb{P}_{a,k}$, the following relations hold:

$$
\eta_k \eta_{k-1} P_k(v) - F_k^* R_k^{[a,k]}(v) = c_k (b_{a_k}(v) - b_{\alpha_k}(\alpha_{k-1}))(I_q - F_k^* F_k)^{1/2} P_{k-1}(v), \\
\eta_k \eta_{k-1} R_k(v) - P_k^{[a,k]}(v) F_k^* = c_k (b_{a_k}(v) - b_{\alpha_k}(\alpha_{k-1})) R_{k-1}(v)(I_p - F_k^* F_k)^{1/2}.
$$

**Proof.** Let $k \in \mathbb{N}_{1,r}$ and let $v \in \mathbb{C} \setminus \mathbb{P}_{a,k}$. By virtue of (8), (11), and [11, Lemma 1.1.12 and Lemma 3.6.32], we get that (10) is equivalent to the relation

$$
\begin{pmatrix} Q_k(v) \\ R_k(v) \\
\end{pmatrix} \eta_k \eta_{k-1} O_k^{[a,k]}(v) = d_k \frac{1 - \overline{\alpha_k} \alpha_{k-1}}{1 - \overline{\alpha_k} \alpha_k} \Theta_{k-1}(v)
$$

(resp., $\begin{pmatrix} P_k(v) \\ R_k(v) \end{pmatrix} \eta_k \eta_{k-1} Q_k^{[a,k]}(v) = d_k \frac{1 - \overline{\alpha_k} \alpha_{k-1}}{1 - \overline{\alpha_k} \alpha_k} \Xi_{k-1}(v)$),

where $d_k$ is given by (5). Hence, by considering the first column of $\Theta_{k-1}(v)$ and the first row of $\Xi_{k-1}(v)$, using

$$
\eta_k \eta_{k-1} \frac{1 - \overline{\alpha_k} \alpha_{k-1}}{1 - |\alpha_k|^2} (b_{a_k}(v) - b_{\alpha_k}(\alpha_{k-1})) = \frac{1 - \overline{\alpha_k} \alpha_{k-1}}{1 - \overline{\alpha_k} \alpha_k} b_{a_k}(v),
$$

one can finally conclude the assertion. \[\square\]
Theorem 2.2. For each strictly contractive $\alpha$ where ($J$ first (resp., second) SNP pair of rational matrix functions corresponding to ($p \times q$) unitary and Hermitian) then a complex $A$ is called $J$-contractive (resp., strictly $J$-contractive, or $J$-unitary) in the event of

$$J \geq A^*JA \quad \text{(resp., } J > A^*JA, \text{ or } J = A^*JA).$$

In our case we shall use the special $(p + q) \times (p + q)$ signature matrices

$$j_{pq} := \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \quad \text{and} \quad j_{qp} := \begin{pmatrix} I_q & 0 \\ 0 & -I_p \end{pmatrix}.$$ 

Henceforth in this section, $[(O_k)_{k=0}^\infty, (Q_k)_{k=0}^\infty]$ (resp., $[(P_k)_{k=0}^\infty, (R_k)_{k=0}^\infty]$) stands for the first (resp., second) SNP pair of rational matrix functions corresponding to $(\alpha_k, F_k)_{k=0}^\infty$, where $(\alpha_k)_{k=0}^\infty$ is some sequence of points belonging to $\mathbb{D}$ and $(F_k)_{k=0}^\infty$ is some sequence of strictly contractive $p \times q$ matrices. Further, we use the notations given by (8) and (11).

**Theorem 2.2.** For each $k \in \mathbb{N}_0 \tau$ and each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$,

$$\Theta_k(v) = \frac{1 - |\alpha_k|^2}{1 - \alpha_k v} \hat{\Theta}_0(v) \hat{\Theta}_1(v) \cdots \hat{\Theta}_k(v),$$

$$\Xi_k(v) = \frac{1 - |\alpha_k|^2}{1 - \alpha_k v} \hat{\Xi}_0(v) \hat{\Xi}_1(v) \cdots \hat{\Xi}_k(v).$$

Moreover, if $v \in \mathbb{D}$ (resp., $v \in \mathbb{T}$) then the matrix $\frac{1 - \alpha_k v}{\sqrt{1 - |\alpha_k|^2}} \Theta_k(v)$ is $j_{pq}$-contractive (resp., $j_{pq}$-unitary) and $\frac{1 - \alpha_k v}{\sqrt{1 - |\alpha_k|^2}} \Xi_k(v)$ is $j_{qp}$-contractive (resp., $j_{qp}$-unitary).

**Proof.** Let $k \in \mathbb{N}_0 \tau$ and let $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$. We prove the above expressions with respect to $\Theta_k(v)$. A proof concerning $\Xi_k(v)$ is similar. As a consequence of (10) and the choice of $O_0$, $P_0$, $Q_0$, and $R_0$ given in (4) and (6) we obtain

$$\Theta_k(v) = \frac{1 - |\alpha_k|^2}{1 - \alpha_k v} \hat{\Theta}_0(v) \hat{\Theta}_1(v) \cdots \hat{\Theta}_k(v). \quad (12)$$
It remains to prove that if \( v \in \mathbb{D} \) (resp., \( v \in \mathbb{T} \)) then
\[
\mathbf{j}_{pq} \geq \left( \frac{1 - \bar{\alpha}_k v}{\sqrt{1 - |\alpha_k|^2}} \Theta_k(v) \right)^* \mathbf{j}_{pq} \left( \frac{1 - \bar{\alpha}_k v}{\sqrt{1 - |\alpha_k|^2}} \Theta_k(v) \right)
\]
(resp., \( \mathbf{j}_{pq} = \left( \frac{1 - \bar{\alpha}_k v}{\sqrt{1 - |\alpha_k|^2}} \Theta_k(v) \right)^* \mathbf{j}_{pq} \left( \frac{1 - \bar{\alpha}_k v}{\sqrt{1 - |\alpha_k|^2}} \Theta_k(v) \right) \)).

Taking [11, Lemma 1.3.13 and Lemma 3.6.32] and (11) into account, this follows immediately from (12) and the fact that if \( v \in \mathbb{D} \) (resp., \( v \in \mathbb{T} \)) then
\[
\mathbf{j}_{pq} \geq \left( b_{\alpha_j}(v) I_p \ 0 \ 0 \right) \left( b_{\alpha_j}(v) I_p \ 0 \right)^* \mathbf{j}_{pq} \left( b_{\alpha_j}(v) I_p \ 0 \ 0 \right)^*
\]
(resp., \( \mathbf{j}_{pq} = \left( b_{\alpha_j}(v) I_p \ 0 \ 0 \right) \left( b_{\alpha_j}(v) I_p \ 0 \right)^* \mathbf{j}_{pq} \left( b_{\alpha_j}(v) I_p \ 0 \ 0 \right)^* \))
for each \( j \in \mathbb{N}_{0,k} \) and some \( u \in \mathbb{T} \).

In view of (8) and some well-known results on \( \mathbf{j}_{pq} \)-contractive matrices (see, e.g., [11, Theorem 1.6.1]), Theorem 2.2 yields particularly the following result.

**Corollary 2.3.** For each \( k \in \mathbb{N}_{0,\tau} \) and each \( v \in \mathbb{D} \cup \mathbb{T} \), the matrices \( Q_k^{[\alpha,k]}(v) \) and \( P_k^{[\alpha,k]}(v) \) are non-singular and the matrices
\[
(Q_k^{[\alpha,k]}(v))^{-1} R_k^{[\alpha,k]}(v), \ b_{\alpha_k}(v)O_k(v)(Q_k^{[\alpha,k]}(v))^{-1}, \ O_k^{[\alpha,k]}(v)(P_k^{[\alpha,k]}(v))^{-1}, \ b_{\alpha_k}(v)(P_k^{[\alpha,k]}(v))^{-1} R_k(v)
\]
are strictly contractive.

Taking Corollary 2.3 into account, the next statement is an easy conclusion of Proposition 2.1 and (10) with the special choice \( v = \alpha_{k-1} \).

**Corollary 2.4.** For every \( k \in \mathbb{N}_{1,\tau} \), the matrices \( Q_k^{[\alpha,k]}(\alpha_{k-1}) \) and \( P_k^{[\alpha,k]}(\alpha_{k-1}) \) are non-singular, the identities
\[
\mathbf{F}_k^* = \eta_k \eta_k^{-1} O_k(\alpha_{k-1}) (Q_k^{[\alpha,k]}(\alpha_{k-1}))^{-1}, \quad \mathbf{F}_k^* = \eta_k \eta_k^{-1} (P_k^{[\alpha,k]}(\alpha_{k-1}))^{-1} R_k(\alpha_{k-1}),
\]
\[
\left( \mathbf{I}_p - \mathbf{F}_k \mathbf{F}_k^* \right)^{\frac{1}{2}} = \frac{\eta_k \eta_k^{-1}}{c_k} Q_{k-1}^{[\alpha,k-1]}(\alpha_{k-1}) (Q_k^{[\alpha,k]}(\alpha_{k-1}))^{-1},
\]
\[
\left( \mathbf{I}_q - \mathbf{F}_k^* \mathbf{F}_k \right)^{\frac{1}{2}} = \frac{\eta_k \eta_k^{-1}}{c_k} (P_k^{[\alpha,k]}(\alpha_{k-1}))^{-1} P_{k-1}^{[\alpha,k-1]}(\alpha_{k-1})
\]
are satisfied, and particularly \( \mathbf{F}_k = 0 \iff O_k(\alpha_{k-1}) = 0 \iff R_k(\alpha_{k-1}) = 0 \).

Note that, if we use the additional settings \( \alpha_{-1} := 0, \eta_{-1} := -1, \ Q_{-1}^{[\alpha,-1]}(\alpha_{-1}) := \mathbf{I}_p \), and \( P_{-1}^{[\alpha,-1]}(\alpha_{-1}) := \mathbf{I}_q \) then the relations in Corollary 2.4 hold also for \( k = 0 \).
Proposition 2.5. For each \( k \in \mathbb{N}_{0,r} \) and each \( v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k} \),
\[
\Xi_k(v) \begin{pmatrix} 0 & I_q \\ -I_p & 0 \end{pmatrix} \Theta_k(v) = -\eta_k \frac{1-|\alpha_k|^2}{(1-|\alpha_k|^2)^2} B_{\alpha,k}(v) \begin{pmatrix} 0 & I_q \\ -I_p & 0 \end{pmatrix},
\]
where \( \Theta_k(v) \) and \( \Xi_k(v) \) are defined as in (8), the number \( \eta_k \) is given as in (9), and \( B_{\alpha,k} \) is given as in (2) with respect to \( \alpha_0, \alpha_1, \ldots, \alpha_k \).

**Proof.** Let \( k \in \mathbb{N}_{0,r} \) and let \( v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k} \). Furthermore, let \( \eta_{-1} := -1 \). A straightforward calculation yields the relation
\[
\begin{pmatrix} b_{\alpha_k}(v)I_p & 0 \\ 0 & \eta_j \eta_{j-1} I_q \end{pmatrix} \begin{pmatrix} 0 & I_q \\ -I_p & 0 \end{pmatrix} = \eta_j \eta_{j-1} b_{\alpha_k}(v) \begin{pmatrix} 0 & I_q \\ -I_p & 0 \end{pmatrix}, \quad j \in \mathbb{N}_{0,k}.
\]
Using this in combination with the circumstance that [11, Lemma 1.1.12] implies
\[
H_{F_j} \begin{pmatrix} 0 & I_q \\ -I_p & 0 \end{pmatrix} H_{F_j} = \begin{pmatrix} 0 & I_q \\ -I_p & 0 \end{pmatrix}, \quad j \in \mathbb{N}_{0,k},
\]
based on the decomposition of \( \Theta_k(v) \) and \( \Xi_k(v) \) according to Theorem 2.2, one can finally conclude the assertion. \( \square \)

The next result is an easy consequence of Proposition 2.5, (8), and
\[
\begin{pmatrix} 0 & I_q \\ -I_p & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -I_p \\ I_q & 0 \end{pmatrix}.
\]

**Corollary 2.6.** For each \( k \in \mathbb{N}_{0,r} \) and each \( v \in \mathbb{C} \setminus (\mathbb{P}_{\alpha,k} \cup \mathbb{Z}_{\alpha,k}) \), the matrices \( \Theta_k(v), \Xi_k(v) \) are non-singular and
\[
(\Theta_k(v))^{-1} = \frac{(1-|\alpha_k|^2)^2}{\eta_k (1-|\alpha_k|^2) B_{\alpha,k}(v)} \begin{pmatrix} -Q_k^{[\alpha,k]}(v) & R_k^{[\alpha,k]}(v) \\ b_{\alpha_k}(v)O_k(v) & -b_{\alpha_k}(v)P_k(v) \end{pmatrix},
\]
\[
(\Xi_k(v))^{-1} = \frac{(1-|\alpha_k|^2)^2}{\eta_k (1-|\alpha_k|^2) B_{\alpha,k}(v)} \begin{pmatrix} -P_k^{[\alpha,k]}(v) & b_{\alpha_k}(v)R_k(v) \\ O_k^{[\alpha,k]}(v) & -b_{\alpha_k}(v)Q_k(v) \end{pmatrix}.
\]

Taking (8) into account, Corollary 2.6 yields by considering the corresponding block entries and using a continuity argument the identities below.

**Corollary 2.7.** For each \( k \in \mathbb{N}_{0,r} \) and each \( v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k} \),
\[
O_k(v)Q_k(v) = P_k(v)R_k(v), \quad Q_k^{[\alpha,k]}(v)O_k^{[\alpha,k]}(v) = R_k^{[\alpha,k]}(v)P_k^{[\alpha,k]}(v),
\]
\[
P_k^{[\alpha,k]}(v)Q_k(v) = R_k(v)Q_k^{[\alpha,k]}(v), \quad O_k^{[\alpha,k]}(v)P_k(v) = Q_k(v)P_k^{[\alpha,k]}(v),
\]
Proposition 2.10. For each $k \in \mathbb{N}_{0,\tau}$ and each $v \in \mathbb{C} \setminus \mathbb{R}_{\alpha,k}$,

\[
\det Q_k(v) = \left(\frac{1 - |\alpha_k|^2}{1 - \alpha_k v}\right)^{p-q} \det P_k(v),
\]

\[
\det Q_k^{[\alpha,k]}(v) = \left(-\eta_k \frac{1 - |\alpha_k|^2}{1 - \alpha_k v}\right)^{p-q} \det P_k^{[\alpha,k]}(v).
\]

PROOF. To simplify notation, for each $k \in \mathbb{N}_{0,\tau}$ we set

\[
r_k(v) := \frac{\sqrt{1 - |\alpha_k|^2}}{1 - \alpha_k v}, \quad v \in \mathbb{C} \setminus \mathbb{R}_{\alpha,k}.
\]

Because of (1), for each $k \in \mathbb{N}_{0,\tau}$, one can see (cf. [16, Remark 2.6]) that

\[
\det Q_k(v) = (r_k(v))^{p-q} \det P_k(v), \quad v \in \mathbb{C} \setminus \mathbb{R}_{\alpha,k},
\]

(13)
is tantamount to
\[
\det Q_k^{[\alpha,k]}(v) = (-\eta_k r_k(v))^{p-q} \det P_k^{[\alpha,k]}(v), \quad v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}.
\] (14)

Thus, we only have to prove one of them. If \( k = 0 \) then it follows from (4) and (6) by using some standard calculation rules of determinants (see, e.g., [11, Lemma 1.1.8]) that
\[
\det Q_0(v) = \frac{(r_0(v))^p}{\sqrt{\det(I_p - F_0 F_0^*)}} = \frac{(r_0(v))^p}{\sqrt{\det(I_q - F_0 F_0^*)}} = (r_0(v))^{p-q} \det P_0(v)
\]
for each \( v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,0} \). Now we assume that, for some \( k \in \mathbb{N}_{1,\tau} \) and for each \( v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k-1} \), we have \( \det Q_{k-1}(v) = (r_{k-1}(v))^{p-q} \det P_{k-1}(v) \) and hence
\[
\det Q_{k-1}^{[\alpha,k-1]}(v) = (-\eta_{k-1} r_{k-1}(v))^{p-q} \det P_{k-1}^{[\alpha,k-1]}(v)
\] (15)
as well. Let \( v \in \mathbb{D} \cup \mathbb{T} \). By Corollary 2.3 we know that \( Q_{k-1}^{[\alpha,k-1]}(v) \) and \( P_{k-1}^{[\alpha,k-1]}(v) \) are non-singular matrices, so that Corollary 2.7 yields the identity
\[
O_{k-1}(v) (Q_{k-1}^{[\alpha,k-1]}(v))^{-1} = (P_{k-1}^{[\alpha,k-1]}(v))^{-1} R_{k-1}(v).
\]

Therefore, taking advantage of the recursions of SNP pairs of rational matrix functions (see (10)), some elementary calculation rules of determinants, the setting
\[
h_k(v) := \left( \eta_k \eta_{k-1} \frac{r_k(v)}{r_{k-1}(v)} \right)^p,
\]
and (15) we obtain
\[
\det Q_k^{[\alpha,k]}(v) = h_k(v) \frac{\det(I_p + b_{\alpha,k-1}(v) F_k O_{k-1}(v)(Q_{k-1}^{[\alpha,k-1]}(v))^{-1}) \det Q_{k-1}^{[\alpha,k-1]}(v)}{\sqrt{\det(I_p - F_k F_k^*)}}
\]
\[= h_k(v) \frac{\det(I_q + b_{\alpha,k-1}(v) (P_{k-1}^{[\alpha,k-1]}(v))^{-1} R_{k-1}(v) F_k) \det Q_{k-1}^{[\alpha,k-1]}(v)}{\sqrt{\det(I_q - F_k F_k^*)}}
\]
\[= (-\eta_{k-1} r_{k-1}(v))^{p-q} h_k(v) \frac{\det(P_{k-1}^{[\alpha,k-1]}(v) + b_{\alpha,k-1}(v) R_{k-1}(v) F_k)}{\sqrt{\det(I_q - F_k F_k^*)}}
\]
\[= (-\eta_k r_k(v))^{p-q} \det P_k^{[\alpha,k]}(v).
\]

Since \( \det Q_k^{[\alpha,k]} \), \( \det P_k^{[\alpha,k]} \), and \( r_k \) are rational functions, we can conclude (14). Hence, since (14) implies (13), for each \( k \in \mathbb{N}_{0,\tau} \) and each \( v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k} \) the assertion is shown.

In the square matrix case \( p = q \), Proposition 2.10 leads in combination with the first both identities of Corollary 2.7, the statement of regularity pointed out in Corollary 2.3, and a continuity argument to the following relations as well.

**Corollary 2.11.** If \( p = q \) then, for each \( k \in \mathbb{N}_{0,\tau} \) and each \( v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k} \),
\[
\det O_k(v) = \det R_k(v) \quad \text{and} \quad \det O_k^{[\alpha,k]}(v) = \det R_k^{[\alpha,k]}(v).
\]
3. Christoffel-Darboux formulae

In this section, we show that, like in the case of orthogonal rational matrix functions (cf. [16, Section 5] and [24, Section 4]), arbitrary SNP pairs of rational matrix functions fulfill Christoffel-Darboux formulae. In fact, we will see that these formulae can be used to characterize SNP pairs of rational matrix functions (cf. [24, Section 5]). The following results present matrix extensions of the scalar rational case discussed in [7, Sections 4 and 5].

To prove the Christoffel-Darboux formulae, we give first certain auxiliary identities. Here and in the sequel, \([(O_k)_{k=0}^\infty, (Q_k)_{k=0}^\infty]\) (resp., \([(P_k)_{k=0}^\infty, (R_k)_{k=0}^\infty]\)) stands again for the first (resp., second) SNP pair of rational matrix functions corresponding to \((\alpha_k, F_k)_{k=0}^\infty\), where \((\alpha_k)_{k=0}^\infty\) is some sequence of points belonging to \(\mathbb{D}\) and \((F_k)_{k=0}^\infty\) is some sequence of strictly contractive \(p \times q\) matrices. In view of (3), if \(j \in \mathbb{N}_{0,\tau}\) and if \(v, w \in \mathbb{C} \setminus \{1/w\}\), we also put

\[
\tau_{\alpha_j}(v, w) := 1 - b_{\alpha_j}(v)b_{\alpha_j}(w) = \frac{(1 - |\alpha_j|^2)(1 - \overline{v \alpha_j})}{(1 - \overline{v \alpha_j})(1 - \alpha_j \overline{w})}
\]

Lemma 3.1. For all \(j \in \mathbb{N}_{0,\tau-1}\) and \(v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha_{j+1}}\), the following holds:

\[
\tau_{\alpha_j}(w, v) \left( (Q_{j+1}^{[\alpha_{j+1}]}(v))^* Q_{j+1}^{[\alpha_{j+1}]}(w) - (O_{j+1}(v))^* O_{j+1}(w) \right)
= \tau_{\alpha_{j+1}}(w, v) \left( (Q_{j+1}^{[\alpha_{j}]}(v))^* Q_{j+1}^{[\alpha_{j}]}(w) - b_{\alpha_j}(v)b_{\alpha_j}(w)(O_j(v))^* O_j(w) \right),
\]

\[
\tau_{\alpha_j}(v, w) \left( O_{j+1}^{[\alpha_{j+1}]}(v)(O_{j+1}^{[\alpha_{j+1}]}(w))^* - Q_{j+1}(v)(Q_{j+1}(w))^* \right)
= \tau_{\alpha_{j+1}}(v, w) \left( O_{j+1}^{[\alpha_{j}]}(v)(O_{j+1}^{[\alpha_{j}]}(w))^* - b_{\alpha_j}(v)b_{\alpha_j}(w)Q_j(v)(Q_j(w))^* \right),
\]

\[
\tau_{\alpha_j}(v, w) \left( (P_{j+1}^{[\alpha_{j+1}]}(v))^* R_{j+1}^{[\alpha_{j+1}]}(w) - (P_{j+1}(v))^* P_{j+1}(w) \right)
= \tau_{\alpha_{j+1}}(v, w) \left( (P_{j+1}^{[\alpha_{j}]}(v))^* R_{j+1}^{[\alpha_{j}]}(w) - b_{\alpha_j}(v)b_{\alpha_j}(w)(P_j(v))^* P_j(w) \right),
\]

\[
\tau_{\alpha_j}(v, w) \left( P_{j+1}^{[\alpha_{j+1}]}(v)(P_{j+1}^{[\alpha_{j+1}]}(w))^* - R_{j+1}(v)(R_{j+1}(w))^* \right)
= \tau_{\alpha_{j+1}}(v, w) \left( P_{j+1}^{[\alpha_{j}]}(v)(P_{j+1}^{[\alpha_{j}]}(w))^* - b_{\alpha_j}(v)b_{\alpha_j}(w)R_j(v)(R_j(w))^* \right),
\]

\[
\tau_{\alpha_j}(v, w) \left( (Q_{j+1}^{[\alpha_{j+1}]}(v))^* R_{j+1}^{[\alpha_{j+1}]}(w) - (O_{j+1}(v))^* P_{j+1}(w) \right)
= \tau_{\alpha_{j+1}}(v, w) \left( (Q_{j+1}^{[\alpha_{j}]}(v))^* R_{j+1}^{[\alpha_{j}]}(w) - b_{\alpha_j}(v)b_{\alpha_j}(w)(O_j(v))^* P_j(w) \right),
\]

\[
\tau_{\alpha_j}(v, w) \left( O_{j+1}^{[\alpha_{j+1}]}(v)(P_{j+1}^{[\alpha_{j+1}]}(w))^* - Q_{j+1}(v)(R_{j+1}(w))^* \right)
= \tau_{\alpha_{j+1}}(v, w) \left( O_{j+1}^{[\alpha_{j}]}(v)(P_{j+1}^{[\alpha_{j}]}(w))^* - b_{\alpha_j}(v)b_{\alpha_j}(w)Q_j(v)(R_j(w))^* \right).
\]
PROOF. Using the line of argument as stated for [24, Lemma 4.1] in the context of orthogonal rational matrix-valued functions (see also [7, Lemma 4.1]), the assertion follows by a straightforward calculation from (10). We give an example of that by proving the first identity. Let \( j \in \mathbb{N}_{0,\tau-1} \) and let \( v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,j+1} \). Because of (16) we have

\[
\mathfrak{e}_{\alpha,j+1}(w, v) = \mathfrak{e}_{\alpha,j}(w, v) \left( \frac{1 - |\alpha_{j+1}|^2}{1 - |\alpha_j|^2} \right) \left( \frac{1 - \overline{\alpha}_j w}{1 - \overline{\alpha}_{j+1} \overline{\alpha}_j \overline{\alpha}_j w} \right)
\]

and (7) implies \((H_{\mathbb{F}_{j+1}})^* \mathbf{j}_{qp} \mathbf{H}_{\mathbb{F}_{j+1}} = \mathbf{j}_{qp}\) (see, e.g., [11, Lemma 3.6.32]). Thus, by considering the lower \( p \times (q + p) \) row of \((\Xi_{j+1}(v))^*\) and the right \((q + p) \times p\) column of \(\Xi_{j+1}(w)\) we get

\[
\mathfrak{e}_{\alpha,j}(w, v) \left( (Q_{j+1}^{[\alpha,j+1]}(v))^* Q_{j+1}^{[\alpha,j+1]}(w) - (O_{j+1}(v))^* O_{j+1}(w) \right)
\]

\[
= - \mathfrak{e}_{\alpha,j}(w, v) \left( \frac{O_{j+1}(v)}{\overline{\eta}_{j+1} \eta_j Q_{j+1}^{[\alpha,j+1]}(v)} \right)^* \mathbf{j}_{qp} \left( \frac{O_{j+1}(w)}{\overline{\eta}_{j+1} \eta_j Q_{j+1}^{[\alpha,j+1]}(w)} \right)
\]

\[
= - \mathfrak{e}_{\alpha,j+1}(w, v) \left( b_{\alpha_j}(v) O_{j}(v) \right)^* \mathbf{j}_{qp} \left( b_{\alpha_j}(w) O_{j}(w) \right)
\]

\[
+ \mathfrak{e}_{\alpha,j+1}(w, v) \left( (Q_j^{[\alpha,j]}(v))^* Q_j^{[\alpha,j]}(w) - b_{\alpha_j}(v) b_{\alpha_j}(w) (O_{j}(v))^* O_{j}(w) \right),
\]

which is the first identity. \( \square \)

**Theorem 3.2.** For all \( k \in \mathbb{N}_{0,\tau} \) and all \( v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k} \), the following Christoffel-Darboux formulae hold:

\[
(1 - \overline{b}_{\alpha_k}(v) b_{\alpha_k}(w)) \sum_{j=0}^{k} (O_j(v))^* O_j(w)
\]

\[
= (Q_k^{[\alpha,k]}(v))^* Q_k^{[\alpha,k]}(w) - b_{\alpha_k}(v) b_{\alpha_k}(w) (O_k(v))^* O_k(w) - \frac{1 - |\alpha_k|^2}{(1 - \overline{\alpha}_k \alpha_k)(1 - \overline{\alpha}_k w)} \mathbf{I}_p,
\]

\[
(1 - \overline{b}_{\alpha_k}(v) b_{\alpha_k}(w)) \sum_{j=0}^{k} Q_j(v) (Q_j(w))^*
\]

\[
= O_k^{[\alpha,k]}(v) (O_k^{[\alpha,k]}(w))^* - b_{\alpha_k}(v) b_{\alpha_k}(w) Q_k(v) (O_k(w))^* + \frac{1 - |\alpha_k|^2}{(1 - \overline{\alpha}_k \alpha_k)(1 - \overline{\alpha}_k w)} \mathbf{I}_p,
\]

\[
(1 - \overline{b}_{\alpha_k}(v) b_{\alpha_k}(w)) \sum_{j=0}^{k} (P_j(v))^* P_j(w)
\]

\[
= (R_k^{[\alpha,k]}(v))^* R_k^{[\alpha,k]}(w) - b_{\alpha_k}(v) b_{\alpha_k}(w) (P_k(v))^* P_k(w) + \frac{1 - |\alpha_k|^2}{(1 - \overline{\alpha}_k \alpha_k)(1 - \overline{\alpha}_k w)} \mathbf{I}_q,
\]

\[
(1 - \overline{b}_{\alpha_k}(v) b_{\alpha_k}(w)) \sum_{j=0}^{k} R_j(v) (R_j(w))^*
\]

\[
= P_k^{[\alpha,k]}(v) (P_k^{[\alpha,k]}(w))^* - b_{\alpha_k}(v) b_{\alpha_k}(w) R_k(v) (R_k(w))^* - \frac{1 - |\alpha_k|^2}{(1 - \overline{\alpha}_k \alpha_k)(1 - \overline{\alpha}_k w)} \mathbf{I}_q,
\]
(1 - b_{\alpha_k}(v)b_{\alpha_k}(w)) \sum_{j=0}^{k} (O_j(v))^*P_j(w) = (Q_k^{[\alpha,k]}(v))^*R_k^{[\alpha,k]}(w) - b_{\alpha_k}(v)b_{\alpha_k}(w)(O_k(v))^*P_k(w),

(1 - b_{\alpha_k}(v)b_{\alpha_k}(w)) \sum_{j=0}^{k} Q_j(v)(R_j(w))^* = O_k^{[\alpha,k]}(v)(P_k^{[\alpha,k]}(w))^* - b_{\alpha_k}(v)b_{\alpha_k}(w)Q_k(v)(R_k(w))^*.

PROOF. In the case \( k = 0 \), the formulae follow immediately from the initial conditions (4) and (6) by using some elementary properties of strictly contractive matrices (see, e.g., [11, Lemma 1.1.12]). Using Lemma 3.1, the proof follows by induction. As an example we prove the first Christoffel-Darboux formula. Let \( v, w \in \mathbb{C}\setminus \mathbb{P}_{\alpha,0} \). According to (4) and (1), we have

\[
(O_0(v))^*O_0(w) = \frac{1-|\alpha_0|^2}{(1-\alpha_0\overline{v})(1-\alpha_0\overline{w})} F_0(I_q - F_0^*F_0)^{-1} F_0^* \]

and

\[
(Q_0^{[\alpha,0]}(v))^*Q_0^{[\alpha,0]}(w) = \frac{1-|\alpha_0|^2}{(1-\alpha_0\overline{v})(1-\alpha_0\overline{w})} (I_p - F_0^*F_0)^{-1}. \]

Hence, because

\[
F_0(I_q - F_0^*F_0)^{-1} F_0^* = (I_p - F_0^*F_0)^{-1} F_0^* F_0^* - (I_p - F_0^*F_0)^{-1} (F_0^*F_0^* - I_p + I_p) = -I_p + (I_p - F_0^*F_0)^{-1},
\]

we get

\[
(1 - b_{\alpha_0}(v)b_{\alpha_0}(w)) \sum_{j=0}^{0} (O_j(v))^*O_j(w) = \frac{1-|\alpha_0|^2}{(1-\alpha_0\overline{v})(1-\alpha_0\overline{w})} F_0(I_q - F_0^*F_0)^{-1} F_0^* b_{\alpha_0}(v)b_{\alpha_0}(w)(O_0(v))^*O_0(w) = (Q_0^{[\alpha,0]}(v))^*Q_0^{[\alpha,0]}(w) - b_{\alpha_0}(v)b_{\alpha_0}(w)(O_0(v))^*O_0(w) - \frac{1-|\alpha_0|^2}{(1-\alpha_0\overline{v})(1-\alpha_0\overline{w})} I_p.
\]

Thus, for the case \( k = 0 \) the first identity is verified. Now let \( k \in \mathbb{N}_{1,\tau} \) and we assume that, for all \( \ell \in \mathbb{N}_{0,k-1} \) and all \( v, w \in \mathbb{C}\setminus \mathbb{P}_{\alpha,\ell} \), the formula

\[
(1 - b_{\alpha_\ell}(v)b_{\alpha_\ell}(w)) \sum_{j=0}^{\ell} (O_j(v))^*O_j(w) = (Q_\ell^{[\alpha,\ell]}(v))^*Q_\ell^{[\alpha,\ell]}(w) - b_{\alpha_\ell}(v)b_{\alpha_\ell}(w)(O_\ell(v))^*O_\ell(w) - \frac{1-|\alpha_\ell|^2}{(1-\alpha_\ell\overline{v})(1-\alpha_\ell\overline{w})} I_p.
\]
is already proved. Therefore, taking (16) into account, an application of the first equality in Lemma 3.1 implies

\[
(1 - \frac{1}{b_{\alpha_k}(v)}b_{\alpha_k}(w)) \sum_{j=0}^{k} (O_j(v))^\ast O_j(w)
\]

\[
= \frac{\xi_{\alpha_k}(w, v)}{\xi_{\alpha_{k-1}}(w, v)} (1 - \frac{1}{b_{\alpha_{k-1}}(v)}b_{\alpha_{k-1}}(w)) \sum_{j=0}^{k-1} (O_j(v))^\ast O_j(w) + \xi_{\alpha_k}(w, v)(O_k(v))^\ast O_k(w)
\]

\[
= \frac{\xi_{\alpha_k}(w, v)}{\xi_{\alpha_{k-1}}(w, v)} \left( (Q_{k-1}^{[\alpha, k-1]}(v))^\ast Q_{k-1}^{[\alpha, k-1]}(w) - \frac{1}{b_{\alpha_{k-1}}(v)}b_{\alpha_{k-1}}(w)(O_{k-1}(v))^\ast O_{k-1}(w) \right)
\]

\[
- \frac{\xi_{\alpha_k}(w, v)}{\xi_{\alpha_{k-1}}(w, v)} \frac{1 - |\alpha_{k-1}|^2}{(1 - \alpha_{k-1}w)(1 - \alpha_{k-1}v)} I_p + \xi_{\alpha_k}(w, v)(O_k(v))^\ast O_k(w)
\]

\[
= (Q_k^{[\alpha, k]}(v))^\ast Q_k^{[\alpha, k]}(w) - (O_k(v))^\ast O_k(w) - \frac{1}{(1 - \alpha_k v)(1 - \alpha_k w)} I_p
\]

\[
+ (O_k(v))^\ast O_k(w) - \frac{1}{(1 - \alpha_k v)(1 - \alpha_k w)} I_p
\]

\[
= (Q_k^{[\alpha, k]}(v))^\ast Q_k^{[\alpha, k]}(w) - \frac{1}{(1 - \alpha_k v)(1 - \alpha_k w)} I_p
\]

firstly for \(v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha, k}\) satisfying \(\tau w \neq 1\). Applying a continuity argument one can get that this identity is actually fulfilled for all \(v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha, k}\). \(\square\)

Using the same argumentation as for orthogonal rational matrix functions (cf. [16, Section 7]), we may conclude from the (first and fourth) Christoffel-Darboux formulæ in Theorem 3.2 with the special choice \(v = w\) that the following holds (cf. Corollary 2.3).

**Corollary 3.3.** Let \(k \in \mathbb{N}_{1, r}\). For all \(v \in \mathbb{D} \cup \mathbb{T}\), the matrices \(Q_k^{[\alpha, k]}(v)\) and \(P_k^{[\alpha, k]}(v)\) are non-singular and \(O_k(v)(Q_k^{[\alpha, k]}(v))^{-1}\) and \((P_k^{[\alpha, k]}(v))^{-1}R_k(v)\) are strictly contractive matrices.

**Remark 3.4.** Let \(k \in \mathbb{N}_{1, r}\). Since \(X^{[\alpha, k]}(\alpha_k) = 0 \iff X \in \bar{\mathcal{R}}_{\alpha, k-1}^{p \times q}\) if \(X \in \mathcal{R}_{\alpha, k-1}^{p \times q}\), it follows from Corollary 3.3 that \(Q_k \in \mathcal{R}_{\alpha, k}^{p \times p}\ \bar{\mathcal{R}}_{\alpha, k}^{p \times q}\) and \(P_k \in \mathcal{R}_{\alpha, k}^{q \times q}\ \bar{\mathcal{R}}_{\alpha, k-1}^{q \times q}\). Furthermore, \(O_k \in \bar{\mathcal{R}}_{\alpha, k-1}^{q \times p}\) is possible (cf. Remark 2.9) and Theorem 3.2 implies

\[
O_k \in \bar{\mathcal{R}}_{\alpha, k-1}^{q \times p} \iff \sum_{j=0}^{k} Q_j(v)(Q_j(\alpha_k))^\ast = \frac{1}{1 - \alpha_k v} I_p, \quad v \in \mathbb{C} \setminus \mathbb{P}_{\alpha, k},
\]

\[
\iff \sum_{j=0}^{k} Q_j(\alpha_k)(Q_j(\alpha_k))^\ast = \frac{1}{1 - |\alpha_k|^2} I_p
\]

\[
\iff \sum_{j=0}^{k} R_j(v)(Q_j(\alpha_k))^\ast = 0, \quad v \in \mathbb{C} \setminus \mathbb{P}_{\alpha, k},
\]

\[
\iff \sum_{j=0}^{k} R_j(\alpha_k)(Q_j(\alpha_k))^\ast = 0.
\]
Similarly, the case \( R_k \in \mathcal{R}_{\alpha,k-1}^{q \times p} \) is possible and Theorem 3.2 shows

\[
R_k \in \mathcal{R}_{\alpha,k-1}^{q \times p} \iff \sum_{j=0}^{k} (P_j(\alpha_k))^\ast P_j(v) = \frac{1}{1 - \alpha_k v} I_q, \quad v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k},
\]

\[
\iff \sum_{j=0}^{k} (P_j(\alpha_k))^\ast P_j(\alpha_k) = \frac{1}{1 - |\alpha_k|^2} I_q
\]

\[
\iff \sum_{j=0}^{k} (P_j(\alpha_k))^\ast O_j(v) = 0, \quad v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k},
\]

\[
\iff \sum_{j=0}^{k} (P_j(\alpha_k))^\ast O_j(\alpha_k) = 0.
\]

**Remark 3.5.** Let \( k \in \mathbb{N}_{1,r} \). Based on the fifth or the sixth formulae in Theorem 3.2 (cf. the reasoning demonstrated in [16, Lemma 6.5]), for each \( v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k} \), one can derive that

\[
P_k^{[\alpha,k]}(v) O_k(v) = R_k(v) Q_k^{[\alpha,k]}(v) \quad \text{and} \quad O_k^{[\alpha,k]}(v) P_k(v) = Q_k(v) R_k^{[\alpha,k]}(v).
\]

In the remaining part of this section we study now an inverse question to Theorem 3.2. Roughly speaking, we shall see that the realization of Christoffel-Darboux formulae is in a way also a sufficient condition for systems of rational matrix functions to be SNP sequences. First some useful information on the identities stated in Lemma 3.1.

**Lemma 3.6.** Let \( \tau \in \mathbb{N} \) or \( \tau = \infty \) and let \( (\alpha_k)_{k=0}^r \) be a sequence of points belonging to \( \mathbb{D} \). Further, let \( j \in \mathbb{N}_{0,\tau-1} \) and let \( O_k \in \mathcal{R}_{\alpha,k}^{q \times p} \), \( Q_k \in \mathcal{R}_{\alpha,k}^{p \times q} \), \( P_k \in \mathcal{R}_{\alpha,k}^{p \times r} \), \( R_k \in \mathcal{R}_{\alpha,k}^{q \times p} \) for \( k \in \{j, j+1\} \). The following statements are equivalent:

(i) The first (resp., third, or fifth) identity of Lemma 3.1 holds for all \( v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,j+1} \).

(ii) The second (resp., fourth, or sixth) identity of Lemma 3.1 is satisfied for all points \( v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,j+1} \).

**Proof.** Taking (1) into account, the assertion follows by a straightforward calculation similar as presented for [24, Lemma 5.2] in the context of orthogonal rational matrix-valued functions (see also [7, Lemma 5.2]). As an example, we demonstrate this fact on the basis of the first and the second identities in Lemma 3.1. If we fix the number \( v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,j+1} \) then, in view of (16) and forming the associated rational matrix functions with respect to the other variable \( w \) and the underlying \( j + 3 \) points \( \alpha_0, \alpha_1, \ldots, \alpha_{j+1}, \alpha_j \) in virtue of (1), one can see that the first identity of Lemma 3.1 is equivalent to the equality

\[
(b_{\alpha_j}(w) - b_{\alpha_j}(v)) \left( Q_{j+1}(w) Q_{[\alpha,j+1]}^{[\alpha,j+1]}(v) - O_{j+1}^{[\alpha,j+1]}(w) O_{j+1}(v) \right)
\]

\[
= (b_{\alpha_{j+1}}(w) - b_{\alpha_{j+1}}(v)) \left( b_{\alpha_j}(w) Q_j(w) Q_{[\alpha,j]}^{[\alpha,j]}(v) - b_{\alpha_j}(v) O_j^{[\alpha,j]}(w) O_j(v) \right)
\]

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for all \(v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,j+1}\). Since, by fixing now the number \(w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,j+1}\) and forming the associated rational matrix functions in virtue of (1), this relation is equivalent to

\[
- \mathfrak{e}_{\alpha,j}(v, w) \left( Q_{j+1}(v)(Q_{j+1}(w))^* - O_{j+1}^{[\alpha,j+1]}(v)(O_{j+1}^{[\alpha,j+1]}(w))^* \right)
\]

\[
= - \mathfrak{e}_{\alpha,j+1}(v, w) \left( b_{\alpha,j}(v)\overline{b_{\alpha,j}(w)} Q_{j}(v)(Q_{j}(w))^* - O_{j}^{[\alpha,j]}(v)(O_{j}^{[\alpha,j]}(w))^* \right)
\]

for all \(v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,j+1}\). Therefore, we obtain finally the equivalence of the first and the second identity of Lemma 3.1. \(\square\)

**Lemma 3.7.** Let \(\tau \in \mathbb{N}\) or \(\tau = \infty\) and let \((\alpha_k)_{k=0}^\tau\) be a sequence of points belonging to \(\mathbb{D}\). Further, let \((O_k)_{k=0}^\tau\), \((Q_k)_{k=0}^\tau\), \((P_k)_{k=0}^\tau\), and \((R_k)_{k=0}^\tau\) be sequences of rational matrix functions such that \(O_0\), \(Q_0\), \(P_0\), and \(R_0\) are defined by (4) and (6) and that \(O_k \in \mathcal{R}_{\alpha,k}^{q \times p}\), \(Q_k \in \mathcal{R}_{\alpha,k}^{p \times p}\), \(P_k \in \mathcal{R}_{\alpha,k}^{q \times q}\), and \(R_k \in \mathcal{R}_{\alpha,k}^{q \times q}\) for \(k \in \mathbb{N}_{1,\tau}\). The following statements are equivalent:

(i) For all \(j \in \mathbb{N}_0,\tau-1\) and all \(v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,j+1}\), the first (resp., second, third, fourth, fifth, or sixth) identity of Lemma 3.1 is fulfilled.

(ii) For all \(k \in \mathbb{N}_{1,\tau}\) and all \(v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}\), the first (resp., second, third, fourth, fifth, or sixth) Christoffel-Darboux formula in Theorem 3.2 holds.

**Proof.** By using the same arguments as in the proof of Theorem 3.2, one can inductively show that (i) implies (ii). It remains to verify that (ii) implicates (i). This can be done similarly as explained for [24, Lemma 5.3] in the context of orthogonal rational matrix-valued functions (see also [7, Lemma 5.3]). Exemplarily, we show this with respect to the first identity of Lemma 3.1 and the first Christoffel-Darboux formula in Theorem 3.2. As already explained in the proof of Theorem 3.2 in detail, (4) leads to the validity of the first Christoffel-Darboux formula in Theorem 3.2 for \(k = 0\) and all \(v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,0}\). Moreover, for \(k \in \mathbb{N}_{1,\tau}\), the first formula in Theorem 3.2 is satisfied for all \(v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}\) if and only if

\[
\left(1 - \overline{b_{\alpha,j}(v)} b_{\alpha,j}(w)\right) \sum_{j=0}^{k-1} (O_j(v))^*O_j(w)
\]

\[
= \left(Q_k^{[\alpha,k]}(v)\right)^*Q_k^{[\alpha,k]}(w) - (O_k(v))^*O_k(w) - \frac{1 - |\alpha_k|^2}{(1-\alpha_k\overline{v})(1-\alpha_kw)}I_p
\]

for all \(v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}\). Consequently, for all \(j \in \mathbb{N}_{0,\tau-1}\) and all \(v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,j+1}\), from (16) and the first formula in Theorem 3.2 it follows

\[
\mathfrak{e}_{\alpha,j}(w, v) \left(\left(Q_{j+1}^{[\alpha,j+1]}(v)\right)^*Q_{j+1}^{[\alpha,j+1]}(w) - (O_{j+1}(v))^*O_{j+1}(w)\right)
\]

\[
= \mathfrak{e}_{\alpha,j}(w, v) \left(\left(Q_{j+1}^{[\alpha,j+1]}(v)\right)^*Q_{j+1}^{[\alpha,j+1]}(w) - (O_{j+1}(v))^*O_{j+1}(w)\right)
\]

\[
- \mathfrak{e}_{\alpha,j}(w, v) \frac{1 - |\alpha_{j+1}|^2}{(1-\alpha_{j+1}\overline{v})(1-\alpha_{j+1}w)}I_p + \mathfrak{e}_{\alpha,j}(w, v) \frac{1 - |\alpha_{j+1}|^2}{(1-\alpha_{j+1}\overline{v})(1-\alpha_{j+1}w)}I_p
\]

\[
= \mathfrak{e}_{\alpha,j+1}(w, v) \sum_{\ell=0}^{j} (O_\ell(v))^*O_\ell(w) + \mathfrak{e}_{\alpha,j+1}(w, v) \frac{1 - |\alpha_j|^2}{(1-\alpha_j\overline{v})(1-\alpha_jw)}I_p
\]

\[
= \mathfrak{e}_{\alpha,j+1}(w, v) \left(\left(Q_{j}^{[\alpha,j]}(v)\right)^*Q_{j}^{[\alpha,j]}(w) - \overline{b_{\alpha,j}(v)} b_{\alpha,j}(w) (O_j(v))^*O_j(w)\right).
\]
Theorem 3.8. Let $\tau \in \mathbb{N}_0$ or $\tau = \infty$, let $(\alpha_k)_{k=0}^\tau$ be a sequence of points belonging to $\mathbb{D}$, and let $(O_k)_{k=0}^\tau$, $(Q_k)_{k=0}^\tau$, $(P_k)_{k=0}^\tau$, and $(R_k)_{k=0}^\tau$ be sequences of rational matrix functions such that the following five conditions are satisfied:

(I) $O_k \in \mathcal{R}_{\alpha,k}^{q \times p}$, $Q_k \in \mathcal{R}_{\alpha,k}^{p \times q}$, $P_k \in \mathcal{R}_{\alpha,k}^{p \times q}$, and $R_k \in \mathcal{R}_{\alpha,k}^{q \times p}$ for each $k \in \mathbb{N}_0, \tau$.

(II) If $k \in \mathbb{N}_0, \tau$ then the first or the second Christoffel-Darboux formula in Theorem 3.2 is fulfilled for all $v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$.

(III) If $k \in \mathbb{N}_0, \tau$ then the third or the fourth Christoffel-Darboux formula in Theorem 3.2 is fulfilled for all $v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$.

(IV) If $k \in \mathbb{N}_0, \tau$ then the fifth or the sixth Christoffel-Darboux formula in Theorem 3.2 is fulfilled for all $v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$.

(V) $Q_0(\alpha_0) \geq 0$ and $P_0(\alpha_0) \geq 0$ as well as the relations

\[
\frac{\eta_k \eta_{k-1}}{1 - \alpha_k \alpha_{k-1}} (Q_k^{[\alpha,k]}(\alpha_{k-1})^* Q_{k-1}^{[\alpha,k-1]}(\alpha_{k-1}) \geq 0,
\]

\[
\frac{\eta_k \eta_{k-1}}{1 - \alpha_k \alpha_{k-1}} P_k^{[\alpha,k-1]}(\alpha_{k-1})(P_{k-1}^{[\alpha,k]}(\alpha_{k-1})^* \geq 0
\]

are fulfilled for each $k \in \mathbb{N}_1, \tau$.

Then, for each $k \in \mathbb{N}_0, \tau$ and each $v \in \mathbb{D} \cup \mathbb{T}$, the matrices $Q_k^{[\alpha,k]}(v)$ and $P_k^{[\alpha,k]}(v)$ are non-singular and if we put

\[
F_k := \eta_k \eta_{k-1} (O_k(\alpha_{k-1})(Q_k^{[\alpha,k]}(\alpha_{k-1})^{-1})^*, \quad k \in \mathbb{N}_0, \tau,
\]

where $\alpha_{-1} := 0$ and $\eta_{-1} := -1$, then $[(O_k)_{k=0}^\tau, (Q_k)_{k=0}^\tau]$ is the first SNP pair of rational matrix functions corresponding to $(\alpha_k, F_k)_{k=0}^\tau$ and $[(P_k)_{k=0}^\tau, (R_k)_{k=0}^\tau]$ is the second SNP pair of rational matrix functions corresponding to $(\alpha_k, F_k)_{k=0}^\tau$.

Proof. First, we remark that in view of the conditions (I), (II), (III), and (IV) an application of Lemma 3.7 and Lemma 3.6 yields that all Christoffel-Darboux formula stated in Theorem 3.2 are satisfied for all $k \in \mathbb{N}_0, \tau$ and all $v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$. We consider now the case $k = 0$. From the first and the fifth Christoffel-Darboux formulae in Theorem 3.2 it follows

\[
(O_0(v))^* O_0(w) = (Q_0^{[\alpha_0]}(v))^* Q_0^{[\alpha_0]}(w) - \frac{1 - |\alpha_0|^2}{1 - \alpha_0 \bar{w}(1 - \alpha_0 w) P_0} I_p
\]

(17)

and

\[
(P_0(w))^* O_0(v) = (R_0^{[\alpha_0]}(w))^* Q_0^{[\alpha_0]}(v)
\]

(18)

for all $v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,0}$. In particular, we can infer from (17) that the matrix $Q_0^{[\alpha,0]}(v)$ is non-singular for each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,0}$ and that by setting

\[
F_0 := -\eta_0 (O_0(0)(Q_0^{[\alpha,0]}(0))^{-1})^*
\]

(19)
a strictly contractive $p \times q$ matrix is defined. Similarly, using the fourth formula stated in Theorem 3.2 one can see that the matrix $P_0^{[α,0]}(v)$ is non-singular for each $v ∈ \mathbb{C} \setminus \mathbb{P}_{α,0}$. Because of (I) there is a $Q ∈ \mathbb{C}^{p×p}$ such that

$$Q_0(v) = \frac{\sqrt{1-|α_0|^2}}{1-\overline{α_0}v}Q$$

(20)

for each $v ∈ \mathbb{C} \setminus \mathbb{P}_{α,0}$. Since (17) and (19) leads particularly to

$$I_p - F_0 F_0^* = (1 - |α_0|^2) \left(\frac{Q_0^{[α,0]}(0)(Q_0^{[α,0]}(0))^*}{1-\overline{α_0}v}\right)^{-1},$$

in view of (20), (1), and $Q_0(α_0) ≥ 0$ we get $Q = (I_p - F_0 F_0^*)^{-\frac{1}{2}}$, i.e.

$$Q_0(v) = \frac{\sqrt{1-|α_0|^2}}{1-\overline{α_0}v}(I_p - F_0 F_0^*)^{-\frac{1}{2}}$$

(21)

for each $v ∈ \mathbb{C} \setminus \mathbb{P}_{α,0}$. Moreover, (18) and (1) result in

$$P_0^{[α,0]}(w)O_0(v) = R_0(w)Q_0^{[α,0]}(v)$$

(22)

for all $v, w ∈ \mathbb{C} \setminus \mathbb{P}_{α,0}$. Consequently, by virtue of (19) we have

$$F_0 = -\overline{η_0} \left( (P_0^{[α,0]}(0))^{-1}R_0(0) \right)^*$$

which implies in combination with (22), (21), (1), and some elementary properties of strictly contractive matrices (see, e.g., [11, Lemma 1.1.12]) then

$$O_0(v) = (P_0^{[α,0]}(0))^{-1}R_0(0)Q_0^{[α,0]}(v)$$

$$= (-\overline{η_0}F_0^*) \left( -\overline{η_0} \frac{\sqrt{1-|α_0|^2}}{1-\overline{α_0}v} (I_p - F_0 F_0^*)^{-\frac{1}{2}} \right)$$

$$= \frac{\sqrt{1-|α_0|^2}}{1-\overline{α_0}v} (I_q - F_0^* F_0)^{-\frac{1}{2}} F_0^*$$

(23)

for each $v ∈ \mathbb{C} \setminus \mathbb{P}_{α,0}$. Thus, the initial condition (4) is proved. Particularly for $τ = 0$ it is shown that $[(O_k)_{k=0}^\tau, (Q_k)_{k=0}^\tau]$ is the first SNP pair of rational matrix functions corresponding to $(α_k, F_k)_{k=0}^\tau$. Now let $τ ∈ \mathbb{N}$ or $τ = ∞$ and let $k ∈ \mathbb{N}_{1,τ}$. Since the assumptions include that the first and the fourth Christoffel-Darboux formula in Theorem 3.2 hold, we get that the matrices $Q_k^{[α,k]}(v)$ and $P_k^{[α,k]}(v)$ are non-singular for each $v ∈ \mathbb{D} \cup \mathbb{T}$ and that

$$F_k := \overline{η_k}F_{k-1} \left( O_k(α_{k-1})(Q_k^{[α,k]}(α_{k-1}))^{-1} \right)^*$$

(24)

is a strictly contractive $p \times q$ matrix (note Corollary 3.3). From (V) it follows

$$\frac{η_kη_{k-1}}{1-α_kα_{k-1}}Q_k^{[α,k]}(α_{k-1})(Q_k^{[α,k]}(α_{k-1}))^{-1} ≥ 0.$$ 

(25)
In view of $b_{\alpha,k-1}(\alpha_{k-1}) = 0$, Lemma 3.7, and Lemma 3.1 we obtain
\[
\begin{align*}
(Q^{[\alpha,k]}_k(\alpha_{k-1}))^*Q^{[\alpha,k]}_k(v) - (O_k(\alpha_{k-1}))^*O_k(v) \\
= (1-b_{\alpha,k-1}(\alpha_{k-1})b_{\alpha,k-1}(v)) \left((Q^{[\alpha,k]}_k(\alpha_{k-1}))^*Q^{[\alpha,k]}_k(v) - (O_k(\alpha_{k-1}))^*O_k(v) \right) \\
= (1-b_{\alpha,k}(\alpha_{k-1})b_{\alpha,k}(v))(Q^{[\alpha,k-1]}_{k-1}(\alpha_{k-1}))^*Q^{[\alpha,k-1]}_{k-1}(v)
\end{align*}
\]
for each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$. In particular, (26), (24), (16), and (25) lead to
\[
(I_p - F_k^*F_k)^{\frac{1}{2}} = \left( I_p - \left( (Q^{[\alpha,k]}_k(\alpha_{k-1}))^* \right)^{-1}(O_k(\alpha_{k-1}))^*O_k(\alpha_{k-1})(Q^{[\alpha,k]}_k(\alpha_{k-1}))^{-1} \right)^{\frac{1}{2}}
= \left( \frac{(1-|\alpha|^2)(1-|\alpha_{k-1}|^2)}{1-\alpha^*_k\alpha_{k-1}}Q_k^*Q_k \right)^{\frac{1}{2}}
= \frac{\eta_k \overline{\eta}_{k-1}}{c_k}Q^{[\alpha,k-1]}_{k-1}(\alpha_{k-1})(Q^{[\alpha,k]}_k(\alpha_{k-1}))^{-1},
\]
where $Q_k := Q^{[\alpha,k-1]}_{k-1}(\alpha_{k-1})(Q^{[\alpha,k]}_k(\alpha_{k-1}))^{-1}$ and where $c_k$ is the nonzero number defined as in Proposition 2.1 (for technical reasons). With a view to (1), the relation (26) implies
\[
Q_k(v)Q^{[\alpha,k]}_k(\alpha_{k-1}) - O^{[\alpha,k]}_k(v)O_k(\alpha_{k-1}) = (b_{\alpha,k}(v) - b_{\alpha,k}(\alpha_{k-1}))Q_{k-1}(v)Q^{[\alpha,k-1]}_{k-1}(\alpha_{k-1})
\]
for each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$. Consequently, an application of (24) and (27) yields
\[
\begin{align*}
\eta_k \overline{\eta}_{k-1}Q_k(v) - O^{[\alpha,k]}_k(v)F_k^* \\
= \eta_k \overline{\eta}_{k-1}Q_k(v) - \eta_k \overline{\eta}_{k-1}O^{[\alpha,k]}_k(v)O_k(\alpha_{k-1})(Q^{[\alpha,k]}_k(\alpha_{k-1}))^{-1} \\
= \eta_k \overline{\eta}_{k-1}(b_{\alpha,k}(v) - b_{\alpha,k}(\alpha_{k-1}))Q_{k-1}(v)Q^{[\alpha,k-1]}_{k-1}(\alpha_{k-1})(Q^{[\alpha,k]}_k(\alpha_{k-1}))^{-1} \\
= c_k(b_{\alpha,k}(v) - b_{\alpha,k}(\alpha_{k-1}))Q_{k-1}(v)(I_p - F_k^*F_k)^{\frac{1}{2}}
\end{align*}
\]
for each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$. In addition, by the sixth Christoffel-Darboux formula stated in Theorem 3.2, Lemma 3.7, and $b_{\alpha,k-1}(\alpha_{k-1}) = 0$ it follows that
\[
O^{[\alpha,k]}_k(v)(P^{[\alpha,k]}_k(\alpha_{k-1}))^* - Q_k(v)(R_k(\alpha_{k-1}))^* = (1-b_{\alpha,k}(v)\overline{b}_{\alpha,k}(\alpha_{k-1}))O^{[\alpha,k-1]}_{k-1}(v)(P^{[\alpha,k-1]}_{k-1}(\alpha_{k-1}))^*
\]
for each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$. In view of (1), for each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$, one can conclude
\[
P^{[\alpha,k]}_k(\alpha_{k-1})O_k(v) - R_k(\alpha_{k-1})Q^{[\alpha,k]}_k(v) = (b_{\alpha,k}(v) - b_{\alpha,k}(\alpha_{k-1}))P^{[\alpha,k-1]}_{k-1}(\alpha_{k-1})O_{k-1}(v).
\]
Thus, we see $P^{[\alpha,k]}_k(\alpha_{k-1})O_k(v) = R_k(\alpha_{k-1})Q^{[\alpha,k]}_k(\alpha_{k-1})$ so that (24) results in
\[
F_k = \overline{\eta}_{k-1}(P^{[\alpha,k]}_k(\alpha_{k-1}))^{-1}R_k(\alpha_{k-1})
\]
and similar to (27) we attain
\[
(I_q - F_k^*F_k)^{\frac{1}{2}} = \frac{\eta_k \overline{\eta}_{k-1}}{c_k}P^{[\alpha,k]}_{k-1}(\alpha_{k-1})^{-1}P^{[\alpha,k-1]}_{k-1}(\alpha_{k-1}^{-1}).
\]
To summarize the considerations after (28), in analogy to (28) we obtain
\[
\eta_k - O_{k-1}(v) - F^*_{k}Q^{[\alpha,k]}(v) = c_k(b_{\alpha_k}(v) - b_{\alpha_k}(\alpha_{k-1}))(I_q - F^*_{k}F_{k})\frac{1}{2}O_{k-1}(v) \tag{29}
\]
for each \( v \in \mathbb{C} \setminus P_{\alpha,k} \). In the end, by virtue of (I), (21), (23), (28), (29), and Proposition 2.1 we get that \([O_{k}]_{k=0}^{\tau}, [Q_{k}]_{k=0}^{\tau}\) is the first SNP pair of rational matrix functions corresponding to \((\alpha_k, F_{k})_{k=0}^{\tau}\). A similar argumentation can be used for \([P_{k}]_{k=0}^{\tau}, [R_{k}]_{k=0}^{\tau}\). □

4. Descriptions of problems of Nevanlinna-Pick type in the non-degenerate case

In the present section, we show that SNP pairs of rational matrix functions can be used to describe the solution set of interpolation problems of Nevanlinna-Pick type for matrix-valued Schur functions. The characterization of the solution set as a linear fractional matrix transformation is by now very well known. In fact, the characterization of many far-reaching generalizations of this case are known too and many different approaches to directional or tangential Nevanlinna-Pick type interpolation problems for possibly operator-valued problems with or without extra constraints and with or without allowing a finite number of poles inside the disk are available. For all of these the reader is referred to the vast literature. Since we only include it for illustrative purposes we restrict ourselves to the simplest non-degenerate case in order not to deviate too far from the main aim of this paper.

Recall that a function \( V: \mathbb{D} \to \mathbb{C}^{p \times q} \) is called a \( p \times q \) Schur function (in \( \mathbb{D} \)) if \( V \) is holomorphic in \( \mathbb{D} \) and if its values \( V(w) \) are contractive for all \( w \in \mathbb{D} \). The notation \( S_{p \times q}(\mathbb{D}) \) will stand for the set of all \( p \times q \) Schur functions (in \( \mathbb{D} \)).

The interpolation problem in question is pointed out in the following.

Problem (MNP):
Let \( m \in \mathbb{N}_0 \), let \( l_0, l_1, \dotsc, l_m \in \mathbb{N}_0 \), let \( \beta_0, \beta_1, \dotsc, \beta_m \) be mutually distinct points in \( \mathbb{D} \), and let \( V_{kt} \) be a complex \( p \times q \) matrix for all \( t \in \mathbb{N}_0, l_k \) and all \( k \in \mathbb{N}_0,m \). Find necessary and sufficient conditions for the existence of a \( p \times q \) Schur function \( V \) satisfying
\[
\frac{1}{t!}V(t)(\beta_k) = V_{kt}, \quad t \in \mathbb{N}_0, l_k \), \( k \in \mathbb{N}_0,m \). \tag{30}
\]
Furthermore, describe the set \( \mathcal{S}_{\Delta} \) of all \( p \times q \) Schur functions \( V \) fulfilling (30).

With regard to Problem (MNP), we denote the given data by \( \Delta \), i.e.
\[
\Delta := \left\{ \left( \beta_k, l_k, (V_{kt})_{t=0}^{l_k} \right)_{k=0}^{m} \right\}. \tag{31}
\]
Based on a function \( V \in S_{p \times q}(\mathbb{D}) \) we define similarly
\[
\Delta[V] := \left\{ \left( \beta_k, l_k, \left( \frac{1}{t!}V(t)(\beta_k) \right)_{t=0}^{l_k} \right)_{k=0}^{m} \right\}.
\]
Furthermore, we put
\[
n := m + \sum_{k=0}^{m} l_k.
\]
The generalized Schwarz-Pick-Potapov block matrix (with respect to the data $\Delta$) of size $(n+1)p \times (n+1)p$ (resp., $(n+1)q \times (n+1)q$) is defined as

$$P_\Delta := (P_{jk})_{j,k=0}^m \quad \text{(resp., } \tilde{P}_\Delta := (\tilde{P}_{jk})_{j,k=0}^m \text{)},$$

where the complex $((l_j+1)p \times (l_k+1)p)$ (resp., $((l_j+1)q \times (l_k+1)q)$) matrices

$$P_{jk} := (\eta_{jk})_{s=0,1,\ldots,l_j t=0,1,\ldots,l_k} \quad \text{(resp., } \tilde{P}_{jk} := (\tilde{\eta}_{jk})_{s=0,1,\ldots,l_j t=0,1,\ldots,l_k} \text{), } j,k \in \mathbb{N}_{0,m},$$

are determined by the entries

$$P_{st}^{(jk)} := \sum_{r=0}^{\min\{s,t\}} \left( \begin{array}{c} s+t-r \choose r \end{array} \right) \beta_j^{s-r} \tilde{\beta}_k^{t-r} \frac{1}{(s-r)! (t-r)! (1-\beta_j \bar{\beta}_k)^{s+t-r+1}} I_p$$

$$- \sum_{\ell=0}^{s} \sum_{h=0}^{t} \sum_{r=0}^{\min\{r,h\}} \frac{(h+\ell-r)!}{(\ell-r)! (h-r)!} \beta_j^{h-r} \tilde{\beta}_k^{r-h} \frac{1}{(1-\beta_j \bar{\beta}_k)^{h+\ell-r+1}} V_{j-s \ell} V_{k,t-h}.$$

(resp.,

$$\tilde{P}_{st}^{(jk)} := \sum_{r=0}^{\min\{s,t\}} \left( \begin{array}{c} s+t-r \choose r \end{array} \right) \tilde{\beta}_j^{s-r} \beta_k^{t-r} \frac{1}{(s-r)! (t-r)! (1-\beta_j \bar{\beta}_k)^{s+t-r+1}} I_q$$

$$- \sum_{\ell=0}^{s} \sum_{h=0}^{t} \sum_{r=0}^{\min\{r,h\}} \frac{(h+\ell-r)!}{(\ell-r)! (h-r)!} \beta_j^{h-r} \tilde{\beta}_k^{r-h} \frac{1}{(1-\beta_j \bar{\beta}_k)^{h+\ell-r+1}} V_{j-s \ell}^* V_{k,t-h}.$$

It is well known (see, e.g., [13, Section 5 in Chapter X], [4, Section 1.1 in Chapter 1], or [23, Theorem 2.1]), that there exists a $V \in \mathcal{S}_{p \times q}(\mathbb{D})$ fulfilling (30) if and only if $P_\Delta \geq 0$ (resp., $\tilde{P}_\Delta \geq 0$). The considerations below are concentrated on the non-degenerate case, i.e. the situation that $P_\Delta > 0$ (resp., $\tilde{P}_\Delta > 0$). In particular, we will see that the solution set $\mathcal{S}_\Delta$ of Problem (MNP) can be parameterized then by linear fractional matrix transformations, where elements of special SNP pairs of rational matrix functions are involved.

With the interpolation points $\beta_0, \beta_1, \ldots, \beta_m$ in the given data $\Delta$ we form a sequence $(\alpha_k)_{k=0}^n$ in which $\beta_j$ appears according to its multiplicity $l_j + 1$ times for each $j \in \mathbb{N}_{0,m}$. For instance, we can choose $\alpha_k := \gamma_k$ with

$$\gamma_k := \beta_j \quad \text{if } \quad j + \sum_{r=0}^{j-1} l_r \leq k \leq \sum_{r=0}^{j} l_r, \quad j \in \mathbb{N}_{0,m}. \quad (32)$$

However, in the following, it is not essential that equal points are successors, i.e. for an arbitrary bijective mapping $\varphi$ of $\{0, 1, \ldots, n\}$ onto itself we can put

$$\alpha_k := \gamma_{\varphi(k)}, \quad k \in \mathbb{N}_{0,n}. \quad (33)$$

In the sequel, $[(O_k)_{k=0}^n, (Q_k)_{k=0}^n]$ (resp., $[(P_k)_{k=0}^n, (R_k)_{k=0}^n]$) stands for the first (resp., second) SNP pair of rational matrix functions corresponding to $(\alpha_k, F_k)_{k=0}^n$ with some sequence $(F_k)_{k=0}^n$ of strictly contractive $p \times q$ matrices.
Remark 4.1. In view of Corollary 2.3 and some elementary properties of strictly contractive matrices (see, e.g., [11, Remark 1.1.2 and part (a) of Lemma 1.1.13]) one can conclude that, for all $S \in \mathcal{S}_{p \times q}(\mathbb{D})$ and all $w \in \mathbb{D}$, the matrices $P_n^{[\alpha,n]}(w) + b_{\alpha, n}(w)R_n(w)S(w)$ and $Q_n^{[\alpha,n]}(w) + b_{\alpha, n}(w)S(w)O_n(w)$ are both non-singular. Moreover, Corollary 2.3 implies in combination with the second identity in Corollary 2.7 that by setting

$$V_0(w) := O_n^{[\alpha,n]}(w)(P_n^{[\alpha,n]}(w))^{-1}, \quad w \in \mathbb{D},$$

a function belonging to $\mathcal{S}_{p \times q}(\mathbb{D})$ is well-defined, where

$$V_0(w) = (Q_n^{[\alpha,n]}(w))^{-1}P_n^{[\alpha,n]}(w), \quad w \in \mathbb{D}.$$

Now we apply some basics on linear fractional matrix transformations (see, e.g., [26, 12, 11, 2]). Like in [11, Section 1.6] we use (occasionally for short) the following notation. If $A \in \mathbb{C}^{p \times p}$, $B \in \mathbb{C}^{p \times q}$, $C \in \mathbb{C}^{q \times p}$, $D \in \mathbb{C}^{q \times q}$, and

$$\Theta := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{(resp., } \Xi := \begin{pmatrix} D & C \\ B & A \end{pmatrix} \text{)}$$

and if $X \in \mathbb{C}^{p \times q}$ such that $\det(CX + D) \neq 0$ (resp., $\det(XC + A) \neq 0$) holds then we put

$$\mathfrak{S}_{\Theta}(X) := (AX + B)(CX + D)^{-1} \quad \text{(resp., } \mathfrak{T}_{\Xi}(X) := (XC + A)^{-1}(XD + B) \text{)}.$$

Lemma 4.2. Let $S \in \mathcal{S}_{p \times q}(\mathbb{D})$. Then the function $V : \mathbb{D} \to \mathbb{C}^{p \times q}$ given by

$$V(w) := \left( O_n^{[\alpha,n]}(w) + b_{\alpha, n}(w)Q_n(w)S(w) \right) \left( P_n^{[\alpha,n]}(w) + b_{\alpha, n}(w)R_n(w)S(w) \right)^{-1}$$

belongs to $\mathcal{S}_{p \times q}(\mathbb{D})$, it admits for each $w \in \mathbb{D}$ the representation

$$V(w) = \left( Q_n^{[\alpha,n]}(w) + b_{\alpha, n}(w)S(w)O_n(w) \right)^{-1} \left( P_n^{[\alpha,n]}(w) + b_{\alpha, n}(w)S(w)P_n(w) \right),$$

and $\Delta[V] = \Delta[V_0]$, where $V_0$ is the matrix function given by (34). Moreover, the matrices $-P_n(w) + O_n(w)V(w)$ and $-Q_n(w) + V(w)R_n(w)$ are non-singular for each $w \in \mathbb{D} \setminus \mathbb{Z}_{\alpha,n}$, whereby the function $S$ can be recovered via

$$S(w) = \frac{1}{b_{\alpha, n}(w)}(R_n^{[\alpha,n]}(w) - Q_n^{[\alpha,n]}(w)V(w))^{-1} \left( -P_n(w) + O_n(w)V(w) \right)^{-1}, \quad (36)$$

and

$$S(w) = \frac{1}{b_{\alpha, n}(w)}(-Q_n(w) + V(w)R_n(w))^{-1} \left( O_n^{[\alpha,n]}(w) - V(w)P_n^{[\alpha,n]}(w) \right). \quad (37)$$

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Proof. Let \( w \in \mathbb{D} \). In addition to the definition of \( V \), we use in the following

\[
W(w) := \left( Q_n^{[\alpha,n]}(w) + b_{\alpha_n}(w)S(w)O_n(w) \right)^{-1} \left( R_n^{[\alpha,n]}(w) + b_{\alpha_n}(w)S(w)P_n(w) \right).
\]

In view of Remark 4.1 and the choice of \( V \) (resp., \( W \)) this matrix-valued function is not only well-defined but also holomorphic in \( \mathbb{D} \), where we can also write

\[
V(w) = \mathcal{E}_{\Theta(w)}(S(w)) \quad \text{(resp., } \, W(w) = \mathcal{E}_{\Xi(w)}(S(w)) \text{)},
\]

by using (35) with

\[
\Theta(w) := \frac{1-\alpha_n w}{\sqrt{1-|\alpha_n|^2}} \begin{pmatrix} b_{\alpha_n}(w)Q_n(w) & O_n^{[\alpha,n]}(w) \\ b_{\alpha_n}(w)R_n(w) & P_n^{[\alpha,n]}(w) \end{pmatrix}
\]

\[
(\text{resp., } \Xi(w) := \frac{1-\alpha_n w}{\sqrt{1-|\alpha_n|^2}} \begin{pmatrix} b_{\alpha_n}(w)P_n(w) & b_{\alpha_n}(w)O_n(w) \\ R_n^{[\alpha,n]}(w) & Q_n^{[\alpha,n]}(w) \end{pmatrix}).
\]

Due to Theorem 2.2 the matrix \( \Theta(w) \) is \( j_{pq} \)-contractive (resp., \( \Xi(w) \) is \( j_{qp} \)-contractive). Therefore, from [11, Theorem 1.6.1] we see that \( V(w) \) (resp., \( W(w) \)) is a contractive \( p \times q \) matrix, since \( S(w) \) is a contractive \( p \times q \) matrix. Hence, \( V \) (resp., \( W \)) belongs to \( S_{p \times q}(\mathbb{D}) \). Moreover, taking into account that \( V \) and \( W \) are particularly holomorphic in \( \mathbb{D} \) and that Proposition 2.5 yields in combination with [11, Proposition 1.6.1] the identity

\[
\mathcal{E}_{\Theta(v)}(S(v)) = \mathcal{E}_{\Xi(v)}(S(v)), \quad v \in \mathbb{D} \setminus \mathbb{Z}_{\alpha,n},
\]

by a continuity argument one can find that

\[
V(w) = \left( Q_n^{[\alpha,n]}(w) + b_{\alpha_n}(w)S(w)O_n(w) \right)^{-1} \left( R_n^{[\alpha,n]}(w) + b_{\alpha_n}(w)S(w)P_n(w) \right).
\]

Furthermore, by setting \( N(w) := P_n^{[\alpha,n]}(w) + b_{\alpha_n}(w)R_n(w)S(w) \), Remark 4.1 leads in combination with Corollary 2.7 to the relation

\[
V(w) - V_o(w) = \left( Q_n^{[\alpha,n]}(w) \right)^{-1} \left( Q_n^{[\alpha,n]}(w)O_n^{[\alpha,n]}(w) - b_{\alpha_n}(w)Q_n^{[\alpha,n]}(w)Q_n(w)S(w) - R_n^{[\alpha,n]}(w)P_n^{[\alpha,n]}(w) - b_{\alpha_n}(w)R_n^{[\alpha,n]}(w)R_n(w)S(w) \right)(N(w))^{-1}.
\]

\[
= -\eta_n \frac{1 - |\alpha_n|^2}{(1-\alpha_n w)^2} B_{\alpha,n}(w) \left( Q_n^{[\alpha,n]}(w) \right)^{-1} S(w) \left( N(w) \right)^{-1}.
\]

(38)

Since the Blaschke product \( B_{\alpha,n} \) has a zero of order \( l_j + 1 \) at the point \( \beta_j \) for each \( j \in \mathbb{N}_{0,m} \) (note (32), (33), (2), and (3)), one can finally conclude

\[
V^{(t)}(\beta_k) = V^{(t)}(\beta_k), \quad t \in \mathbb{N}_{0,d}, \quad k \in \mathbb{N}_{0,m},
\]

i.e. \( \Delta^{[V]} = \Delta^{[V]} \). The rest of the assertion follows from Corollary 2.6 using a basic result on inverting linear fractional matrix transformations (see, e.g., [11, Proposition 1.6.2]).
Lemma 4.3. If \([\tilde{O}_k]_{k=0}^{n}, \tilde{Q}_k)_{k=0}^{n}\) (resp., \((\tilde{P}_k)_{k=0}^{n}, \tilde{R}_k)_{k=0}^{n}\)) is the first (resp., the second) SNP pair of rational matrix functions corresponding to \((\alpha_k, \tilde{F}_k)_{k=0}^{n}\), with some sequence \((\tilde{F}_k)_{k=0}^{n}\) of strictly contractive \(p \times q\) matrices, so that \(\Delta[\tilde{V}_0] = \Delta[\tilde{V}_0]\), where \(\tilde{V}_0\) is defined as in (34) and the function \(\tilde{V}_0\) similarly by

\[
\tilde{V}_0(w) := \tilde{O}_n^{[a,n]}(w)(\tilde{P}_n^{[a,n]}(w))^{-1}, \quad w \in \mathbb{D},
\]

then the equality \(\tilde{F}_k = F_k\) holds for each \(k \in N_{0,n}\).

**Proof.** By virtue of (4), (6), and (1) we have for each \(w \in \mathbb{D}\) the relations

\[
\tilde{F}_0 = \tilde{O}_0^{[a,0]}(w)(\tilde{P}_0^{[a,0]}(w))^{-1}, \quad F_0 = O_0^{[a,0]}(w)(P_0^{[a,0]}(w))^{-1},
\]

and so \(\tilde{F}_0 = F_0\) follows evidently if \(n = 0\). Now let \(n > 0\). In view of the recursions defining SNP pairs of rational matrix functions, (1), and the condition \(\Delta[\tilde{V}_0] = \Delta[\tilde{V}_0]\) we obtain that the values of the matrix-valued functions

\[
\left(O_{n-1}^{[a,n-1]} + b_{a,n-1}\tilde{Q}_n-1\tilde{F}_n\right)\left(P_{n-1}^{[a,n-1]} + b_{a,n-1}\tilde{R}_n-1\tilde{F}_n\right)^{-1},
\]

\[
\left(O_{n-1}^{[a,n-1]} + b_{a,n-1}\tilde{Q}_n-1\tilde{F}_n\right)\left(P_{n-1}^{[a,n-1]} + b_{a,n-1}\tilde{R}_n-1\tilde{F}_n\right)^{-1}
\]

and their derivatives up to the order \(l_j\) at the point \(\beta_j\) for each \(j \in N_{0,m}\) coincide. Because of Lemma 4.2, a successive continuation of this procedure yields that, for each \(k \in N_{1,n}\), the values of the matrix-valued functions

\[
\left(O_{k-1}^{[a,k-1]} + b_{a,k-1}\tilde{Q}_{k-1}\tilde{F}_k\right)\left(P_{k-1}^{[a,k-1]} + b_{a,k-1}\tilde{R}_{k-1}\tilde{F}_k\right)^{-1},
\]

\[
\left(O_{k-1}^{[a,k-1]} + b_{a,k-1}\tilde{Q}_{k-1}\tilde{F}_k\right)\left(P_{k-1}^{[a,k-1]} + b_{a,k-1}\tilde{R}_{k-1}\tilde{F}_k\right)^{-1}
\]

and their derivatives up to the order \(r_{j,k} - 1\) at any point \(\beta_j\) contained in the sequence \((\alpha_{\ell})_{\ell=0}^{k}\) (where \(r_{j,k}\) stands for the number how many times) coincide and in particular that

\[
O_0^{[a,0]}(\alpha_0)(\tilde{F}_0^{[a,0]}(\alpha_0))^{-1} = O_0^{[a,0]}(\alpha_0)(P_0^{[a,0]}(\alpha_0))^{-1}.
\]

In the following, by induction on \(k\), we verify that \(\tilde{F}_k = F_k\) holds for each \(k \in N_{0,n}\). In the case of \(k = 0\), the equalities (40) and (41) supply immediately

\[
\tilde{F}_0 = F_0.
\]

Now let \(k \in N_{1,n}\) and we assume that \(\tilde{F}_j = F_j\) is already proved for each \(j \in N_{0,k-1}\). In view of this induction assumption, the definition of SNP pairs of rational matrix functions, and the considerations below of the proof we obtain that the values of the functions

\[
\left(O_{k-1}^{[a,k-1]} + b_{a,k-1}\tilde{Q}_{k-1}\tilde{F}_k\right)\left(P_{k-1}^{[a,k-1]} + b_{a,k-1}\tilde{R}_{k-1}\tilde{F}_k\right)^{-1},
\]

\[
\left(O_{k-1}^{[a,k-1]} + b_{a,k-1}\tilde{Q}_{k-1}\tilde{F}_k\right)\left(P_{k-1}^{[a,k-1]} + b_{a,k-1}\tilde{R}_{k-1}\tilde{F}_k\right)^{-1}
\]
\[(O_{k-1}^{[\alpha,k-1]} + b_{\alpha_k-1} Q_{k-1} F_k)(P_{k-1}^{[\alpha,k-1]} + b_{\alpha_k-1} R_{k-1} F_k)^{-1}\]

and their derivatives up to the order \(r_{j,k} - 1\) at any point \(\beta_j\) contained in the sequence \((\alpha_{\ell})_{\ell=0}^k\) coincide on the one hand and on the other hand, by setting

\[N_k := P_{k-1}^{[\alpha,k-1]} + b_{\alpha_k-1} R_{k-1} F_k \quad \text{and} \quad \tilde{N}_k := Q_{k-1}^{[\alpha,k-1]} + b_{\alpha_k-1} \tilde{F}_k O_{k-1}.\]

Remark 4.1, Lemma 4.2, and Corollary 2.7 provide for each \(w \in D\) the identity

\[(O_{k-1}^{[\alpha,k-1]}(w) + b_{\alpha_k-1}(w)Q_{k-1}(w)F_k)(P_{k-1}^{[\alpha,k-1]}(w) + b_{\alpha_k-1}(w)R_{k-1}(w)F_k)^{-1}\]

\[= \left(\tilde{N}_k(w)^{-1} \left(Q_{k-1}^{[\alpha,k-1]}(w)O_{k-1}^{[\alpha,k-1]}(w) + b_{\alpha_k-1}(w)\tilde{F}_k O_{k-1}(w)O_{k-1}^{[\alpha,k-1]}(w)\right) + b_{\alpha_k-1}(w)Q_{k-1}^{[\alpha,k-1]}(w)Q_{k-1}(w)F_k + \left(b_{\alpha_k-1}(w)\right)^2 \tilde{F}_k O_{k-1}(w)Q_{k-1}(w)F_k\]

\[- b_{\alpha_k-1}(w)Q_{k-1}(w)R_{k-1}(w)F_k - \left(b_{\alpha_k-1}(w)\right)^2 \tilde{F}_k R_{k-1}(w)F_k - R_{k-1}^{[\alpha,k-1]}(w)F_{k-1}^{[\alpha,k-1]}(w) - b_{\alpha_k-1}(w)\tilde{F}_k F_{k-1}(w)F_{k-1}^{[\alpha,k-1]}(w)(N_k(w))^{-1}\]

\[\eta_{k-1} \frac{1 - |\alpha_{k-1}|^2}{(1 - |\alpha_{k-1}|)^2} B_{\alpha_k-1}(w) \left(\tilde{N}_k(w)^{-1}(\tilde{F}_k - F_k)(N_k(w))^{-1}\right).\]

Since \(\eta_{k-1}(1 - |\alpha_{k-1}|^2) \neq 0\) and since the Blaschke product \(B_{\alpha_{k-1}}\) has in view of (32), (33), (2), and (3) only \(k\) zeroes (at \(\alpha_0, \alpha_1, \ldots, \alpha_{k-1}\)), one can finally conclude \(\tilde{F}_k = F_k\). \(\square\)

For the reader’s convenience, below we state briefly the Schur-Nevanlinna-Potapov algorithm (SNP algorithm for short) for \(p \times q\) Schur functions studied in [15, Section 5] (see also [9]). Let \(S \in \mathcal{S}_{p \times q}(D)\) and let \(\zeta_0, \zeta_1, \zeta_2, \ldots \in D\). We set \(S_0 := S\) and \(S_0 := S_0(\zeta_0)\). If \(S_0\) is a strictly contractive \(p \times q\) matrix then for each \(w \in D\) the matrix \(I_q - S_0^* S_0(w)\) is non-singular (cf. [11, Remark 1.1.2 and Lemma 1.1.13]) and we can define the matrix function (holomorphic in \(D\))

\[S_1 := \frac{1}{b_{\zeta_0}}(I_p - S_0 S_0^*)^{-\frac{1}{2}}(S_0 - S_0)(I_q - S_0^* S_0)^{-1}(I_q - S_0^* S_0)^{\frac{1}{2}}\]

and recursively if for \(k \in \mathbb{N}_0\) the function \(S_k\) is already defined and

\[S_k := S_k(\zeta_k)\] (42)

is a strictly contractive \(p \times q\) matrix then

\[S_{k+1} := \frac{1}{b_{\zeta_k}}(I_p - S_k S_k^*)^{-\frac{1}{2}}(S_k - S_k)(I_q - S_k^* S_k)^{-1}(I_q - S_k^* S_k)^{\frac{1}{2}},\] (43)

where \(b_{\zeta_k}\) is given as in (3) concerning \(\zeta_k\). If \(S \in \mathcal{S}_{p \times q}(D)\) and \(\zeta_0, \zeta_1, \zeta_2, \ldots \in D\) so that the SNP algorithm can be carried out at least \(r\) times (that is after obtaining \(S_r\) and \(S_r\)) then \((S_k)_{k=0}^r\) given by (42) is called the sequence of SNP parameters associated with \([S, (\zeta_k)_{k=0}^r]\).
Using (7), (35), and [11, Lemma 1.1.12], the relation (43) can be written as

\[ b_{\zeta_k}(w)S_{k+1}(w) = \mathcal{S}_{s_{k}}(S_k(w)), \quad w \in \mathbb{D}. \]

Thus, for each \( w \in \mathbb{D} \), [11, Proposition 1.6.2 and Lemma 3.6.32] imply

\[ S_k(w) = \mathcal{S}_{(H_{-s_k})^{-1}}(b_{\zeta_k}(w)S_{k+1}(w)) = \mathcal{S}_{\Phi_k}(S_{k+1}(w)) \]

with

\[ \Phi_k(w) := H_{s_k} \left( \begin{pmatrix} b_{\zeta_k}(w)I_p & 0 \\ 0 & I_q \end{pmatrix} \right) \]

and, by virtue of [11, Proposition 1.6.3], consequently

\[ S(w) \equiv S_0(w) = \mathcal{S}_{\Phi_0}(\mathcal{S}_{\Phi_1}(\cdots (\mathcal{S}_{\Phi_k}(S_{k+1}(w))) \cdots )) \quad (44) \]

The algorithm defines \( p \times q \) Schur functions \( S_0, S_1, S_2, \ldots \) (cf. (43) and [11, Theorem 1.6.1]). It breaks down after the \( r \)-th step (that means after obtaining \( S_r \) and \( S_r \)) if and only if \( I_p - S_r^*S_r \) (resp., \( I_q - S_rS_r^* \)) is a singular matrix.

Now we are able to prove the already announced parameterization of the solution set \( S_\Delta \) of Problem (MNP) in the non-degenerate case by linear fractional matrix transformations in terms of SNP pairs of rational matrix functions.

**Theorem 4.4.** Let \( \Delta \) be a given data set (31) whereby \( P_\Delta > 0 \) (resp., \( \widetilde{P}_\Delta > 0 \)). Further, let \( V_\bullet \in S_\Delta \), let \( (\alpha_k)_{k=0}^n \) be given by (32) and (33), and let \( (S_k)_{k=0}^n \) be the sequence of SNP parameters associated with \( [V_\bullet, (\alpha_k)_{k=0}^n] \). If \( V \in S_{p \times q}(\mathbb{D}) \) then the following are equivalent:

(i) \( V \in S_\Delta \).

(ii) \( (S_k)_{k=0}^n \) is the sequence of SNP parameters associated with \( [V, (\alpha_k)_{k=0}^n] \).

In addition, if \( [(O_k)_{k=0}^n, (Q_k)_{k=0}^n] \) (resp., \( [(P_k)_{k=0}^n, (R_k)_{k=0}^n] \)) stands for the first (resp., second) SNP pair of rational matrix functions corresponding to \( (\alpha_k, F_k)_{k=0}^n \), where \( F_0 := S_0 \) and \( F_k := -\eta_{k-1}s_k \) for each \( k \in \mathbb{N}_{1,n} \), then the statement (i) is also equivalent to:

(iii) There is an \( S \in S_{p \times q}(\mathbb{D}) \) so that \( V \) admits, for all \( w \in \mathbb{D} \), the description

\[ V(w) = \left( O_n^{[a,n]}(w) + b_{a_n}(w)Q_n(w)S(w) \right) \left( P_n^{[a,n]}(w) + b_{a_n}(w)R_n(w)S(w) \right)^{-1}. \]

(iv) There is an \( \tilde{S} \in S_{p \times q}(\mathbb{D}) \) so that \( V \) admits, for all \( w \in \mathbb{D} \), the description

\[ V(w) = \left( O_n^{[a,n]}(w) + b_{a_n}(w)\tilde{S}(w)O_n(w) \right)^{-1} \left( P_n^{[a,n]}(w) + b_{a_n}(w)\tilde{S}(w)P_n(w) \right). \]

Moreover, if (i) is satisfied then \( S = \tilde{S} \) and the \( p \times q \) Schur function \( S \) can be recovered, for each \( w \in \mathbb{D} \setminus \mathbb{Z}_{a,n} \), via the formulae (36) and (37).
\textbf{Proof.} Let \( w \in \mathbb{D} \). Note firstly that a combination of [15, Corollary 2.7 and Theorem 5.5] with [23, Corollary 3.2] implies that the SNP algorithm can be carried out at least \( n + 1 \) times for some \( V \in \mathcal{S}_{p \times q}(\mathbb{D}) \) and for any choice of points \( \zeta_0, \zeta_1, \ldots, \zeta_{n+1} \in \mathbb{D} \) if and only if the matrix \( \mathbf{P}_\Delta^{[n]} \) (resp., \( \mathbf{P}_\Delta^{[q]} \)) is non-singular. Thus, by the assumption \( \mathbf{P}_\Delta > 0 \) (resp., \( \mathbf{P}_\Delta > 0 \)), for any solution of Problem (MNP) the SNP algorithm can be carried out (at least) \( n + 1 \) times. Consequently, we can suppose in the following a given \( n \) times for some \( V \). Consequently, we can suppose in the following a given \( n \times q \) Schur function \( V \) for which the SNP algorithm can be carried out (at least) \( n + 1 \) times. In particular (cf. (42) and (44)), we find SNP parameters \( \tilde{\mathbf{S}}_0, \tilde{\mathbf{S}}_1, \ldots, \tilde{\mathbf{S}}_n \) associated with the pair \([V, (\alpha_k)_{n=0}^n]\) which are strictly contractive \( p \times q \) matrices and a \( p \times q \) Schur function \( V_{n+1} \) such that

\[
V(w) = \mathcal{S}_{\Phi(w)}(V_{n+1}(w))
\] (45)

holds, where

\[
\Phi(w) := \prod_{k=0}^{n} \tilde{\Phi}_k(w) := \tilde{\Phi}_0(w)\tilde{\Phi}_1(w) \cdots \tilde{\Phi}_n(w)
\]

and (by using (7))

\[
\tilde{\Phi}_k(w) := \mathbf{H}_{\tilde{\mathbf{S}}_k}\left(\begin{array}{cc} b_{\alpha_k}(w)I_p & 0 \\ 0 & I_q \end{array}\right), \quad k \in \mathbb{N}_{0,n}.
\]

By virtue of (7) and (9), with \( \eta_{-1} := -1 \) one can also write

\[
\Phi(w) = \left(\prod_{k=0}^{n} \mathbf{H}_{\eta_{k-1}\tilde{\mathbf{S}}_k}\left(\begin{array}{cc} b_{\alpha_k}(w)I_p & 0 \\ 0 & I_q \end{array}\right)\right)\left(\begin{array}{cc} I_p & 0 \\ 0 & -\eta_n I_q \end{array}\right).
\] (46)

We define now \([\tilde{\mathcal{O}}_k]_{n=0}^n, \tilde{\mathcal{Q}}_k^n_{n=0}\) (resp., \([\tilde{\mathcal{R}}_k]_{k=0}^n, \tilde{\mathcal{P}}_k^n_{k=0}\)) as the first (resp., second) SNP pair of rational matrix functions corresponding to \((\alpha_k, \tilde{\mathbf{F}}_{k=0}^n)\) with \( \tilde{\mathbf{F}}_k := -\eta_{k-1}\tilde{\mathbf{S}}_k \) for each \( k \in \mathbb{N}_{0,n} \) subject to Section 2. Thus, setting

\[
\tilde{\mathcal{O}}_k(w) := \mathbf{H}_{\tilde{\mathbf{F}}_k}\left(\begin{array}{cc} b_{\alpha_k}(w)I_p & 0 \\ 0 & \eta_k\eta_{k-1} I_q \end{array}\right), \quad k \in \mathbb{N}_{0,n}.
\]

Theorem 2.2 yields the identity

\[
\begin{pmatrix} b_{\alpha_n}(w)\tilde{\mathcal{Q}}_n(w) & \tilde{\mathcal{O}}_n^{[n]}(w) \\ b_{\alpha_n}(w)\tilde{\mathcal{R}}_n(w) & \tilde{\mathcal{P}}_n^{[n]}(w) \end{pmatrix} = \frac{1-|\alpha_n|^2}{1-\alpha_n w} \tilde{\mathcal{O}}_0(w)\tilde{\mathcal{O}}_1(w) \cdots \tilde{\mathcal{O}}_n(w),
\]

while (46) implies

\[
\Phi(w) = \left(\prod_{k=0}^{n} \tilde{\mathcal{O}}_k(w)\right)\left(\begin{array}{cc} I_p & 0 \\ 0 & -\eta_n I_q \end{array}\right).
\]

Therefore, by (45), Remark 4.1, (35), and [11, Proposition 1.6.3], we see that

\[
V(w) = \left(\tilde{\mathcal{O}}_n^{[n]}(w) + b_{\alpha_n}(w)\tilde{\mathcal{Q}}_n(w)S(w)\right)\left(\tilde{\mathcal{P}}_n^{[n]}(w) + b_{\alpha_n}(w)\tilde{\mathcal{R}}_n(w)S(w)\right)^{-1}
\] (47)
with \( S(w) := -\eta_n V_{n+1}(w) \). Obviously, \( S \in \mathcal{S}_{p \times q}(\mathbb{D}) \) because \( V_{n+1} \in \mathcal{S}_{p \times q}(\mathbb{D}) \). Moreover, via the construction of the matrix functions \( \hat{O}_n \) and \( \hat{P}_n \), Lemma 4.2 provides that the \( p \times q \) Schur function \( \hat{V}_0 \) given as in (39) fulfills \( \Delta^{[V]} = \Delta^{[V_0]} \). In particular, since \( V_\bullet \in \mathcal{S}_\Delta \), the considerations above supply that

\[
V_\bullet(w) = \left( \hat{O}_n^{[\alpha,n]}(w) + b_\alpha(w)Q_n(w)S_\bullet(w) \right) \left( P_n^{[\alpha,n]}(w) + b_\alpha(w)R_n(w)S_\bullet(w) \right)^{-1}
\]

for some \( S_\bullet \in \mathcal{S}_{p \times q}(\mathbb{D}) \) and that the \( p \times q \) Schur function \( V_0 \) given as in (34) fulfills the identity \( \Delta^{[V]} = \Delta^{[V_0]} \). Consequently, if \( V \in \mathcal{S}_\Delta \) then

\[
\Delta^{[V]} = \Delta^{[V_0]} = \Delta^{[V_\bullet]}
\]

and hence Lemma 4.3 yields \( \tilde{F}_k = F_k \) (i.e. \( \tilde{S}_k = S_k \)) for each \( k \in \mathbb{N}_{0,n} \). Therefore, (i) implicates (ii). Furthermore, if (ii) holds then (47) implies that \( V \) admits the representation stated in (iii) with \( S(w) := -\eta_n V_{n+1}(w) \). If (iii) holds then from Lemma 4.2 one can get

\[
\Delta^{[V]} = \Delta^{[V_0]} = \Delta^{[V_\bullet]},
\]

i.e. \( V \in \mathcal{S}_\Delta \). Thus, the statements (i), (ii), and (iii) are equivalent. Based on it, the remaining part of the assertion is an easy consequence of Lemma 4.2.

Observe that the equivalence of (i) and (ii) in Theorem 4.4 is closely related to [11, Corollary 3.8.1] which is a matrix version of Schur’s classical result that, for each \( l \in \mathbb{N} \), there is a one-to-one correspondence between the first \( l \) Taylor coefficients of a Schur function at the point zero and the first \( l \) corresponding Schur parameters. Clearly, applying appropriate conformal mappings of the open unit disk \( \mathbb{D} \) onto itself, one can obtain a similar result with respect to arbitrary points \( \beta_0, \beta_1, \ldots, \beta_m \in \mathbb{D} \). Nevertheless, it seems to be really hard and unwieldy to derive the equivalence of (i) and (ii) directly from [11, Corollary 3.8.1], since the underlying sequence \( (\alpha_k)_{k=0}^{n} \) has only to fulfill (33) and hence the points \( \alpha_0, \alpha_1, \ldots, \alpha_n \) are not strictly in the order as \( (\gamma_k)_{k=0}^{n} \) defined by (32). Furthermore, in the particular case of the finite Taylor coefficient problem at zero, i.e. if \( m := 0 \) and if \( \beta_0 := 0 \) in Problem (MNP) is chosen, the equivalence of (i), (iii), and (iv) in Theorem 4.4 leads to [14, Theorem 14].

For a fixed point \( w \in \mathbb{D} \), now we study the geometric structure of the set

\[
\mathcal{R}_{\Delta,w} := \{ V(w) : V \in \mathcal{S}_\Delta \}
\]

in the non-degenerate situation. Clearly, we have

\[
\mathcal{R}_{\Delta,\beta_k} = \{ V_{k_0} \}, \quad k \in \mathbb{N}_{0,m},
\]

where \( V_{k_0} \) is the prescribed value in Problem (MNP). In general, as in similar kind of interpolation problems (cf. [22, Section 2]), the set \( \mathcal{R}_{\Delta,w} \) is a so-called Weyl matrix ball \( \mathcal{R}(M; L, R) \) with certain complex \( p \times q \) matrix \( M \), complex \( p \times p \) matrix \( L \), and complex \( q \times q \) matrix \( R \), i.e. the set of all complex \( p \times q \) matrices \( X \) fulfilling \( X = M + LKR \) for some contractive \( p \times q \) matrix \( K \). We present now appropriate parameters.
Proposition 4.5. Under the assumptions of Theorem 4.4, if \( w \in \mathbb{D} \) is fixed then the set \( \mathfrak{R}_{\Delta, w} \) in (48) can be described by

\[
\mathfrak{R}_{\Delta, w} = \mathfrak{r}\left( M_{n,w}, |B_{\alpha,n}(w)|L_{n,w}^{\frac{1}{2}}, R_{n,w}^{\frac{1}{2}} \right),
\]

where

\[
M_{n,w} := \frac{|1 - \alpha_n|}{1 - |\alpha_n|^2} L_{n,w} \left( (Q_n^{[\alpha,n]}(w))^* P_n^{[\alpha,n]}(w) - |b_{\alpha,n}(w)|^2 (O_n(w))^* P_n(w) \right),
\]

\[
L_{n,w} := \frac{1 - |\alpha_n|^2}{|1 - \alpha_n w|^2} \left( (Q_n^{[\alpha,n]}(w))^* Q_n^{[\alpha,n]}(w) - |b_{\alpha,n}(w)|^2 (O_n(w))^* O_n(w) \right)^{-1},
\]

\[
R_{n,w} := \frac{1 - |\alpha_n|^2}{|1 - \alpha_n w|^2} \left( (P_n^{[\alpha,n]}(w))^* P_n^{[\alpha,n]}(w) - |b_{\alpha,n}(w)|^2 R_n(w)(R_n(w))^* \right)^{-1}.
\]

Moreover, the complex \( p \times q \) matrix \( M_{n,w} \) admits also the representation

\[
M_{n,w} = \frac{|1 - \alpha_n|}{1 - |\alpha_n|^2} \left( O_n^{[\alpha,n]}(w) (P_n^{[\alpha,n]}(w))^* - |b_{\alpha,n}(w)|^2 Q_n(w)(R_n(w))^* \right) R_{n,w}.
\]

Proof. The assertion can be deduced from a general result on linear fractional transformations (cf. [9, Appendix], [22, Theorem 1 in §2], and [12, Chapter 6]). Essentially, following the line of argument as in the proof of [14, Theorem 16], based on Theorem 4.4, Corollary 2.3, and Corollary 2.6 we get the assertion. \( \square \)

Remark 4.6. Let \( w \in \mathbb{D} \) be fixed. If we set \( S_n(w) := -b_{\alpha,n}(w)O_n(w)(Q_n^{[\alpha,n]}(w))^{-1} \) in view of Proposition 4.5, Corollary 2.3, and Corollary 2.7 then

\[
M_{n,w} = \left( Q_n^{[\alpha,n]}(w) + b_{\alpha,n}(w)(S_n(w))^* O_n(w) \right)^{-1} \left( P_n^{[\alpha,n]}(w) + b_{\alpha,n}(w)(S_n(w))^* P_n(w) \right),
\]

\[
M_{n,w} = \left( Q_n^{[\alpha,n]}(w) + b_{\alpha,n}(w)Q_n(w)(S_n(w))^* \right) \left( P_n^{[\alpha,n]}(w) + b_{\alpha,n}(w)R_n(w)(S_n(w))^* \right)^{-1},
\]

\[
L_{n,w} = \frac{1 - |\alpha_n|^2}{|1 - \alpha_n w|^2} \left( Q_n^{[\alpha,n]}(w) \right)^{-1} \left( I_q - (S_n(w))^* S_n(w) \right)^{-1} \left( Q_n^{[\alpha,n]}(w)^{-1} \right)^*,
\]

\[
R_{n,w} = \frac{1 - |\alpha_n|^2}{|1 - \alpha_n w|^2} \left( (P_n^{[\alpha,n]}(w))^{-1} \right)^* \left( I_q - S_n(w)(S_n(w))^* \right)^{-1} \left( P_n^{[\alpha,n]}(w)^{-1} \right).\]

In particular (keep in mind Proposition 2.10 and [11, Lemma 1.1.8]), we get

\[
det L_{n,w} = det R_{n,w}. \tag{50}
\]

Note that the center \( M \) of a matrix ball \( \mathfrak{r}(M; L, R) \) is uniquely determined, but not the semi-radii \( L \) and \( R \) (see, e.g., [11, Corollary 1.5.1 and Theorem 1.5.2]). Consequently, the concrete order of the points in the underlying sequence \( (\alpha_j)_{j=1}^n \) subject to (32) and (33) does obviously not have an influence on the shape of the matrix \( M_{n,w} \) given by Proposition 4.5. In view of (50) and [11, Theorem 1.5.2] one can see that also the choice of the matrices \( L_{n,w} \) and \( R_{n,w} \) according to Proposition 4.5 includes this property.
Remark 4.7. If $w \in \mathbb{D}$ and if $S_n$ stands for the constant function on $\mathbb{D}$ with value $(S_n(w))^*$, where $S_n(w)$ is defined as in Remark 4.6, then Theorem 4.4 and Remark 4.6 show that, by

$$V_n(v) := (Q_n^{[\alpha, n]}(v) + b_{\alpha_n}(w)S_n(v)O_n(v))^{-1} (R_n^{[\alpha, n]}(v) + b_{\alpha_n}(w)S_n(v)P_n(v)), \quad v \in \mathbb{D},$$

a $p \times q$ Schur function $V_n$ is defined which belongs to $S_{\Delta}$ and which satisfies additionally the identity $V_n(w) = M_{n,w}$. Moreover, if we define the matrix functions $V_\bullet$ and $S_\bullet$ via

$$V_\bullet(w) := M_{n,w} \quad \text{and} \quad S_\bullet(w) := (S_n(w))^*, \quad w \in \mathbb{D},$$

then $V_\bullet$ is continuous, satisfies $V_\bullet(\beta_k) = V_{k0}$ for all $k \in \mathbb{N}_{0,m}$, and admits the representation

$$V_\bullet(w) = (Q_n^{[\alpha, n]}(w) + b_{\alpha_n}(w)S_\bullet(w)O_n(w))^{-1} (R_n^{[\alpha, n]}(w) + b_{\alpha_n}(w)S_\bullet(w)P_n(w))$$

for each $w \in \mathbb{D}$. However, since the matrix function $S_\bullet$ is not holomorphic in $\mathbb{D}$ (in particular $S_\bullet \not\in S_{p \times q}(\mathbb{D})$), in view of Theorem 4.4 it follows $V_\bullet \not\in S_{\Delta}$.

Remark 4.8. By Theorem 3.2 and (16) we see that, for each $w \in \mathbb{D}$, the matrices $M_{n,w}$, $L_{n,w}$, and $R_{n,w}$ stated in Proposition 4.5 can be recovered via

$$\frac{1}{1-|w|^2}M_{n,w} = L_{n,w}\left(\sum_{j=0}^{n} (O_j(w))^*P_j(w)\right) = \left(\sum_{j=0}^{n} Q_j(w)(R_j(w))^*\right)R_{n,w},$$

$$L_{n,w} = (I_p + (1-|w|^2)\sum_{j=0}^{n} (O_j(w))^*O_j(w))^{-1}, \quad R_{n,w} = (I_q + (1-|w|^2)\sum_{j=0}^{n} R_j(w)(R_j(w))^*)^{-1}.$$

Remark 4.9. Let the assumptions of Theorem 4.4 be fulfilled, let $k \in \mathbb{N}_{0,m}$, and let the underlying sequence $(\alpha_j)_{j=0}^n$ be satisfying the additional condition $\alpha_n = \beta_k$. Clearly (note (49) and Proposition 4.5), we have

$$V_{k0} = 0 \iff M_{n,\beta_k} = 0,$$

where in the case of $n = 0$ from (42), (4), and (6) it follows that $V_{00} = 0$ if and only if $O_0$ (resp., $R_0$) is the complex $q \times p$ matrix function with value 0. Moreover, if $n \geq 1$ then, by virtue of Remark 4.8 and Remark 3.4, we see that

$$V_{k0} = 0 \iff O_n \in \mathcal{R}_{q \times p}^{q \times p} \iff R_n \in \mathcal{R}_{q \times p}^{q \times p}.$$

Let $k \in \mathbb{N}_{0,m}$. Since $\mathcal{R}_{\Delta, \beta_k}$ given by (48) contains (at least) the required value $V_{k0}$ from Problem (MNP), this set is extraneous. But following the idea of [33, Section 6] concerning complex-valued functions, if we consider in the non-degenerate case instead the set

$$\mathcal{R}_{\Delta, \beta_k} := \left\{ \frac{1}{(\ell_k+1)!} V^{(\ell_k+1)}(\beta_k) : V \in S_{\Delta} \right\}$$

then $\mathcal{R}_{\Delta, \beta_k}$ fills a (nontrivial) matrix ball as well, where based on the matrices $L_{n,\beta_k}$ and $R_{n,\beta_k}$ defined in Proposition 4.5 one can calculate semi-radii of this matrix ball. This will be emphasized by the concluding statement of this paper.
Proposition 4.10. Let the assumptions of Theorem 4.4 be fulfilled. Let \( k \in \mathbb{N}_{0,m} \), let the underlying sequence \( (\alpha_j)_{j=0}^n \) be satisfying the additional condition \( \alpha_n = \beta_k \), and let the complex matrices \( L_{n,\beta_k} \) and \( R_{n,\beta_k} \) be defined as in Proposition 4.5. Furthermore, let
\[
M'_{n,\beta_k} := \frac{1}{(l_k+1)!} V^{(l_k+1)}(\beta_k),
\]
where the rational matrix-valued function \( V_o \) is given by (34). Then
\[
\mathcal{R}_{\Delta,\beta_k} = \mathcal{R}\left(M'_{n,\beta_k}; \frac{1}{(1-|\beta_k|^2)^{l_k+1}} \left( \prod_{j=0}^m |b_{\beta_j}(\beta_k)|^{l_j+1} \right) \mathcal{L}_{n,\beta_k}^\frac{1}{2}, \mathcal{R}_{n,\beta_k}^\frac{1}{2} \right).
\]

Proof. The definition of \( L_{n,\beta_k} \) (resp., \( R_{n,\beta_k} \)) implies by virtue of \( \alpha_n = \beta_k \), \( b_{\alpha_n}(\alpha_n) = 0 \), and the polar decomposition of matrices the existence of a unitary \( p \times p \) matrix \( V \) (resp., unitary \( q \times q \) matrix \( U \)) such that the relation
\[
\frac{1}{\sqrt{1-|\beta_k|^2}} \left( Q_n^{[\alpha_n]}(\beta_k) \right)^{-1} = \mathcal{L}_{n,\beta_k}^\frac{1}{2} V \left( \text{resp.,} \frac{1}{\sqrt{1-|\beta_k|^2}} \left( P_n^{[\alpha_n]}(\beta_k) \right)^{-1} = \mathcal{U} R_{n,\beta_k}^\frac{1}{2} \right) \tag{51}
\]
holds. Moreover, taking into account Theorem 4.4, (32), (33), (2), \( b_{\alpha_n}(\alpha_n) = 0 \), \( \alpha_n = \beta_k \), and the fact that by setting \( g := b_{\alpha_n}^{l_k+1} \) it follows
\[
\frac{1}{(l_k+1)!} g^{(l_k+1)}(\alpha_n) = \frac{1}{(\eta_n(\alpha_n^2 - 1))^{l_k+1}},
\]
based on (38), a straightforward calculation yields
\[
\frac{1}{(l_k+1)!} V^{(l_k+1)}(\beta_k) = \frac{1}{(l_k+1)!} V_o^{(l_k+1)}(\beta_k)
\]
\[
= \frac{1}{(\eta_n(\beta_k^2 - 1))^{l_k+2}} \prod_{j=0}^m (b_{\beta_j}(\beta_k))^{l_j+1} \left( Q_n^{[\alpha_n]}(\beta_k) \right)^{-1} S(\beta_k) \left( P_n^{[\alpha_n]}(\beta_k) \right)^{-1}
\]
for each \( V \in \mathcal{S}_\Delta \) with some \( S \in \mathcal{S}_{p \times q}(\mathbb{D}) \) (and any \( S \in \mathcal{S}_{p \times q}(\mathbb{D}) \) can appear). Therefore, in view of (51), one can conclude the assertion. \( \square \)

References

