Orthogonal Laurent polynomials and quadrature formulas for unbounded intervals: II. Interpolatory rules

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Abstract

We study the convergence of quadrature formulas for integrals over the positive real line with an arbitrary (possibly complex) distribution function. The nodes of the quadrature formulas are the zeros of orthogonal Laurent polynomials with respect to an auxiliary distribution function and a certain nesting. The quadratures are called interpolatory (product) formulas. The class of functions for which convergence holds is characterized in terms of the moments of the auxiliary distribution function. We also include the convergence analysis of related two-point Padé-type approximants to the Stieltjes transform of the given distribution function. Finally, some illustrative numerical examples are also given.

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1 Introduction

In this paper we shall be mainly concerned with the estimation of integrals

$$I(f,\mu) = \int_a^b f(x)\mu(x)\,\mathrm{d}x, \quad 0 \le a < b \le +\infty \tag{1.1}$$

with μ L_1-Lebesgue-integrable and f at least Riemann-integrable whose singularities can only be the origin and/or infinity.

To approximate $I(f,\mu)$ we use quadrature rules of the form

$$I_n(f,\mu) = \sum_{j=1}^{n} A_{jn} f(x_{jn}) \tag{1.2}$$

(which are called *product integration rules* — see [22]), whose nodes $\{x_{jn}\}_1^n$ are preassigned in (a,b) and the weights or coefficients $\{A_{jn}\}_1^n$ are determined by requiring that (1.2) is exact for functions f in a linear space of dimension at least n. Thus, when the moment integrals

$$c_k = \int_a^b x^k \mu(x)\,\mathrm{d}x, \quad k = 0,1,\dots$$

exist and are easily computed, then the weights are defined by imposing that

$$I_n(P,\mu) = I(P,\mu), \quad \forall P \in \Pi_{n-1} \tag{1.3}$$

or equivalently by requiring $I_n(x^k,\mu) = I(x^k,\mu) = c_k$, $k = 0,1,\dots,n-1$ (we use the notation Π_k to denote the space of polynomials of degree at most k for k a nonnegative integer, while Π will denote the space of all polynomials).

In fact, μ could also be a weight function on (a,b), i.e., $\mu(x) > 0$ a.e. on (a,b). In this case, by an appropriate choice of the nodes $\{x_{jn}\}$, formulas of the form (1.2) can be found that integrate exactly all polynomials up to a degree that is much higher than $n-1$. When the formulas have the highest possible degree of exactness in the set of polynomials that can be obtained in this way, they are called *Gaussian formulas*. However if μ is not a "standard" weight function (see e.g., [13]), the calculation of the Gaussian formulas requires a long computational process, with possible numerical instability problems. For this reason, formula (1.2) satisfying (1.3) with "easily computable nodes" could be desirable even when μ is a weight function. Such quadratures should exhibit some nice properties so that their accuracy and efficiency can be assured. Concerning the latter properties, we define the following. A sequence of rules $\{I_n(f,\mu)\}_1^\infty$ like (1.2) is said to be *convergent in a class* $\mathcal{A}$ iff $\lim_{n\to\infty} I_n(f,\mu) = I(f,\mu)$ for all $f \in \mathcal{A}$. Obviously, it seems natural to make the class $\mathcal{A}$ as large as possible. On the other hand, a sequence $\{I_n(f,\mu)\}_1^\infty$ is said to be (numerically) *stable* if there exists a positive constant M (independent of n) such that

$$\sum_{j=1}^{n} |A_{jn}| \leq M, \quad n = 1, 2, \ldots \tag{1.4}$$

Condition (1.4) means that the possible roundoff errors in the evaluation of $f(x_{jn})$ remain under control during the computation. On the other hand, it should be also noticed that the success of the rules (1.2) strongly depends on the smoothness of the integrand f and the appropriate choice of the nodes $\{x_{jn}\}_1^n$. Thus, when f is assumed to be smooth on (a,b), formulas $I_n(f,\mu)$ satisfying (1.3) give excellent results by taking $\{x_{jn}\}$ as the zeros of polynomials orthogonal with respect to an appropriate weight function (see e.g., the excellent paper [22] when dealing with a finite interval and [23] for the unbounded case). However, the situation changes drastically when f posseses singularities near (a,b). This is precisely the starting point of this paper where we will assume that the integrand f may exhibit singularities at the origin and/or at infinity. Thus it seems quite reasonable that our proposed rules integrate exactly the rational functions with all their poles at the origin and/or at infinity, i.e., functions of the form:

$$L(x) = \sum_{j=p}^{q} \alpha_j x^j, \quad p, q \in \mathbb{Z}, \quad p \leq q.$$

They are called Laurent polynomials. For our purposes, it should be also assumed that now the integrals

$$c_k = \int_a^b x^k \mu(x)\, \mathrm{d}x, \quad 0 \leq a < b \leq +\infty \tag{1.5}$$

exist and that they can be easily computed, not only for the positive integers, but for all integers $k = 0, \pm1, \pm2, \ldots$. For p and q nonnegative integers, we shall write

$$\Lambda_{-p,q} = \operatorname{span}\{x^j : -p \leq j \leq q\},$$

(observe that $\Lambda_{0,n} = \Pi_n$), while Λ represents the space of all Laurent polynomials (L-polynomials for short).

When taking p and q such that $p + q = n - 1$, ($n \in \mathbb{N}$ fixed) and since $\Lambda_{-p,q}$ is a Chebyshev system of dimension n on (a, b) $(0 \leq a < b \leq +\infty)$ [20], we can, starting from n distinct nodes $x_{1n}, \ldots, x_{nn}$ on (a, b), determine weights $A_{1n}, \ldots, A_{nn}$ such that

$$I_n(L, \mu) = \sum_{j=1}^{n} A_{jn} L(x_{jn}) = I(L, \mu), \quad \forall\, L \in \Lambda_{-p,q}. \tag{1.6}$$

Furthermore, it can be seen that an equivalent definition is

$$I_n(f, \mu) = \int_a^b L_{n-1}(f, x)\mu(x)\, \mathrm{d}x \tag{1.7}$$

with $L_{n-1}(f, \cdot)$ the unique L-polynomial in $\Lambda_{-p,q}$, interpolating f at the nodes $\{x_{jn}\}_{j=1}^{n}$. Hence, such rules are sometimes referred to as "interpolatory rules" in $\Lambda_{-p,q}$. Like in the polynomial case, an adequate choice of the nodes $\{x_{jn}\}$ still remains to be a crucial problem. This is the main aim of the paper: providing easily computable nodes $\{x_{jn}\}$ such that the rules $I_n(f, \mu)$ given by (1.6) are stable and convergent in a class of functions as large as possible. In this respect, the case of a finite interval $[a, b]$, $0 \leq a < b < +\infty$ was previously studied by the authors [6,2,3]. So, here we will deal with integrals (1.1) on $(0, \infty)$. The paper is organized as follows. In order to make it self contained, we shall review in Section 2 the most relevant properties of certain sequences of orthogonal L-polynomials along with the quadrature formulas based upon their zeros (Gauss-type formulas) providing with an extension of a previous result about convergence of these quadratures as done in [5], which represents the first part of this work. The main results are given in Section 3. Here some results by Sloan et al. [22,23] are extended to the case of Laurent polynomials and unbounded intervals. An Erdős-Turán-type theorem is also produced. In Section 4, as a consequence of the convergence results, certain rational functions (two-point Padé-type approximants) associated with the Stieltjes-transform of μ are studied. Finally in Section 5, some illustrative examples are discussed.

2 Preliminary results. Gauss-type quadrature formulas

Throughout this section, φ will denote a "strong" distribution function (see [16]) on $(0, \infty)$, i.e., a real valued, bounded, nondecreasing function with infinitely many points of increase on any interval $[a, b] \subset \mathbb{R}^+ = [0, \infty)$ and so that the following integrals ("strong moments") satisfy

$$d_n = \int_0^\infty x^n \, \mathrm{d}\varphi(x) < \infty, \quad n \in \mathbb{Z}. \tag{2.1}$$

From (2.1) it can be easily seen that the moment function $d(t) = \int_0^\infty x^t \, \mathrm{d}\varphi(x)$ associated with φ is defined for any $t \in \mathbb{R}$. As already said, the zeros of orthogonal polynomials play a fundamental role in the construction of formulas $I_n(f, \mu)$, satisfying (1.3). When dealing with integrals (1.1) so that the moments (1.5) exist for any integer k, we shall see in Section 3 that a similar role is now played by certain sequences of Laurent polynomials orthogonal with respect to the distribution φ.

For this purpose, let us start from a nondecreasing sequence $\{p(n)\}_{n=0}^\infty$ of nonnegative integers such that $0 \leq p(n) \leq n$ and $s(n) = p(n) - p(n-1) \in \{0, 1\}$ and let $\mathcal{L}_n = \Lambda_{-p(n),q(n)}$ with $q(n) = n - p(n)$, and $\mathcal{L} = \Lambda_{-p,q} = \bigcup_{n=0}^\infty \mathcal{L}_n$ where $p = \lim_{n\to\infty} p(n)$ and $q = \lim_{n\to\infty} q(n)$. Note that $\dim(\mathcal{L}_n) = n + 1$, $\mathcal{L}_n \subset \mathcal{L}_{n+1}$ and that at least one of p or q has to be infinite. Thus if $p = 0$ ($p(n) = 0$ and $q(n) = n$ for each n), then $\mathcal{L} = \Pi$, while for $p = q = \infty$, it results in $\mathcal{L} = \Lambda$.

Now, from (2.1), we can define an inner product over Λ as usual:

$$\langle f, g \rangle = \int_0^\infty f(x)\overline{g(x)} \, \mathrm{d}\varphi(x), \quad \forall f, g \in \Lambda, \tag{2.2}$$

so that by applying the Gram-Schmidt orthogonalization process to the basis $\{x^j : -p(n) \leq j \leq q(n)\}$ of $\mathcal{L}_n$ an orthogonal basis $\{V_0, V_1, \ldots, V_n\}$ for $\mathcal{L}_n$ can be obtained so that $V_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ and $V_n \perp \mathcal{L}_{n-1}$ (i.e., $\langle V_n, L \rangle = 0, \forall\, L \in \mathcal{L}_{n-1}, \mathcal{L}_{-1} = \emptyset$).

Observe that V_n is uniquely defined up to a multiplicative factor. When the process is repeated for each n, an essentially unique sequence $\{V_n\}_0^\infty$ is obtained that will be called a sequence of orthogonal L-polynomials with respect to the distribution φ and the "ordering" induced by $\{p(n)\}_0^\infty$. In the case that $\langle V_n, V_n \rangle = 1$, $\{V_n\}_0^\infty$ will be called "orthonormal". Further properties on general sequences of orthogonal L-polynomials have been recently given in [9]. As indicated in [9] and [5], the following can be easily proved.

Theorem 2.1 *Let $\{V_n\}_0^\infty$ be a sequence of orthogonal L-polynomials as defined above. Then*

(1) For each $n \geq 1$, V_n has exactly n distinct zeros $x_{1n}, \ldots, x_{nn}$ in $(0, \infty)$.

(2) Let $I_n(f,\, \mathrm{d}\varphi) = \sum_{j=1}^{n} \lambda_{jn} f(x_{jn})$ *be the interpolating rule in* $\mathcal{L}_{n-1}$ *for* $I(f,\, \mathrm{d}\varphi) = \int_0^\infty f(x)\, \mathrm{d}\varphi(x)$ *based upon the zeros of* V_n *then*
(a) $\lambda_{jn} > 0,\ j = 1, \ldots, n$
(b) $I_n(R,\, \mathrm{d}\varphi) = I(R,\, \mathrm{d}\varphi),\ \forall R \in \mathcal{L}_n \cdot \mathcal{L}_{n-1}$

Remark 2.2 Observe that $\dim(\mathcal{L}_n \cdot \mathcal{L}_{n-1}) = \dim \Lambda_{-(p(n)+p(n-1)),q(n)+q(n-1)} = 2n$. For this reason, the quadrature formula will be called to be *Gauss-type* formulas and denoted by $I_n^{GT}(f,\, \mathrm{d}\varphi)$. Furthermore, positivity of the weights automatically implies stability of the sequence $\{I_n^{GT}(f,\, \mathrm{d}\varphi)\}_1^\infty$. This fact along with the property that for each n $I_n^{GT}(f,\, \mathrm{d}\varphi)$ represents a Riemann-Stieltjes sum for $I(f,\, \mathrm{d}\varphi)$, see [4], turns out to be essential when considering convergence of our quadrature rules later on. For details see Theorems 2.5-2.6 and Corollary 2.7 in [5].

As clearly exposed in [5], convergence of the quadrature formulas $\{I_n^{GT}(f,\, \mathrm{d}\varphi)\}_1^\infty$ directly leads to the so-called *Strong Stieltjes Moment Problem* so that it being determinate becomes essential to guarantee convergence in the class of the Riemann-Stieltjes integrable functions with respect to $\mathrm{d}\varphi$. Taking this into account the authors in [5] made use of the known Carlemans condition for the determination of the strong moment problem [1],

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{d_n}} = \infty \quad \text{if} \quad \lim_{n\to\infty} q(n) = \infty, \tag{2.3}$$

or

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{d_{-n}}} = \infty \quad \text{if} \quad \lim_{n\to\infty} p(n) = \infty. \tag{2.4}$$

Thus, the class of functions for which convergence of the sequence $\{I_n^{GT}(f,\, \mathrm{d}\varphi)\}$ holds was characterized in terms of the moments $\{d_k\}$ of the distribution function φ, by means of two entire functions associated with the distribution φ, namely

$$Q(y) = \sum_{k=0}^{\infty} \frac{y^k}{d_k}, \quad y \in \mathbb{C} \tag{2.5}$$

$$M(y) = \sum_{k=0}^{\infty} \frac{y^k}{d_{-k}}, \quad y \in \mathbb{C}. \tag{2.6}$$

and the set

$$C_h(\mathbb{R}^+) = \{f : \mathbb{R}^+ \to \mathbb{C} : f \text{ is continuous and } \lim_{x\to\infty} f(x)h(x) = 0\},$$

with h being a bounded continuous function on $\mathbb{R}^+$. The following result (see Theorem 16 in [19]) will be needed.

Theorem 2.3 *[19] Let a positive function $p(x)$ on $\mathbb{R}$ have the property that*

$$\limsup_{r\to\infty} \frac{E(r)}{r} > 0,$$

where $E(r) = \sharp\{x \in \mathbb{R} : |x| < r, p(x) > 0\}$, where as usual $\sharp A$ denote cardinal of A. If the polynomials are dense in $C_p(\mathbb{R})$, then they are also dense in $C_{(1+|t|)^s p(t)}(\mathbb{R})$, for any $s > 0$.

Remark 2.4 It can be checked immediately that the same conclusion holds for $C_{|t|^s p(t)}(\mathbb{R})$, for any $s > 0$. It is enough to observe that

$$|t|^s p(t) < (1 + |t|)^s p(t), \quad t \in \mathbb{R}.$$

Now, proceeding in a similar way as in [5] and making use of Theorem 2.3 we can prove the following,

Theorem 2.5 *Let h be given by $\frac{x^\rho}{Q(x)}$ or $\frac{x^{\tilde{\rho}}}{M(x)}$, with $\rho,\ \tilde{\rho} > 0$. Then the space Π of all polynomials is dense in the class $C_h(\mathbb{R}^+)$ with respect to the weighted uniform norm $\|\cdot\|_{\infty,h}$, where $\|f\|_{\infty,h} = \|fh\|_\infty = \sup_{x\geq 0} |f(x)h(x)|$, if the conditions (2.3) or (2.4) are respectively satisfied.*

Moreover, for $x \in (0, \infty)$, we define

$$H(x) = \frac{x^{\rho-\tilde{\rho}}}{Q(x)M(1/x)}, \tag{2.7}$$

and set

$$C_H(0, \infty) = \{f : (0, \infty) \to \mathbb{C} : f \text{ is continuous and } \lim_{x\to\infty} f(x)H(x) = \lim_{x\to 0^+} f(x)H(x) = 0\}.$$

In [5], the authors proved a result concerning density of the space of Laurent polynomials for $\tilde{\rho} = \rho = 0$. Similarly, one can obtain

Theorem 2.6 *The space Λ of all L-polynomials is dense in the class $C_H(0, \infty)$ with H given by (2.7) with respect to the weighted uniform norm $\|\cdot\|_\infty^H$, where $\|f\|_\infty^H = \|fH\|_\infty = \sup_{x \geq 0} |f(x)H(x)|$, if both conditions (2.3) and (2.4) are satisfied.*

Theorems 2.5-2.6 are the basis to study the rate of convergence of $\{I_n^{GF}(f, \mathrm{d}\varphi)\}$ when f is assumed to be smooth on $(0, \infty)$, which is the fundamental ingredient to be used in the subsequent section. On the other hand, by analyzing more in depth Remark 4.1 in [5], we can produce a result similar to Theorem 4.1 in [5], but now given in a more general framework and more adjusted to the aim of this paper which consists of enlarging the class of functions where convergence of our quadrature rules can be assured. Indeed, one has.

Theorem 2.7 *Let φ be a strong distribution function on $\mathbb{R}^+$ with moment sequence $d_n = \int_0^\infty x^n \,\mathrm{d}\varphi(x)$, $n \in \mathbb{Z}$ and let $\{I_n^{GT}(f, \mathrm{d}\varphi)\}_1^\infty$ be the sequence of Gauss-type formulas as introduced in Theorem 2.1. Then the following is true.*

(1) Assume that $\lim_{n\to\infty} q(n) = \infty$ and $\sum_{n=1}^\infty d_n^{-\frac{1}{2n}} = +\infty$. If

$$\sum_{k=[\rho]+1}^{\infty} \frac{d(k-\rho)}{d_k} < \infty, \quad \rho > 0 \tag{2.8}$$

where $d(t)$ is the moment function and $[x]$ denotes the integer part of x, then

$$\lim_{n\to\infty} I_n^{GT}(f, \mathrm{d}\varphi) = I(f, \mathrm{d}\varphi)$$

for any function f locally Riemann-Stieltjes integrable with respect to $\mathrm{d}\varphi$ satisfying for sufficiently large x

$$|f(x)| \leq C\frac{Q(x)}{x^\rho}, \quad C > 0. \tag{2.9}$$

(Q as defined in (2.5).)

(2) Assume that $\lim_{n\to\infty} p(n) = \infty$ and $\sum_{n=1}^\infty d_{-n}^{-\frac{1}{2n}} = +\infty$. If

$$\sum_{k=[\tilde{\rho}]+1}^{\infty} \frac{d(\tilde{\rho}-k)}{d_{-k}} < \infty, \quad \tilde{\rho} > 0. \tag{2.10}$$

Then

$$\lim_{n\to\infty} I_n^{GT}(f, \mathrm{d}\varphi) = I(f, \mathrm{d}\varphi)$$

for any function f Riemann-Stieltjes integrable with respect to $\mathrm{d}\varphi$ *on* $[a,\infty)$, $(a > 0)$ *satisfying for sufficiently small* x

$$|f(x)| \le \tilde{C} M(1/x) x^{\tilde{\rho}}, \quad \tilde{C} > 0. \tag{2.11}$$

(M as defined in (2.6).)

(3) If assumption in (1) and (2) hold simultaneously, then

$$\lim_{n\to\infty} I_n^{GT}(f,\,\mathrm{d}\varphi) = I(f,\,\mathrm{d}\varphi)$$

for any function f Riemann-Stieltjes integrable with respect to $\mathrm{d}\varphi$ *on* $[a,b], 0 < a < b < \infty$, *satisfying (2.9)-(2.11).*

PROOF. In order to deduce (1) by the Remark 4.2 in [5], it is sufficient to show that the integral $I(f,\varphi)$ exists, with $f(x) = \frac{Q(x)}{1+x^\rho}$.

Now, taking into account that Q is an entire function, it holds

$$\begin{aligned}
\int_0^\infty \frac{Q(x)}{1+x^\rho}\,\mathrm{d}\varphi(x) &= \lim_{b\to\infty} \int_0^b \frac{Q(x)}{1+x^\rho}\,\mathrm{d}\varphi(x) \\
&= \lim_{b\to\infty} \int_0^b \sum_{k=0}^{\infty} \frac{x^k}{d_k} \frac{\mathrm{d}\varphi}{1+x^\rho} \\
&= \lim_{b\to\infty} \sum_{k=0}^{\infty} \int_0^b \frac{x^k}{d_k} \frac{\mathrm{d}\varphi}{1+x^\rho} \\
&= \lim_{b\to\infty} \left[\sum_{k=0}^{[\rho]} \int_0^b \frac{x^k}{d_k} \frac{\mathrm{d}\varphi}{1+x^\rho} + \sum_{k=[\rho]+1}^{\infty} \int_0^b \frac{x^k}{d_k} \frac{\mathrm{d}\varphi}{1+x^\rho} \right] \\
&= \lim_{b\to\infty} [I_1(b) + I_2(b)]
\end{aligned}$$

Let us first study,

$$I_1(b) = \sum_{k=0}^{[\rho]} \int_0^b \frac{x^k}{d_k} \frac{\mathrm{d}\varphi}{1+x^\rho}.$$

To this end observe that $h_k(x) = \dfrac{x^k}{1+x^\rho} \le h_k(\alpha)$, $\alpha^\rho = \frac{k}{\rho-k}$, $k = 0, 1, \ldots, [\rho]$, for any $x \in \mathbb{R}^+$. Hence it holds

$$I_1(b) \le S \sum_{k=0}^{[\rho]} \frac{d_0}{d_k} < \infty$$

with S a certain positive constant. Now, let us consider

$$I_2(b) = \sum_{k=[\rho]+1}^{\infty} \frac{1}{d_k} \int_0^b \frac{x^k}{1+x^\rho} \, \mathrm{d}\varphi \le \sum_{k=[\rho]+1}^{\infty} \frac{1}{d_k} \int_0^b x^{k-\rho} \, \mathrm{d}\varphi,$$

Thus, by (2.8)

$$I_2(b) \le \sum_{k=[\rho]+1}^{\infty} \frac{d(k-\rho)}{d_k} < \infty.$$

Finally, take into account that, for $x > 0$ sufficiently large one has,

$$0 < \chi \le \frac{x^\rho}{1+x^\rho} \le 1.$$

(2) This follows immediately from (1), since the negative moments $\{d_{-k}\}_{k\ge 0}$ enable us to define the positive moments for the distribution $-\varphi(\dfrac{1}{t})$.

(3) In order to prove (3), we first consider the auxiliary functions

$$f_0(x) = \begin{cases} f(x) - L(x) & x \in (0,1] \\ 0 & x > 1 \end{cases}$$

$$f_\infty(x) = \begin{cases} 0 & x \in (0,1] \\ f(x) - L(x) & x > 1 \end{cases}$$

where L is a L-polynomial satisfying that if $f \in \mathcal{C}^m(0,\infty)$, then $L \in \mathcal{L}_m$, and

$$f^{(k)}(1) = L^{(k)}(1), \quad k = 0, 1, \ldots, m.$$

In case f is not a continuous function in $(0, \infty)$, we proceed as in Proposition 3.9 in [5], i.e. we have $L(t) = f(1)$.

Therefore, $f_0, f_\infty \in \mathcal{C}^m(0, \infty)$, so that the regularity of function f is preserved and this fact will be required in the forthcoming estimation. Clearly, one has

$$f = f_0 + f_\infty + L,$$

Hence,

$$I(f,\, \mathrm{d}\varphi) = I(f_0,\, \mathrm{d}\varphi) + I(f_\infty,\, \mathrm{d}\varphi) + I(L,\, \mathrm{d}\varphi),$$

Thus, we must prove that

$$I(f_0,\, \mathrm{d}\varphi) < \infty \quad \text{and} \quad I(f_\infty,\, \mathrm{d}\varphi) < \infty,$$

which can be assured taking into account that both f_0 and f_∞ satisfy (2.9) and (2.11), respectively. □

From this theorem we can also estimate the error of the quadrature rule i.e. $E_n(f) = I(f,\, \mathrm{d}\varphi) - I_n(f,\, \mathrm{d}\varphi)$, when f is a continuous function. Indeed, it holds

Corollary 2.8 *Under the same hypothesis as in Theorem 2.7 with f a continuous function on $(0, \infty)$, we have.*

(1) Assume that (2.8) holds and f satisfies (2.9). Then

$$|E_n(f)| \leq \left(2I(\frac{Q(x)}{x^\rho},\, \mathrm{d}\varphi) + \epsilon_n \right) m_{q(n)+q(n-1)}(f,\, \frac{x^\rho}{Q(x)}),$$

where $m_n(f, h) = \min_{p \in \Pi_n} \|f - p\|_{\infty, h}$ and $\{\epsilon_n\}_{n \geq 0} \subset (0, \infty)$ such that $\lim_{n\to\infty} \epsilon_n = 0$.

(2) Assume that (2.10) holds and f satisfies (2.11). Then

$$|E_n(f)| \leq \left(2I(M(\frac{1}{x})x^{\tilde{\rho}},\, \mathrm{d}\varphi) + \tilde{\epsilon}_n \right) m_{p(n)+p(n-1)}(\tilde{f},\, \frac{x^{\tilde{\rho}}}{M(x)})$$

where $\tilde{f}(x) = f(\frac{1}{x})$ and $\{\tilde{\epsilon}_n\}_{n \geq 0} \subset (0, \infty)$ such that $\lim_{n\to\infty} \tilde{\epsilon}_n = 0$.

(3) Assume that (2.8)-(2.10) hold and f satisfies (2.9)-(2.11). Then,

$$|E_n(f)| \leq \left(2I(Q(x)M(\frac{1}{x})x^{\tilde{\rho}-\rho},\, \mathrm{d}\varphi) + \nu_n\right) M_{2n-1}(f,H),$$

where $M_{2n-1}(f,H) = \min_{L\in\mathcal{L}_{2n-1}} \|f-L\|_\infty^H$ with H given by (2.7) and $\{\nu_n\}_{n\geq 0} \subset (0,\infty)$ such that $\lim_{n\to\infty} \nu_n = 0$.

Remark 2.9 Conditions (2.8) and (2.10) are satisfied by the so-called Freud-type distributions (weights) studied by López-Martínez in [18]; which are of the form $\mathrm{d}\varphi(t) = \omega(t)\,\mathrm{d}t$ such that

$$\omega(t) = t^\nu e^{-\tau(t)}\,, \quad t \in (0,\infty)$$

with $\nu \in \mathbb{R}$ and τ a continuous function on $(0,\infty)$, such that there exist $\gamma, \tilde{\gamma} > \frac{1}{2}$ and $s, \tilde{s} > 0$ fulfilling

$$\lim_{t\to 0^+} (\tilde{s}t)^{\tilde{\gamma}}\tau(t) = \tilde{A} > 0,$$

and

$$\lim_{t\to+\infty} \frac{\tau(t)}{(st)^\gamma} = A > 0\,.$$

Since we are only concerned with the asymptotic behaviour of $\frac{d(k-\rho)}{d_k}$ as $k \to \infty$, we can restrict ourselves to study the behaviour of the moment function $d(t)$ for the corresponding Freud weights, namely

$$\omega_{\gamma,\theta}(t) = t^\theta e^{-rt^\gamma}, \quad t \in (0,\infty), \quad r > 0, \quad \theta > -1, \quad \gamma \geq \frac{1}{2}..$$

yielding,

$$d(t) = \frac{1}{\gamma}\frac{\Gamma(\frac{t+\theta+1}{\gamma})}{r^{\frac{t+1+\theta}{\gamma}}}, \quad t \geq 0,$$

where Γ is the Gamma function which, as is well known (see [11]), admits the following asymptotic expansions,

$$\Gamma(z) = e^{-z}z^{z-\frac{1}{2}}(2\pi)^{\frac{1}{2}}[1 + \frac{B_1}{z} + \frac{B_2}{z} + \cdots], \quad z \to \infty, \quad |\mathrm{Arg}(z)| < \pi.$$

B_k, $k = 1, 2, \ldots$ being the Bernoulli's numbers.

Thus, for sufficiently large k, one can write

$$\frac{d(k-\rho)}{d_k} \approx \gamma^{\rho/\gamma} \left(\frac{r}{k+\theta+1-\rho} \right)^{\frac{\rho}{\gamma}}, \ k \to \infty.$$

Therefore,

$$\sum_{[\rho]+1}^{\infty} \frac{d(k-\rho)}{d_k} < \infty \quad \Leftrightarrow \quad \frac{\rho}{\gamma} > 1, \quad \left(\text{and since } \gamma \geq \frac{1}{2} \right) \quad \Leftrightarrow \quad \rho > \gamma \geq \frac{1}{2}.$$

Here it should be also observed that θ does not play an essential role in the convergence of the series.

3 Convergence of the interpolatory quadratures for a possibly signed measure

In this section we shall be mainly concerned with the convergence and stability of the sequence of quadrature formulas

$$I_n(f, \mu) = \sum_{j=1}^{n} A_{jn} f(x_{jn}) \tag{3.1}$$

of interpolatory type in $\mathcal{L}_{n-1}$, $n = 1, 2, \ldots$ to the integral

$$I(f, \mu) = \int_0^{\infty} f(x) \mu(x) \, \mathrm{d}x \tag{3.2}$$

where in general μ does not have a constant sign on $\mathbb{R}^+$ (it could even take complex values). Clearly, in this situation it becomes meaningless to deal with Gauss-type formulas since it can not be assured that μ generates an inner product (2.2) that enables us to speak about orthogonality. So, a special choice of the nodes $\{x_{jn}\}_1^n$ in (3.1) will be made so that the results by Sloan et all [22,23] for the polynomial case are to be partially extended to the L-polynomials. On the other hand, previous results by the authors involving L-polynomials and concerning integrals (1.1) but on a finite interval $[a,b]$ $0 \leq a < b < +\infty$ are now completed for the interval $(0,\infty)$.

In the sequel we will deal with an auxiliary distribution φ. Furthermore, for the sake of simplicity, we can assume without loss of generality that φ is absolutely continuous so that $\mathrm{d}\varphi(x) = \omega(x)\,\mathrm{d}x$, ω being a weight function on $(0,\infty)$. Thus for the corresponding orthogonality we shall refer to ω, instead of $\mathrm{d}\varphi$.

We now prove the main result of this Section.

Theorem 3.1 *Let us assume that μ satisfies*

$$\int_0^\infty \frac{|\mu(x)|^2}{\omega(x)}\,\mathrm{d}x < +\infty. \tag{3.3}$$

Let $I_n(f,\mu) = \sum_{j=1}^n A_{jn} f(x_{jn})$ denote the n-point interpolatory rule in $\mathcal{L}_{n-1}$ whose nodes $\{x_{jn}\}_{j=1}^n$ are the zeros of the nth orthogonal L-polynomial with respect to ω associated with $\{p(n)\}_{n\geq 0}$. Assume that either

$$\lim_{n\to\infty} q(n) = \infty, \quad \sum_{n=1}^\infty d_n^{-\frac{1}{2n}} = +\infty \quad \textit{and} \quad \sum_{k=[\rho]+1}^\infty \frac{d(k-\rho)}{d_k} < +\infty, \quad \rho > 0, \tag{3.4}$$

or

$$\lim_{n\to\infty} p(n) = \infty, \quad \sum_{n=1}^\infty d_{-n}^{-\frac{1}{2n}} = +\infty \quad \textit{and} \quad \sum_{k=[\tilde\rho]+1}^\infty \frac{d(\tilde\rho-k)}{d_{-k}} < +\infty, \quad \tilde\rho > 0, \tag{3.5}$$

where for $t \in \mathbb{R}$, $d(t) = \int_0^\infty x^t \omega(x)\,\mathrm{d}x$ denotes the moment function associated with ω. Then if (3.4) holds and f is any function locally Riemann-Stieltjes integrable with respect to ω, such that

$$|f(x)| \leq C\sqrt{\frac{Q(x)}{x^\rho}}, \quad C > 0 \tag{3.6}$$

for sufficiently large x with Q given by (2.5), it follows that

$$\int_0^\infty f(x)\mu(x)\,\mathrm{d}x < \infty, \tag{3.7}$$

$$\lim_{n\to\infty} I_n(f) = \int_0^\infty f(x)\mu(x)\,\mathrm{d}x \tag{3.8}$$

and

$$\lim_{n\to\infty} \sum_{j=1}^{n} |A_{jn}| f(x_{jn}) = \int_0^\infty f(x)|\mu(x)|\,\mathrm{d}x. \tag{3.9}$$

On the other hand, if (3.5) holds, then (3.7)-(3.8)-(3.9) follow for any function f Riemann-Stieltjes integrable with respect to ω on $[a,\infty)$, $a>0$, such that

$$|f(x)| \leq \tilde{C}\sqrt{M(1/x)x^{\tilde{\rho}}}, \quad \tilde{C} > 0, \tag{3.10}$$

for sufficiently small x and M given by (2.6).

Moreover, if (3.4)-(3.5) hold, then (3.7)-(3.8)-(3.9) follow for any function f locally Riemann-Stieltjes integrable with respect to ω on $[a,b]$, $0<a<b<\infty$, such that (3.6) and (3.10) are both satisfied

PROOF.

(3.7) can be easily deduced making use of Schwarz's inequality along with (3.3) and taking into account that by Theorem 2.7

$$\int_0^\infty f^2(x)\,\omega(x)\,\mathrm{d}x < \infty.$$

Now, recall that $\{p(n)\}_0^\infty$ is a given nondecreasing sequence of nonnegative integers such that $0 \leq p(n) \leq n$ and $s(n) = p(n) - p(n-1) \in \{0,1\}$, giving rise to the nested sequence of L-polynomial subspaces

$$\mathcal{L}_n = \operatorname{span}\{x^j : -p(n) \leq j \leq q(n)\}, \quad n = 0,1,\ldots, \quad q(n) = n - p(n).$$

Thus if $p = \lim_{n\to\infty} p(n) = \infty$ or $q = \lim_{n\to\infty} q(n) = \infty$, then by either (3.4) or (3.5) respectively it follows that (see [1]) the space $\Lambda_{-p,q} = \bigcup_0^\infty \mathcal{L}_n = \mathcal{L}$ is dense in

$$L_2(\omega) = \{f : [0,\infty) \to \mathbb{C} : f \quad \text{measurable and} \quad \int_0^\infty |f(x)|^2 \omega(x)\,\mathrm{d}x < +\infty\}$$

with respect to the norm induced by ω, i.e.,

$$\langle L, D\rangle_\omega = \int_0^\infty L(x)\overline{D(x)}\omega(x)\,\mathrm{d}x \quad \text{and} \quad \|L\|_\omega = \sqrt{\langle L, L\rangle_\omega}.$$

Let $\{V_n\}_0^\infty$ be the corresponding sequence of orthonormal L-polynomials as introduced in Section 2. Let $\{x_{jn}\}_1^n$ be the zeros of V_n and denote by $A_{jn}(\psi)$ the weights of the n-point interpolatory quadrature formula $I_n(f,\psi)$ in $\mathcal{L}_{n-1}$ based upon these zeros for the integral $I(f,\psi)$, i.e.,

$$I_n(f,\psi) = \sum_{j=1}^n A_{jn}(\psi)L(x_{jn}) = I(L,\psi) = \int_0^\infty L(x)\,\psi(x)\,\mathrm{d}x, \quad \forall L \in \mathcal{L}_{n-1}.$$

Now taking into account (3.3), it follows that $\frac{\mu}{\omega} \in L_2(\omega)$. Hence, it admits the Fourier expansion,

$$\frac{\mu(x)}{\omega(x)} = \sum_{k=0}^\infty b_k V_k(x). \tag{3.11}$$

Recalling that

$$A_{jn}(\mu) = \int_0^\infty L_{jn}(x)\mu(x)\,\mathrm{d}x$$

with $L_{jn} \in \mathcal{L}_{n-1}$ and $L_{jn}(x_{kn}) = \delta_{jk}$, $1 \le j,k \le n$, we can write

$$A_{jn}(\mu) = \int_0^\infty L_{jn}(x)\frac{\mu(x)}{\omega(x)}\omega(x)\,\mathrm{d}x = \sum_{k=0}^\infty b_k \int_0^\infty L_{jn}(x)V_k(x)\omega(x)\,\mathrm{d}x.$$

As $L_{jn} \in \mathcal{L}_{n-1}$, by virtue of orthogonality, it follows that

$$A_{jn}(\mu) = \sum_{k=0}^{n-1} b_k \int_0^\infty L_{jn}(x) V_k(x) \omega(x) \, \mathrm{d}x.$$

On the other hand, for $0 \leq k \leq n-1$, $L_{jn} V_k \in \mathcal{L}_n \cdot \mathcal{L}_{n-1}$. Hence

$$A_{jn}(\mu) = \sum_{k=0}^{n-1} b_k \lambda_{jn} V_k(x_{jn}) \tag{3.12}$$

where $\{\lambda_{jn}\}_{j=1}^n$ are the positive weights of the n-point Gauss-type formula for ω so that (3.12) follows from Theorem 2.1. Thus (3.12) can be written as

$$A_{jn}(\mu) = \lambda_{jn} S_{n-1}^{\mu/\omega}(x_{jn}) \tag{3.13}$$

where $S_{n-1}^{\mu/\omega}$ denotes the $(n-1)$th partial sum in (3.11).

In the rest of the proof we shall assume that both conditions (3.4)-(3.5) hold and we shall prove (3.9). The proof of (3.8) can be done in a similar way, and the same can be said when only one of the conditions either (3.4) or (3.5) is satisfied. As a result, for a given positive number ε, we shall see that there exists a positive integer $n_0(\varepsilon)$ so that for $n > n_0(\varepsilon)$ it holds that

$$\left| \int_0^\infty f(x) |\mu(x)| \, \mathrm{d}x - \sum_{j=1}^n |A_{jn}(\mu)| f(x_{jn}) \right| < \varepsilon$$

for any function f satisfying (3.6) and (3.10).

For this purpose, take a function σ satisfying (3.3) which will be more precisely defined later on, then we have

$$\begin{aligned} \left| \int_0^\infty f(x) |\mu(x)| \, \mathrm{d}x - \sum_{j=1}^n |A_{jn}(\mu)| f(x_{jn}) \right| \leq & \left| \int_0^\infty f(x) |\mu(x)| \, \mathrm{d}x - \int_0^\infty f(x) |\sigma(x)| \, \mathrm{d}x \right| \\ & + \left| \sum_{j=1}^n |A_{jn}(\sigma)| f(x_{jn}) - \sum_{j=1}^n |A_{jn}(\mu)| f(x_{jn}) \right| \end{aligned}$$

$$+\left|\int_0^\infty f(x)|\sigma(x)|\,\mathrm{d}x-\sum_{j=1}^n|A_{jn}(\sigma)|f(x_{jn})\right|.$$

Setting $\|f\|_\infty^g=\sup_{x\geq 0}|f(x)g(x)|$ where $g^2(x)=\frac{x^{\rho-\tilde\rho}}{Q(x)M(\frac{1}{x})}$ and using the linearity of the weights $A_{jn}(\mu)$ with respect to μ, we arrive at

$$\begin{aligned}
&\left|\int_0^\infty f(x)|\mu(x)|\,\mathrm{d}x-\sum_{j=1}^n|A_{jn}(\mu)|f(x_{jn})\right|\\
&\quad\leq\|f\|_\infty^g\left\{\int_0^\infty|\mu(x)-\sigma(x)|\frac{\mathrm{d}x}{g(x)}\;+\sum_{j=1}^n|A_{jn}(\mu-\sigma)|\frac{1}{g(x_{jn})}\right\}\\
&\qquad+\left|\int_0^\infty f(x)|\sigma(x)|\,\mathrm{d}x-\sum_{j=1}^n|A_{jn}(\sigma)|f(x_{jn})\right|\\
&\quad\leq\|f\|_\infty^g\left\{\int_0^\infty\left|\frac{\mu(x)}{\omega(x)}-\frac{\sigma(x)}{\omega(x)}\right|\frac{\omega(x)}{g(x)}\,\mathrm{d}x\;+\sum_{j=1}^n\lambda_{jn}\left|S_{n-1}^{\frac{\mu}{\omega}-\frac{\sigma}{\omega}}(x_{jn})\right|\frac{1}{g(x_{jn})}\right\}\\
&\qquad+\left|\int_0^\infty f(x)|\sigma(x)|\,\mathrm{d}x-\sum_{j=1}^n\lambda_{jn}|S_{n-1}^{\sigma/\omega}(x_{jn})|f(x_{jn})\right|
\end{aligned}$$

Observe that the last inequality follows from (3.13). On the other hand, setting $\tilde\mu=\mu/\omega$ and $\tilde\sigma=\sigma/\omega$, then by (3.3) we see that $\tilde\mu$ and $\tilde\sigma$ belong to $L_2(\omega)$. Hence, making use of Schwarz's inequality, it follows that

$$\begin{aligned}
&\left|\int_0^\infty f(x)|\mu(x)|\,\mathrm{d}x-\;\sum_{j=1}^n|A_{jn}(\mu)|f(x_{jn})\right|\\
&\quad\leq\|f\|_\infty^g\left\{\|\tilde\mu-\tilde\sigma\|_2 I(\frac{1}{g^2},\omega)^{1/2}+\left(\sum_{j=1}^n\lambda_{jn}|S_{n-1}^{\tilde\mu-\tilde\sigma}(x_{jn})|^2\right)^{1/2}\left(\sum_{j=1}^n\lambda_{jn}\frac{1}{g^2(x_{jn})}\right)^{1/2}\right\}\\
&\qquad+\left|\sum_{j=1}^n\lambda_{jn}|S_{n-1}^{\tilde\sigma}(x_{jn})|f(x_{jn})-\int_0^\infty f(x)|\sigma(x)|\,\mathrm{d}x\right|.
\end{aligned}$$

Since $|S_{n-1}^{\tilde{\mu}-\tilde{\sigma}}(x)|^2 \in \mathcal{L}_{n-1} \cdot \mathcal{L}_{n-1} \subset \mathcal{L}_n \cdot \mathcal{L}_{n-1}$, by Theorem 2.1 and Bessel's inequality ([7]) one has

$$\left(\sum_{j=1}^{n} \lambda_{jn} |S_{n-1}^{\tilde{\mu}-\tilde{\sigma}}(x_{jn})|^2 \right)^{1/2} = \|S_{n-1}^{\tilde{\mu}-\tilde{\sigma}}\|_\omega \leq \|\tilde{\mu} - \tilde{\sigma}\|_\omega.$$

Taking into account that $\mathcal{L}$ is dense in $L_2(\omega)$, there exists an L-polynomials $R \in \mathcal{L}$ such that $\|R - \tilde{\mu}\|_2 < \varepsilon$. Take now $\tilde{\sigma} = R$, i.e., $\sigma = R\omega$. Then for $n > \tilde{n}_0$ it holds that $S_{n-1}^{\tilde{\sigma}} = S_{n-1}^{R} = R$, and

$$\begin{aligned} &\left| \int_0^\infty f(x)|\mu(x)| \,\mathrm{d}x - \sum_{j=1}^{n} |A_{jn}(\mu)| f(x_{jn}) \right| \\ &\quad \leq \varepsilon \|f\|_\infty^g \left\{ (I(\frac{1}{g^2}, \omega))^{1/2} + \left(\sum_{j=1}^{n} \lambda_{jn} \frac{1}{g^2(x_{jn})} \right)^{1/2} \right\} \\ &\quad + \left| \sum_{j-1}^{n} \lambda_{jn} |R(x_{jn})| f(x_{jn}) - \int_0^\infty f(x)|R(x)|\omega(x) \,\mathrm{d}x \right|. \end{aligned} \tag{3.14}$$

Finally, by Theorem 2.7, since $|R|f$ and $\dfrac{1}{g^2}$ satisfy (2.9) and (2.11), one has

$$\lim_{n\to\infty} \sum_{j=1}^{n} \lambda_{jn} \frac{1}{g^2(x_{jn})} = I(\frac{1}{g^2}, \omega)$$

and

$$\lim_{n\to\infty} \sum_{j=1}^{n} \lambda_{jn} |R(x_{jn})| f(x_{jn}) = \int_0^\infty f(x)|R(x)|\omega(x) \,\mathrm{d}x.$$

Hence, the first summand in the right-hand member of inequality (3.14) remains bounded by $D\varepsilon$, D being a positive constant while the second summand is smaller than ε for any nonnegative integer $n > \hat{n}_0$.

Thus, our proof comes to a conclusion taking $n_0 = \max\{\tilde{n}_0, \hat{n}_0\}$. □

Remark 3.2 Taking $f(x) = 1$ in (3.9) one has

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} |A_{jn}| = \int_0^{\infty} |\mu(x)| \, \mathrm{d}x. \tag{3.15}$$

Therefore the stability of the sequence $\{I_n(f)\}$ is guaranteed. Compare with Theorem 4.1 in [2]. Now, the following error estimation can be easily obtained.

Corollary 3.3 *Under the same assumptions as in Theorem 3.1, if conditions (3.4)-(3.5) both hold, then*

$$|I(f,\mu) - I_n(f,\mu)| \leq \left(2 \int_0^{\infty} \frac{|\mu(x)|}{g(x)} \, \mathrm{d}x + \hat{\epsilon}_n \right) M_{n-1}(f,g),$$

where $\{\hat{\epsilon}_n\}_{n\geq 0} \subset (0,\infty)$ *such that* $\lim_{n\to\infty} \hat{\epsilon}_n = 0$*, and* $M_n(f,g)$ *as given in Corollary 2.8.*

Remark 3.4 Even more, taking into account Corollary 3.10 in [5], one can write

$$M_n(f,g) \leq C_1 m_{p(n-1)}(\tilde{f}_0, h_0, \mathbb{R}^+) + C_2 m_{q(n-1)}(f_\infty, h_\infty, \mathbb{R}^+)$$

with $\tilde{f}_0(x) = f_0(\frac{1}{x})$ and where f_0 and f_∞ as given in the proof of Theorem 2.1. Here $C_1 = \|1\|_{\infty, h_0}$ and $C_2 = \|1\|_{\infty, h_\infty}$, with $h_0^2(t) = \frac{t^{\tilde{\rho}}}{M(t)}$ and $h_\infty^2(t) = \frac{t^{\rho}}{Q(t)}$.

We conclude this section giving an extension of the known Erdős-Turán theorem [12].

Theorem 3.5 *Let* $L_{n-1}(f,\cdot)$ *denote the L-polynomial in* $\mathcal{L}_{n-1}$ *interpolating a function* f *at the zeros* $\{x_{jn}\}_{j=1}^{n}$ *of the* n*th orthogonal L-polynomial for* ω *associated with* $\{p(n)\}_{n\geq 0}$*. Then, if (3.4) and (3.5) are both satisfied, it holds that*

$$\lim_{n\to\infty} \int_0^{\infty} |f(x) - L_{n-1}(f,x)|^2 \omega(x) \, \mathrm{d}x = 0 \tag{3.16}$$

for any function $f \in C_g(0,\infty)$ *with* $g^2(x) = \frac{x^{\rho-\tilde{\rho}}}{Q(x)M(1/x)}$.

PROOF. Let us consider the sequence of linear operators $\{T_n\}_1^\infty$ between Banach spaces defined by

$$T_n: \quad C_g(0,\infty) \to L_2(\omega): \quad f \mapsto T_n(f) = L_{n-1}(f,\cdot).$$

Then we have

(1) $\lim_{n\to\infty} T_n(L) = L,\ L \in \mathcal{L}$.

(2) $\|T_n(f)\|_2^2 = \int_0^\infty (L_{n-1}(f,x))^2\omega(x)\,\mathrm{d}x = \sum_{n=1}^n \lambda_{jn} L_{n-1}^2(f,x_{jn})$
$= \sum_{n=1}^n \lambda_{jn} f^2(x_{jn}) = \sum_{n=1}^n \lambda_{jn} f^2(x_{jn}) g^2(x_{jn}) \frac{1}{g^2(x_{jn})}$
$\leq (\|f\|_\infty^g)^2 \cdot I_n^{GT}\left(\frac{1}{g^2},\omega\right).$

By virtue of Theorem 2.7,

$$\lim_{n\to\infty} I_n^{GT}\left(\frac{1}{g^2},\omega\right) = \int_0^\infty \frac{\omega(x)}{g^2(x)}\,\mathrm{d}x < \infty.$$

Hence, the sequence $\{T_n\}$ is bounded. The proof now follows from the Banach-Steinhaus theorem [17], since $\mathcal{L}$ is dense in $C_g(0,\infty)$. □

Remark 3.6 A similar result to the one in Theorem 3.5 can be also produced for the class of the continuous functions satisfying either (3.8) or (3.11) and when just one of the conditions (3.4) or (3.5) holds.

Remark 3.7 Part (3.8) in Theorem 3.1 could also be obtained as a direct consequence of Theorem 3.5. Indeed,

$$\begin{aligned}\left|\int_0^\infty f(x)\mu(x)\,\mathrm{d}x - \sum_{j=1}^n A_{jn} f(x_{jn})\right| &= \left|\int_0^\infty [f(x) - L_{n-1}(f,x)]\mu(x)\,\mathrm{d}x\right| \\ &= \left|\int_0^\infty [f(x) - L_{n-1}(f,x)]\sqrt{\omega(x)}\frac{\mu(x)}{\sqrt{\omega(x)}}\,\mathrm{d}x\right| \\ &\leq \left[\int_0^\infty [f(x) - L_{n-1}(f,x)]^2\omega(x)\,\mathrm{d}x\right]^{1/2} \left[\int_0^\infty \frac{|\mu(x)|^2}{\omega(x)}\,\mathrm{d}x\right]^{1/2}.\end{aligned}$$

However, this is only valid for a more restrictive class of functions than the one given in Theorem 3.1.

4 An Application to Two-Point Approximation

For a given L^1-integrable function μ on $(0, \infty)$, its Stieltjes transform is defined as

$$F_\mu(z) = \int_0^\infty \frac{\mu(x)}{z - x} \, \mathrm{d}x, \quad z \notin [0, \infty). \tag{4.1}$$

If we assume that

$$\int_0^\infty x^k \, |\mu(x)| \, \mathrm{d}x < \infty,$$

for any integer k and we denote $c_k = \int_0^\infty x^k \, \mu(x) \, \mathrm{d}x$, then it is known that F_μ admits the asymptotic expansions [16]

$$L^0(z) = -\sum_{j=0}^\infty c_{-(j+1)} z^j \quad \text{and} \quad L^\infty(z) = \sum_{j=1}^\infty c_{j-1} z^{-j} \tag{4.2}$$

around $z = 0$ and $z = \infty$, respectively.

The approximation of functions (4.1) (especially when μ is a weight function) by rational functions using expansions (4.2) has been widely studied in the last decades. See e.g. [8,21,18]. In this section, we shall be concerned with certain rational approximants to F_μ closely related to the quadrature rules analyzed in Section 3. Thus, let r and s be nonnegative integers and assume that a quadrature rule $I_n(f, \mu) = \sum_{j=1}^n A_j f(x_j)$, $(x_j \neq 0)$ for the integral $I(f, \mu) = \int_0^\infty f(x)\mu(x) \, \mathrm{d}x$ has been provided which is exact in $\Lambda_{-r,s}$, i.e.,

$$I_n(L, \mu) = I(L, \mu), \quad \forall L \in \Lambda_{-r,s}.$$

Set $F_n(z) = I_n(\frac{1}{z-\cdot}) = \sum_{j=1}^{n} \frac{A_j}{z-x_j}$. Then it is easily proved that

$$L^0(z) - F_n(z) = \sum_{j=r}^{\infty} \tilde{c}_j z^j = O(z^r), \qquad z \to 0^- ,$$
$$L^\infty(z) - F_n(z) = \sum_{j=s+2}^{\infty} c_j^* z^{-j} = O((1/z)^{s+2}), \qquad z \to \infty^- .$$

If $r + s = n - 1$, then the rational function F_n coincides with a so-called two-point Padé-type approximant (2PTA) to the pair (4.2) or to F_μ of order r [10]. It will be denoted by

$$F_n(z) = (r/n)_{(L^0,L^\infty)}(z) = (r/n)_{F_\mu}(z), \quad 0 \le r \le n-1.$$

If $r + s = 2n - 1$, then F_n coincides with the two-point Padé approximant (2PA) to F_μ of order r. Clearly, if $r = 0$ (hence $s = 2n - 1$), we have

$$L^\infty(z) - F_n(z) = O((1/z)^{2n+1}), \qquad z \to \infty^-.$$

So, we see that in this case F_n coincides with the $[n-1/n]_{F_\mu}$ one-point Padé approximant at infinity to F_μ.

Suppose now that we are dealing with a sequence $\{I_n(f,\mu)\}_1^\infty$ of quadrature rules so that for each n, $I_n(f,\mu) = \sum_{j=1}^{n} A_{jn} f(x_{jn})$ is of interpolatory type in $\mathcal{L}_{n-1}$ with distinct nodes on $(0,\infty)$. The aim of this section is to study the convergence of the sequence of 2PTA($r = p(n-1)$, $s = q(n-1)$)

$$F_n(z) = (p(n-1)/n)_{F_\mu}(z) = I_n(\frac{1}{z-\cdot}, \mu), \quad n = 1, 2, \ldots .$$

As in Section 3, the nodes $\{x_{jn}\}_1^n$ are to be chosen as the zeros of the nth orthogonal L-polynomial with respect to an appropriate weight function ω and the ordering induced by the sequence $\{p(n)\}_0^\infty$.

Theorem 4.1 *Let μ be an L^1-integrable function on $(0,\infty)$ and ω an auxiliary weight function on $(0,\infty)$ satisfying*

$$M = \int_0^\infty \frac{|\mu(x)|^2}{\omega(x)} < +\infty \tag{4.3}$$

and whose moments $d_k = \int_0^\infty x^k \omega(x)\,\mathrm{d}x$ *exist for* $k \in \mathbb{Z}$*. Let* $\{p(n)\}_0^\infty$ *be a nondecreasing sequence of nonnegative integers with* $0 \le p(n) \le n$ *and* $s(n) = p(n) - p(n-1) \in \{0,1\}$ *and assume that either*

$$\lim_{n\to\infty}(n - p(n)) = \lim_{n\to\infty} q(n) = \infty \quad \textit{and} \quad \sum_{j=1}^{\infty} d_j^{-\frac{1}{2j}} = +\infty \tag{4.4}$$

or

$$\lim_{n\to\infty} p(n) = \infty \quad \textit{and} \quad \sum_{j=1}^{\infty} d_{-j}^{-\frac{1}{2j}} = +\infty. \tag{4.5}$$

Consider the 2PTA $(p(n-1)/n)_{F_\mu}$ *with* $\{x_{jn}\}$ *the zeros of the nth orthogonal L-polynomial with respect to* ω *associated with* $\{p(n)\}_{n\ge 0}$*. Then* $\{(p(n-1)/n)_{F_\mu}\}_{n\ge 1}$ *converges uniformly to* F_μ *on any compact set* $K \subset \mathbb{C}\setminus[0,\infty)$*.*

PROOF. Take $z \in \mathbb{C}\setminus[0,\infty)$, then by Remark 3.7, it follows that

$$\lim_{n\to\infty}(p(n-1)/n)_{F_\mu} = \lim_{n\to\infty} I_n\left(\frac{1}{z-\cdot},\mu\right) = I\left(\frac{1}{z-\cdot},\mu\right) = F_\mu(z).$$

Hence, pointwise convergence is assured. Let K be a compact subset in $\mathbb{C}\setminus[0,\infty)$. Then for any $z \in K$

$$\left|(p(n-1)/n)_{F_\mu}(z)\right| = \left|\sum_{j=1}^{n}\frac{A_{jn}}{z-x_{jn}}\right| \le \sum_{j=1}^{n}\frac{|A_{jn}|}{|z-x_{jn}|} \le \frac{1}{\mathrm{dist}(K,[0,\infty))}\sum_{j=1}^{n}|A_{jn}|.$$

From (3.15) we conclude that $\{(p(n-1)/n)_{F_\mu}(z)\}_{n\ge 1}$ is a normal family in $\mathbb{C}\setminus[0,\infty)$ and the proof is a consequence of the Stieltjes-Vitali Theorem [15]. □

Set $E_n(z) = F_\mu(z) - (p(n-1)/n)_{F_\mu}(z)$, $z \notin [0,\infty)$. Now, making use of Corollary 3.3, we can deduce the following result that provides with an estimation for $E_n(z)$.

Corollary 4.2 *Under the same assumptions as in Theorem 4.1 and taking* $z \in \mathbb{C}\setminus\mathbb{R}^+$*, one* *has*

(1) *If we assume (4.4), then*

$$|E_n(z)| \leq \left(2 \int_0^\infty \frac{|\mu(x)|}{\tilde{h}_0(x)} \, \mathrm{d}x + \tilde{\epsilon}_n \right) m_{p(n-1)}(\frac{t}{zt-1}, h_0, \mathbb{R}^+),$$

with $\tilde{h_0}(t) = h_0(\frac{1}{t})$ *and where* $h_0^2(t) = \frac{t^{\tilde{\rho}}}{M(t)}$ *and* $\{\tilde{\epsilon}_n\}_{n\geq 1} \subset (0,\infty)$ *such that* $\lim_{n\to\infty} \tilde{\epsilon}_n = 0$.

(2) *If we assume (4.5), then*

$$|E_n(z)| \leq \left(2 \int_0^\infty \frac{|\mu(x)|}{h_\infty(x)} \, \mathrm{d}x + \epsilon_n \right) m_{q(n-1)}(\frac{1}{z-t}, h_\infty, \mathbb{R}^+),$$

where $h_\infty^2(t) = \frac{t^\rho}{Q(t)}$ *and* $\{\epsilon_n\}_{\geq 1} \subset (0,\infty)$ *such that* $\lim_{n\to\infty} \epsilon_n = 0$.

(3) *If we assume (4.4)-(4.5), then*

$$|E_n(z)| \leq \left(2 \int_0^\infty \frac{|\mu(x)|}{\tilde{h}_0(x) h_\infty(x)} \, \mathrm{d}x + \nu_n \right) \left(C_1 m_{p(n-1)}(\tfrac{t}{zt-1}, h_0, \mathbb{R}^+) + C_2 m_{q(n-1)}(\tfrac{1}{z-t}, h_\infty, \mathbb{R}^+) \right),$$

where $C_1 = \|1\|_{\infty,h_0}, C_2 = \|1\|_{\infty,h_\infty}$, $\tilde{h}_0(t) = h_0(\frac{1}{t})$ *where* $h_0^2(t) = \frac{t^{\tilde{\rho}}}{M(t)}$, $h_\infty^2(t) = \frac{t^\rho}{Q(t)}$ *and* $\{\nu_n\}_{\geq 1} \subset (0,\infty)$ *such that* $\lim_{n\to\infty} \nu_n = 0$.

Conversely, the following expression will be used later on to give an upper bound for the error of the nth interpolatory rule $I_n(f,\mu)$ for an analytic integrand, in terms of the error for 2PTA.

Lemma 4.3 *Let* $\{V_n\}_1^\infty$ *be a sequence of orthonormal L-polynomials with respect to a weight function* ω *and the ordering induced by* $\{p(n)\}_{n\geq 0}$. *Let* $I_n(f,\mu) = \sum_{j=1}^n A_{jn} f(x_{jn})$ *be the interpolatory rule in* $\mathcal{L}_{n-1}$ *whose nodes are the zeros of* V_n. *Then, for* $z \in \mathbb{C}\backslash\mathbb{R}^+$

$$E_n(z) = F_\mu(z) - I_n(\frac{1}{z-\cdot}, \mu) = \frac{1}{z^{s(n)} V_n(z)} \int_0^\infty \frac{x^{s(n)} V_n(x)}{z-x} \mu(x) \, \mathrm{d}x, \tag{4.6}$$

where $s(n) = p(n) - p(n-1)$.

PROOF. Let $R_n(z,x)$ be the L-polynomial in $\mathcal{L}_{n-1}$ interpolating $1/(z-x)$ (z is a parameter) at the above nodes. So,

$$(p(n-1)/n)_{F_\mu}(z) = I_n(\frac{1}{z-\cdot}, \mu) = \int_0^\infty R_n(z,x) \mu(x) \, \mathrm{d}x.$$

On the other hand, it can be easily checked that

$$R_n(z,x) = \frac{1}{z-x}\left(1 - \left(\frac{x}{z}\right)^{s(n)} \frac{V_n(x)}{V_n(z)}\right).$$

Now, the proof follows directly. □

Remark 4.4 Actually, (4.6) is valid for any L-polynomial V_n of the form $V_n(x) = \gamma_n \frac{\prod_{j=1}^{n}(x-x_{jn})}{x^{p(n)}}$, $\gamma_n \neq 0$, $x_{jn} \neq 0$, $x_{jn} \neq x_{k,n}$ if $j \neq k$.

From this lemma, we can deduce the following error bound.

Theorem 4.5 *Assume that μ and ω satisfy (4.3) and set $\delta(z) = \mathrm{dist}(z,[0,\infty))$, $z \notin [0,\infty)$. Then*

$$|E_n(z)| = |F_\mu(z) - (p(n-1)/n)_{F_\mu}(z)| \leq \left(1 + \frac{1}{|z|}\right) \frac{M^{1/2}}{\delta(z)|V_n(z)|}. \tag{4.7}$$

PROOF. Recall that $s(n) = p(n) - p(n-1) \in \{0,1\}$. Assume first $s(n) = 0$. Then by (4.6), for $z \notin [0,\infty)$,

$$\begin{aligned}
|E_n(z)| &= \frac{1}{|V_n(z)|}\left|\int_0^\infty \frac{V_n(x)}{z-x}\mu(x)\,\mathrm{d}x\right| \leq \frac{1}{|V_n(z)|}\int_0^\infty \left|\frac{V_n(x)}{z-x}\right| |\mu(x)|\,\mathrm{d}x \\
&\leq \frac{1}{\delta(z)|V_n(z)|}\int_0^\infty |V_n(x)| \frac{|\mu(x)|}{\sqrt{\omega(x)}}\sqrt{\omega(x)}\,\mathrm{d}x \\
&\leq \frac{1}{\delta(z)|V_n(z)|}\left(\int_0^\infty V_n^2(x)\omega(x)\,\mathrm{d}x\right)^{1/2}\left(\int_0^\infty \frac{|\mu(x)|^2}{\omega(x)}\,\mathrm{d}x\right)^{1/2} \leq \frac{M^{1/2}}{\delta(z)|V_n(z)|} \\
&\leq \left(1 + \frac{1}{|z|}\right)\frac{M^{1/2}}{\delta(z)|V_n(z)|}
\end{aligned}$$

Suppose now that $s(n) = 1$. Again by (4.6) for $z \notin [0,\infty)$,

$$E_n(z) = \frac{1}{zV_n(z)}\int_0^\infty \frac{xV_n(x)}{z-x}\mu(x)\,\mathrm{d}x = \frac{1}{zV_n(z)}\left[\int_0^\infty \frac{zV_n(x)}{z-x}\mu(x)\,\mathrm{d}x - \int_0^\infty V_n(x)\mu(x)\,\mathrm{d}x\right].$$

Proceeding as above it results in the same expression (4.7). □

Finally, from Theorem 4.5 and the Cauchy and Fubini Theorems, the following upper bound for the error in $I_n(f)$ can be found.

Theorem 4.6 *Consider the same conditions as in Theorem 4.5 and let* $I_n(f,\mu) = \sum_{j=1}^{n} A_{jn} f(x_{jn})$ *be the* n*-point interpolatory rule in* $\mathcal{L}_{n-1}$ *whose nodes are the zeros of* V_n*, the* n*th orthogonal L-polynomial for* ω *associated with* $\{p(n)\}_{n\geq 0}$*. Let* f *be an analytic function in a simply connected domain* G *which contains the half line in its interior and whose boundary* Γ *is a rectifiable Jordan curve. Then*

$$|I(f,\mu) - I_n(f,\mu)| \leq \frac{C(f,\Gamma)M^{1/2}}{\inf_{z\in\Gamma}|V_n(z)|} \tag{4.8}$$

$C(f,\Gamma)$ *being a positive constant depending on* f *and* Γ*.*

Remark 4.7 This result was earlier proved by the authors for another choice of the nodes. See Theorem 5.1 in [3].

5 Numerical Examples

Interpolatory quadrature rules exactly integrating L-polynomials have been already considered by the authors in a series of previous papers([2–6,9]). However, until this moment no numerical experiment had been worked out. Thus, the aim of this Section is to numerically illustrate the efficacy of those quadratures rules denoted by L-formulas as opposed to the usual polynomial quadratures. Furthermore, a comparison between the interpolatory L-formulas and the Gauss-type L-formulas will be also made. As partially indicated in Section 1, the integrand f in (1.1) exhibits two essential features, on one hand, the existence of possible singularities at the origin and/or infinity (as it is well known this behavior is extremely hard to be "reproduced" by polynomials) and on the other hand, an oscillating character that is highly difficult to be "simulated" by means of the L-polynomial interpolation. We will try to avoid this last drawback by putting such oscillations into the weight μ.

For our purposes we shall restrict ourselves to integrals (1.1) of the form

$$I(f,\mu) = \int_0^\infty f(x)\mu(x)\,\mathrm{d}x = \int_0^\infty f(x)\,x^\theta e^{-(\alpha x + \frac{\beta}{x})}\,\mathrm{d}x \tag{5.1}$$

where $\theta \in \mathbb{R}$, and $\alpha, \beta \in \mathbb{C}$ such that $\Re(\alpha), \Re(\beta) > 0$. Here f is an improperly Riemann-Stieltjes real integrable function with respect to $x^\theta e^{-(\alpha x + \frac{\beta}{x})}$

In order to compute (5.1) we can restrict, without loss of generality to calculate integrals like

$$\int_0^\infty f(x) \begin{pmatrix} \sin(bx) \\ \cos(bx) \end{pmatrix} x^\theta \, e^{-(ax+\frac{c}{x})} \, \mathrm{d}x \quad \text{and} \quad \int_0^\infty f(x) \begin{pmatrix} \sin(\frac{d}{x}) \\ \cos(\frac{d}{x}) \end{pmatrix} x^\theta \, e^{-(ax+\frac{c}{x})} \, \mathrm{d}x$$

where $\alpha = a + bi$ and $\beta = c + di$.

Thus, we see that complex arithmetic is not any more required. We will concentrate on the situation

$$\int_0^\infty f(x) \sin(\frac{1}{x}) \, \frac{e^{-\frac{1}{2}(x+\frac{1}{x})}}{\sqrt{x}} \, \mathrm{d}x, \tag{5.2}$$

where we have taken $a = c = \frac{1}{2}$, $b = 0$ and $d = 1$.

As a first strategy to estimate (5.2) we will use the classical Gaussian formulas,

I. Gauss-Laguerre quadrature formulas

Since (5.2) can be rewritten as,

$$I(f, \mu) = J(g) = \int_0^\infty g(x) \frac{e^{-\frac{x}{2}}}{\sqrt{x}} \, \mathrm{d}x$$

where $g(t) = f(t) \sin(\frac{1}{t}) e^{-\frac{1}{2t}}$ we can make use of the Gauss-Laguerre quadrature formula associated with the weight function $\frac{e^{-\frac{t}{2}}}{\sqrt{t}}$, denoted by

$$J_n(g) = \sum_{j=1}^{n} l_{jn} \, g(t_{jn}).$$

Here $\{l_{jn}\}$ and $\{t_{jn}\}$ are the weights and nodes corresponding to $\frac{e^{-\frac{t}{2}}}{\sqrt{t}}$ that have been extensively studied.

On the other hand, as an alternative approach, we will make use of the

II. L-quadrature formulas.

Here, two strategies will be considered. First, by writing (5.2) as

$$I(f,\mu) = M(h) = \int_0^\infty h(t)\frac{e^{-\frac{1}{2}(x+\frac{1}{x})}}{\sqrt{x}}\,\mathrm{d}x,$$

where $h(t) = f(t)\sin(\frac{1}{t})$, one can use Gauss-type L-formulas $I_n^{GT}(h)$ introduced in Section 2, which are denoted by

$$I_n^{GT}(h) = \sum_{j=1}^{n} \lambda_{jn}\, h(x_{jn}),$$

$\{\lambda_{jn}\}$ and $\{x_{jn}\}$ being the corresponding L-weight and L-nodes associated with $\frac{e^{-\frac{1}{2}(t+\frac{1}{t})}}{\sqrt{t}}$ with respect to the "ordering" induced by the sequence $\{p(n)\}_{n\in\mathbb{N}}$ which throughout this Section, will be taken as $p(n) = [\frac{n+1}{2}]$.

Secondly, we will directly consider (5.2) and use as an estimation for this integral, the interpolatory L-formulas $I_n(f)$ given by

$$I_n(f) = \sum_{j=1}^{n} A_{jn}\, f(z_{jn}),$$

where $\{z_{jn}\}$ are the zeros of the nth orthogonal L-polynomial with respect to the weight $\frac{e^{-(t+\frac{1}{t})}}{\sqrt{t}}$. It should be observed that according to Theorem 3.1, now $\mu(t) = \sin(\frac{1}{t})\ \frac{e^{-\frac{1}{2}(t+\frac{1}{t})}}{\sqrt{t}}$ and $\omega(t) = \frac{e^{-(t+\frac{1}{t})}}{\sqrt{t}}$. Hence, condition (3.3) i.e.

$$\int_0^\infty \frac{|\mu(x)|^2}{\omega(x)}\,\mathrm{d}x < +\infty,$$

is clearly satisfied.

Now, to compute the weights A_{jn} one needs the "moments" of μ. Therefore we will start from the integrals

$$\mu_n = \int_0^\infty x^n x^\theta \; e^{-(\alpha x+\frac{\beta}{x})} \; \mathrm{d}x, \quad n \in \mathbb{Z}.$$

with $\theta \in \mathbb{R}, \alpha, \beta \in \mathbb{C}$ such that $\Re(\alpha), \Re(\beta) > 0$. Integrating by part, one obtains

$$\mu_{n+2} = \frac{n+2+\theta}{\alpha}\mu_{n+1} + \frac{\beta}{\alpha}\mu_n, \qquad n \in \mathbb{Z}.$$

Hence, when making $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2} + i$, $\theta = -\frac{1}{2}$ and taking real and imaginary parts, it follows

$$\begin{array}{l} c^{(1)}_{n+2} - (2n+3)c^{(1)}_{n+1} - c^{(1)}_n + 2c^{(2)}_n = 0 \\ c^{(2)}_{n+2} - (2n+3)c^{(2)}_{n+1} - 2c^{(1)}_n - c^{(2)}_n = 0, \end{array} , n \in \mathbb{Z}$$

where $c^{(1)}_n = \int_0^\infty x^n \cos(\frac{1}{x}) \; \frac{e^{-\frac{1}{2}(x+\frac{1}{x})}}{\sqrt{x}} \; \mathrm{d}x$ and $c^{(2)}_n = \int_0^\infty x^n \sin(\frac{1}{x}) \; \frac{e^{-\frac{1}{2}(x+\frac{1}{x})}}{\sqrt{x}} \; \mathrm{d}x$, $n \in \mathbb{Z}$. Thus, one only needs to compute $c^{(1)}_0$, $c^{(1)}_1$, $c^{(2)}_0$ and $c^{(2)}_1$.

As for the computation of the L-Gaussian weights $\{\lambda_{jn}\}$ and the L-nodes $\{x_{jn}\}$ and $\{z_{jn}\}$ the effort to be made is rather small since the weight functions we are dealing with belong to the family,

$$\omega(t) = \frac{e^{-\frac{1}{2}(x+\frac{a}{x})}}{\sqrt{x}}, \qquad a > 0, \quad t \in (0, \infty),$$

which have been studied by Ranga [21].

Actually, both families of L-nodes $\{x_{jn}\}$ and $\{z_{jn}\}$, $n = 1, 2, \ldots$ can be easily expressed in terms of the nodes $\{t_{jn}\}$ i.e. in terms of the zeros of the Laguerre polynomials of the order $-1/2$. Furthermore, the L-weights $\{\lambda_{jn}\}$ can be also represented by means of the Christoffel-Laguerre numbers $\{l_{jn}\}$ of order $-1/2$. For details see [21,14]. In short, the extra computational effort to be made reduces to the calculation of the weight $\{A_{jn}\}_{j=1}^n$.

Numerical experiments have been carried out for the following integrands f,

i	$f_i(t)$
1	$\exp[\frac{1}{4t}]$
2	$\sqrt{1+t}$
3	$\log[0.001+t]$
4	$\log[1+8t^2]$
5	$\log[\sqrt{t}+\frac{1}{\sqrt{t}}]$

In the next tables, the corresponding absolute errors for the different quadratures are exposed. They are respectively denoted by $EJ_n(g_i) = J(g_i) - J_n(g_i)$, $E_n^{GT}(h_i) = M(h_i) - I_n^{GT}(h_i)$ and $E_n(f) = I(f_i) - I_n(f_i)$ (recall that $J(g_i) = M(h_i) = I(f_i)$, $i = 1, 2, \ldots, 5$).

As usual n denotes the number of nodes used in the quadrature.

Table 1. **Absolute errors for** $f_1(t) = e^{-\frac{1}{4t}}$

n	$EJ_n(g_1)$	$EI_n^{GT}(h_1)$	$EI_n(f_1)$
2	$0.82E-0$	$0.11E-0$	$0.10E-1$
4	$0.66E-0$	$0.14E-0$	$0.79E-3$
6	$0.68E-0$	$0.24E-1$	$0.17E-3$
8	$0.13E-1$	$0.40E-1$	$0.88E-4$
10	$0.14E-0$	$0.18E-0$	$0.18E-4$
12	$0.22E-1$	$0.14E-1$	$0.37E-7$
14	$0.71E-1$	$0.36E-2$	$0.13E-5$
16	$0.18E-0$	$0.41E-2$	$0.43E-6$
18	$0.19E-0$	$0.24E-2$	$0.48E-7$
20	$0.15E-0$	$0.10E-2$	$0.18E-7$

Table 2. **Absolute errors for** $f_2(t) = \sqrt{1+t}$

n	$EJ_n(g_2)$	$EI_n^{GT}(h_2)$	$EI_n(f_2)$
2	$0.44E-0$	$0.15E-0$	$0.21E-1$
4	$0.52E-0$	$0.22E-1$	$0.16E-2$
6	$0.34E-0$	$0.20E-1$	$0.19E-3$
8	$0.49E-1$	$0.25E-2$	$0.96E-4$
10	$0.40E-1$	$0.33E-2$	$0.29E-4$
12	$0.70E-1$	$0.37E-3$	$0.34E-6$
14	$0.98E-1$	$0.63E-3$	$0.32E-5$
16	$0.11E-0$	$0.66E-4$	$0.19E-5$
18	$0.10E-0$	$0.13E-3$	$0.34E-6$
20	$0.87E-1$	$0.12E-4$	$0.16E-6$

Table 3. **Absolute errors for** $f_3(t) = \log[0.001 + t]$

n	$EJ_n(g_3)$	$EI_n^{GT}(h_3)$	$EI_n(f_3)$
2	$0.57E-0$	$0.64E-1$	$0.49E-1$
4	$0.21E-0$	$0.36E-1$	$0.46E-2$
6	$0.31E-0$	$0.16E-1$	$0.87E-3$
8	$0.53E-1$	$0.55E-2$	$0.53E-3$
10	$0.77E-2$	$0.34E-2$	$0.16E-3$
12	$0.88E-2$	$0.99E-3$	$0.20E-5$
14	$0.34E-1$	$0.74E-3$	$0.22E-4$
16	$0.73E-1$	$0.19E-3$	$0.13E-4$
18	$0.89E-1$	$0.16E-3$	$0.24E-5$
20	$0.93E-1$	$0.40E-4$	$0.13E-5$

Table 4. **Absolute errors for** $f_4(t) = \log[1 + 8t^2]$

n	$EJ_n(g_4)$	$EI_n^{GT}(h_4)$	$EI_n(f_4)$
2	$0.92E-2$	$0.14E-1$	$0.11E-0$
4	$0.59E-0$	$0.83E-1$	$0.11E-1$
6	$0.22E-0$	$0.29E-1$	$0.24E-2$
8	$0.90E-2$	$0.12E-1$	$0.15E-2$
10	$0.93E-1$	$0.62E-2$	$0.43E-3$
12	$0.12E-0$	$0.21E-2$	$0.59E-6$
14	$0.11E-0$	$0.14E-2$	$0.56E-4$
16	$0.86E-1$	$0.41E-3$	$0.28E-4$
18	$0.58E-1$	$0.31E-3$	$0.43E-5$
20	$0.32E-1$	$0.83E-4$	$0.22E-5$

Table 5. **Absolute errors for** $f_5(t) = Log[\sqrt{t} + \frac{1}{\sqrt{t}}]$

n	$EJ_n(g_5)$	$EI_n^{GT}(h_5)$	$EI_n(f_5)$
2	$0.33E-0$	$0.20E-0$	$0.75E-2$
4	$0.30E-0$	$0.45E-1$	$0.87E-3$
6	$0.24E-0$	$0.56E-2$	$0.62E-4$
8	$0.30E-1$	$0.99E-4$	$0.26E-4$
10	$0.21E-1$	$0.12E-3$	$0.10E-4$
12	$0.30E-1$	$0.65E-4$	$0.32E-6$
14	$0.53E-1$	$0.31E-4$	$0.54E-6$
16	$0.69E-1$	$0.30E-4$	$0.45E-6$
18	$0.71E-1$	$0.93E-5$	$0.77E-7$
20	$0.65E-1$	$0.29E-5$	$0.14E-7$

As it could be expected, one can see from the previous tables that the presence of singularities at the origin and/or the infinity along with the oscillations generated by $\sin(\frac{1}{x})$ makes the convergence of the classical Gauss-Laguerre rule extremely slow. As for the Gauss-type L-quadrature, the presence of an oscillating factor in the integrand make the results not to be so good as those given in [4](see also [9]). Fortunately, these drawbacks could be overcome by using the interpolatory L-quadrature.

References

[1] E. Aldén. A survey of weak and strong moment problems with generalizations. Preprint Series 2, Dept. Math. Umeå, 1985.

[2] A. Bultheel, C. Díaz-Mendoza, P. González-Vera, and R. Orive. Quadrature on the half line and two-point Padé approximants to Stieltjes functions. Part II: Convergence. *J. Comput. Appl. Math.*, 77:53–76, 1997.

[3] A. Bultheel, C. Díaz-Mendoza, P. González-Vera, and R. Orive. Quadrature on the half line and two-point Padé approximants to Stieltjes functions. Part III: The unbounded case. *J. Comput. Appl. Math.*, 87:95–117, 1997.

[4] A. Bultheel, C. Díaz-Mendoza, P. González-Vera, and R. Orive. On the convergence of certain Gauss-type quadrature formulas for unbounded intervals. *Math. Comp.*, 69:721–747, 2000.

[5] A. Bultheel, C. Díaz-Mendoza, P. González-Vera, and R. Orive. Orthogonal Laurent polynomials and quadrature formulas for unbounded intervals: Part I. Gauss-type formulas. *Rocky Mountain J. Math.*, 33, 2003. Accepterd. To appear in June 2003.

[6] A. Bultheel, P. González-Vera, and R. Orive. Quadrature on the half line and two-point Padé approximants to Stieltjes functions. Part I: Algebraic aspects. *J. Comput. Appl. Math.*, 65:57–72, 1995.

[7] E.W. Cheney. *Introduction to approximation theory.* McGraw Hill, 1966.

[8] L. Cochran and S.C. Cooper. Orthogonal Laurent polynomials on the real line. In S.C. Cooper and W.J. Thron, editors, *Continued fractions and orthogonal functions*, pages 47–100, New York, 1994. Marcel Dekker.

[9] C. Díaz-Mendoza, P. González-Vera, and M. Jiménez-Paiz. Stieltjes distributions and orthogonal Laurent polynomials with applications to quadratures and Padé approximation. *Math. Comp.*, 2003. Submitted.

[10] A. Draux. Approximation de type-Padé en deux points. Technical Report ANO-110, Université des Sciences et Techniques de Lille, Lab. d'analyse numérique et d'optimisation, 1983.

[11] A. Erdélyi, editor. *Higher transcendental functions, vol. 3.* Mc Graw-Hill, New York, 1953.

[12] G. Freud. *Orthogonal polynomials.* Pergamon Press, Oxford, 1971.

[13] W. Gautschi. *Numerical analysis, an introduction.* Birkhäuser Verlag, 1997.

[14] B.A. Hagler, W.B. Jones, and W.J. Thron. Orthogonal Laurent polynomials of Jacobi, Hermite and Laguerre types. In W.B. Jones and A. Sri Ranga, editors, *Orthogonal functions, moment theory, and continued fractions, theory and applications*, volume 199 of *Lecture Notes in Pure and Applied Mathematics*, pages 187–208. Marcel Dekker, 1998.

[15] E. Hille. *Analytic function theory*, volume 2. Ginn and Co., Boston, 1962.

[16] W.B. Jones, W.J. Thron, and H. Waadeland. A strong Stieltjes moment problem. *Trans. Amer. Math. Soc.*, 206:503–528, 1980.

[17] V.J. Krylov. *Approximate calculation of integrals.* MacMillan, 1962.

[18] G. López-Lagomasino and A. Martínez-Finkelshtein. Rate of convergence of two-point Padé approximants and logarithmic asymptotics of Laurent-type orthogonal polynomials. *Constr. Approx.*, 11:255–286, 1995.

[19] S.N. Mergelyan. Weighted approximations by polynomials. *Amer. Math. Soc. Transl. Ser. 2*, 10:59–106, 1958.

[20] E.M. Nikishin and V.N. Sorokin. *Rational approximation and orthogonality*, volume 92 of *Transl. Math. Monographs.* Amer. Math. Soc., 1991.

[21] A.S. Ranga. Another quadrature rule of highest algebraic degree. *Numer. Math.*, 28:283–294, 1994.

[22] I.H. Sloan and W.E. Smith. Properties of interpolatory product rules. *SIAM J. Numer. Anal.*, 19:427–442, 1982.

[23] W.E. Smith, I.H. Sloan, and A.H. Optie. Product integration over infinite intervals. I. Rules based on the zeros of Hermite polynomials. *Math. Comp.*, 40:519–545, 1983.