Orthogonal rational functions and rational modifications of a measure

Karl Deckers

Department of Computer Science, Katholieke Universiteit Leuven,
Heverlee, Belgium.
PhD student. Supervisor: Adhemar Bultheel.

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Outline

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2. Preliminaries
3. Measures on $\mathbb{T}$
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5. Measures on I

Motivation

Explicit examples of orthogonal rational functions (ORFs)

- Chebyshev ORFs on $I = [-1, 1]$ w.r.t.

$$w(x) = (1-x)^a(1+x)^b, \quad a, b \in \left\{\pm\frac{1}{2}\right\}.$$

- Takenaka-Malmquist basis on $\mathbb{T} = \{z : |z|^2 = 1\}$ w.r.t. Lebesgue measure

Motivation

Relating I and $\mathbb{T}$

$$\mathring{w}(\theta) = w(\cos\theta)|\sin\theta|, \quad \theta \in [-\pi, \pi], \; z = e^{\mathbf{i}\theta}$$

(a, b)	$w(x)$	$\mathring{w}(\theta)$	$\mathring{\mu}'(z)$
$\left(-\frac{1}{2}, -\frac{1}{2}\right)$	$\left(1-x^2\right)^{-1/2}$	1	$\frac{1}{\mathbf{i}z}$
$\left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(1-x^2\right)^{1/2}$	$\sin^2\theta$	$-\frac{(z^2-1)^2}{4\mathbf{i}z^3}$
$\left(\frac{1}{2}, -\frac{1}{2}\right)$	$\left(\frac{1-x}{1+x}\right)^{1/2}$	$1-\cos\theta$	$-\frac{(z-1)^2}{2\mathbf{i}z^2}$
$\left(-\frac{1}{2}, \frac{1}{2}\right)$	$\left(\frac{1+x}{1-x}\right)^{1/2}$	$1+\cos\theta$	$\frac{(z+1)^2}{2\mathbf{i}z^2}$

$$\phi_n(z; \mathring{\mu}) \underset{\leftarrow}{\overset{\rightarrow}{}} \phi_n(x; w)$$

Orthogonal polynomials

Polynomial modifications

$$\begin{array}{rcl} d\tilde{\mu} & = & (1 \pm \cos\theta) d\mathring{\mu} = |z \pm 1|^2 d\mathring{\mu} \\ d\hat{\mu} & = & \sin^2\theta d\mathring{\mu} = |z^2 - 1|^2 d\mathring{\mu} \end{array}, \quad z = e^{\mathbf{i}\theta}$$

$$\begin{array}{rcl} d\tilde{\mu} & = & |p_m(z)|^2 d\mathring{\mu} \\ & = & |(z-\gamma_1)\cdot\ldots\cdot(z-\gamma_m)|^2 d\mathring{\mu}, \quad |\gamma_j| \leq 1,\ j = 1,\ldots,m \end{array}$$

$$\begin{array}{rcl} \phi_n(z;\tilde{\mu}) \perp_{\tilde{\mu}} \mathcal{P}_{n-1} & \Rightarrow & p_m(z)\phi_n(z;\tilde{\mu}) \perp_{\mathring{\mu}} p_m(z)\mathcal{P}_{n-1} \\ & \Rightarrow & \text{relation } \phi_n(z;\tilde{\mu}) \text{ and } \phi_{n+m}(z;\mathring{\mu}) \end{array}$$

Orthogonal rational functions

Rational modifications

$$\begin{aligned} d\tilde{\mu} &= |r_m(z)|^2 d\mathring{\mu} \\ &= \left| \frac{(z-\gamma_1)\cdot\ldots\cdot(z-\gamma_m)}{(1-\overline{\alpha}_1 z)\cdot\ldots\cdot(1-\overline{\alpha}_m z)} \right|^2 d\mathring{\mu}, \\ & \qquad |\gamma_j| \leq 1, \quad \text{and} \quad |\alpha_j| < 1, \quad j = 1, \ldots, m \end{aligned}$$

$$\begin{aligned} \phi_n(z;\tilde{\mu}) \perp_{\tilde{\mu}} \mathring{\mathcal{L}}_{n-1} &\Rightarrow r_m(z)\phi_n(z;\tilde{\mu}) \perp_{\mathring{\mu}} r_m(z)\mathring{\mathcal{L}}_{n-1} \\ &\Rightarrow \text{relation } \phi_n(z;\tilde{\mu}) \text{ and } \phi_{n+m}(z;\mathring{\mu}) \end{aligned}$$

Canonical basis for $\mathring{\mathcal{L}}$

Blaschke factors and Blaschke products

- canonical basis for $\mathring{\mathcal{L}}_n$:

$$\begin{aligned} \zeta_{\beta_k}(z) &= \eta_{\beta_k}\frac{z-\beta_k}{1-\overline{\beta}_k z}, \quad \eta_{\beta_k} = \begin{cases} \frac{\overline{\beta}_k}{|\beta_k|}, & \beta_k \neq 0 \\ 1, & \beta_k = 0 \end{cases} \\ B_k(z) &= \zeta_{\beta_k}(z)B_{k-1}(z), \quad B_0(z) \equiv 1 \end{aligned}$$

- canonical basis for $\tilde{\mathcal{L}}_m$: $\tilde{B}_k(z) = \zeta_{\alpha_k}(z)\tilde{B}_{k-1}(z)$
- canonical basis for $\hat{\mathcal{L}}_{n+m}$:

$$\hat{B}_k(z) = \begin{cases} \tilde{B}_k(z), & k \leq m \\ \tilde{B}_m(z)B_{k-m}(z), & k > m \end{cases}$$

Definitions

Monic ORFs

$\phi_n(z)$ is called monic iff $\phi_n(z) = 1 \cdot B_n(z) + f_{n-1}(z)$, $f_{n-1} \in \mathring{\mathcal{L}}_{n-1}$ or equivalently $\overline{\phi_n^*}(\beta_n) = 1$ where

$$\phi_n^*(z) = B_n(z)\phi_{n*}(z) \quad \text{and} \quad \phi_{n*}(z) = \overline{\phi_n(1/\overline{z})}.$$

Suppose $\phi_n(z) = a \cdot B_n(z) + f_{n-1}(z)$, then

$$\begin{aligned} \phi_n^*(z) &= B_n(z)\left(\overline{a} \cdot B_{n*}(z) + f_{n-1*}(z)\right) \\ &= \overline{a} + \zeta_{\beta_n}(z) f_{n-1}^*(z), \qquad \left(B_{k*}(z) = B_k^{-1}(z)\right). \end{aligned}$$

Definitions

From now on we assume

- ϕ_k's are monic
- φ_k's are orthonormal, with $\varphi_k = \mathring{\kappa}_k \phi_k$

Reproducing kernel

$$\begin{aligned} \mathring{k}_n(z,u;\mathring{\mu}) &= \sum_{k=0}^{n} \varphi_k(z;\mathring{\mu})\overline{\varphi_k(u;\mathring{\mu})} \\ &= \left|\mathring{\kappa}_{n+1}\right|^2 \frac{\phi^*_{n+1}(z;\mathring{\mu})\overline{\phi^*_{n+1}(u;\mathring{\mu})} - \phi_{n+1}(z;\mathring{\mu})\overline{\phi_{n+1}(u;\mathring{\mu})}}{1-\zeta_{\beta_{n+1}}(z)\overline{\zeta_{\beta_{n+1}}(u)}}, \quad n>0 \end{aligned}$$

Rational modification

Relating monic ORFs

$\phi_n(z;\tilde{\mu}) \perp_{\tilde{\mu}} \mathring{\mathcal{L}}_{n-1}$ and $\phi_{n+m}(z;\mathring{\mu}) \perp_{\mathring{\mu}} \hat{\mathcal{L}}_{n+m-1}$

$$\phi_{n+m}(z;\mathring{\mu}) - \frac{r_m(z)}{\overline{r_m^*}(\beta_n)}\phi_n(z;\tilde{\mu}) \in \hat{\mathcal{L}}_{n+m-1}$$

because $\overline{\phi_{n+m}^*}(\beta_n;\mathring{\mu}) = 1$ and

$$\begin{aligned}[r_m(z)\phi_n(z;\tilde{\mu})]^* &= \hat{B}_{n+m}(z)r_{m*}(z)\phi_{n*}(z;\tilde{\mu}) \\ &= \tilde{B}_m(z)r_{m*}(z)B_n(z)\phi_{n*}(z;\tilde{\mu}) = r_m^*(z)\phi_n^*(z;\tilde{\mu}).\end{aligned}$$

Rational modification

Basis for $\hat{\mathcal{L}}_{n+m-1}$

Consider orthogonal decomposition

$$\hat{\mathcal{L}}_{n+m-1} = r_m \mathring{\mathcal{L}}_{n-1} \oplus \left[r_m \mathring{\mathcal{L}}_{n-1} \right]^{\perp_{\mathring{\mu}}}$$

- $\{r_m \phi_k(z; \tilde{\mu})\}_{k=0}^{n-1}$ is orthogonal basis for $r_m \mathring{\mathcal{L}}_{n-1}$ w.r.t. $\mathring{\mu}$
- $\left\{ \{g_{i,k}(z)\}_{k=0}^{m_i-1} \right\}_{i=1}^{j}$ is basis for $\left[r_m \mathring{\mathcal{L}}_{n-1} \right]^{\perp_{\mathring{\mu}}}$ where

$$g_{i,k}(z) = \left. \frac{\partial^k \hat{k}_{n+m-1}(z, u; \mathring{\mu})}{\partial \overline{u}^k} \right|_{u=\gamma_i}$$

Rational modification

$$\phi_{n+m}(z;\mathring{\mu}) - \frac{r_m(z)}{\overline{r_m^*}(\beta_n)}\phi_n(z;\tilde{\mu}) = \sum_{k=0}^{n-1} \Lambda_k r_m(z)\phi_k(z;\tilde{\mu}) + \sum_{i=1}^{j}\sum_{k=0}^{m_i-1} \lambda_{i,k} g_{i,k}(z)$$

$\phi_{n+m}(z;\mathring{\mu}) \perp_{\mathring{\mu}} r_m(z)\phi_k(z;\tilde{\mu})$ and $r_m(z)\phi_n(z;\tilde{\mu}) \perp_{\mathring{\mu}} r_m(z)\phi_k(z;\tilde{\mu})$, so that $\Lambda_k = 0$.

$$\Rightarrow \phi_{n+m}(z;\mathring{\mu}) - \frac{r_m(z)}{\overline{r_m^*}(\beta_n)}\phi_n(z;\tilde{\mu}) = \sum_{i=1}^{j}\sum_{k=0}^{m_i-1} \lambda_{i,k} g_{i,k}(z)$$

Rational modification

Suppose the zeros γ_i are simple:

$$\phi_{n+m}(z;\mathring{\mu}) - \frac{r_m(z)}{\overline{r_m^*}(\beta_n)}\phi_n(z;\tilde{\mu}) = \sum_{j=1}^{m} \lambda_j \hat{k}_{n+m-1}(z,\gamma_j;\mathring{\mu}).$$

$$\phi_{n+m}(\gamma_i;\mathring{\mu}) = \sum_{j=1}^{m} \lambda_j \hat{k}_{n+m-1}(\gamma_i,\gamma_j;\mathring{\mu}), \quad i = 1,\ldots,m.$$

$$\underline{\lambda} = \mathbf{K}^{-1}\underline{\phi}_{n+m}(\mathring{\mu}).$$

Rational modification

Theorem

Let $d\tilde{\mu} = |r_m(z)|^2 d\mathring{\mu}$ where $r_m \in \tilde{\mathcal{L}}_m \setminus \tilde{\mathcal{L}}_{m-1}$ with simple zeros in $\{\gamma_j\}_{j=1}^m$. Let $\phi_n(z;\tilde{\mu})$ denote the monic ORF in $\mathring{\mathcal{L}}_n$ w.r.t. $\tilde{\mu}$. Similarly, let $\phi_{n+m}(z;\mathring{\mu})$ denote the monic ORF in $\hat{\mathcal{L}}_{n+m}$ w.r.t. $\mathring{\mu}$. Then

$$\frac{r_m(z)}{\overline{r_m^*}(\beta_n)}\phi_n(z;\tilde{\mu}) = \frac{1}{\det \mathbf{K}} \begin{vmatrix} \phi_{n+m}(z;\mathring{\mu}) & \hat{\underline{k}}_{m+n-1}^T(z;\mathring{\mu}) \\ \underline{\phi}_{n+m}(\mathring{\mu}) & \mathbf{K} \end{vmatrix}$$

Computational aspects

Computing the monic ORFs

- computing $\phi_n(z; \mathring{\mu}_m)$ by means of intermediate results; i.e. rational modifications of degree 1:

$$d\mathring{\mu}_1 = \left| \frac{z-\gamma}{1-\overline{\alpha}z} \right| d\mathring{\mu}$$

$$\left(\frac{z-\gamma}{1-\overline{\alpha}z} \right) \phi_n(z; \mathring{\mu}_1)$$

$$= \overline{\eta}_\alpha \frac{1-\gamma\overline{\beta}_n}{1-\alpha\overline{\beta}_n} \left[\phi_{n+1}(z; \mathring{\mu}) - \frac{\phi_{n+1}(\gamma; \mathring{\mu})}{\hat{k}_n(\gamma, \gamma; \mathring{\mu})} \hat{k}_n(z, \gamma; \mathring{\mu}) \right].$$

- distinction between $|\gamma| = 1$ and $|\gamma| < 1$

Computational aspects

$|\gamma| = 1$

$$\frac{(z-\gamma)^2}{1-\overline{\alpha}z}\phi_n(z;\mathring{\mu}_1) = \frac{\overline{\eta}_\alpha \frac{1-\gamma\overline{\beta}_n}{1-\alpha\overline{\beta}_n}}{\begin{vmatrix} \phi_{n+1}(\gamma;\mathring{\mu}) & \phi^*_{n+1}(\gamma;\mathring{\mu}) \\ \phi'_{n+1}(\gamma;\mathring{\mu}) & \phi^{*'}_{n+1}(\gamma;\mathring{\mu}) \end{vmatrix}} \times$$

$$\begin{vmatrix} (z-\gamma)\phi_{n+1}(z;\mathring{\mu}) & \phi_{n+1}(z;\mathring{\mu}) & \phi^*_{n+1}(z;\mathring{\mu}) \\ 0 & \phi_{n+1}(\gamma;\mathring{\mu}) & \phi^*_{n+1}(\gamma;\mathring{\mu}) \\ \left(\frac{1-\overline{\beta}_n z}{1-\overline{\beta}_n\gamma}\right)\phi_{n+1}(\gamma;\mathring{\mu}) & \phi'_{n+1}(\gamma;\mathring{\mu}) & \phi^{*'}_{n+1}(\gamma;\mathring{\mu}) \end{vmatrix},$$

where ϕ' represents the derivative of ϕ.

Computational aspects

$|\gamma| < 1$

$$\frac{(1-\overline{\gamma}z)(z-\gamma)}{1-\overline{\alpha}z}\phi_n(z;\mathring{\mu}_1) = \frac{\overline{\eta}_\alpha \frac{1-\gamma\overline{\beta}_n}{1-\alpha\overline{\beta}_n}}{\begin{vmatrix} \phi_{n+1}(\gamma;\mathring{\mu}) & \phi^*_{n+1}(\gamma;\mathring{\mu}) \\ \overline{\phi^*_{n+1}}(\gamma;\mathring{\mu}) & \overline{\phi_{n+1}}(\gamma;\mathring{\mu}) \end{vmatrix}} \times$$

$$\begin{vmatrix} (1-\overline{\gamma}z)\phi_{n+1}(z;\mathring{\mu}) & c_n(z)\phi_{n+1}(z;\mathring{\mu}) & c_n(z)\phi^*_{n+1}(z;\mathring{\mu}) \\ \left(\frac{1-|\gamma|^2}{1-\overline{\beta}_n\gamma}\right)\phi_{n+1}(\gamma;\mathring{\mu}) & \phi_{n+1}(\gamma;\mathring{\mu}) & \phi^*_{n+1}(\gamma;\mathring{\mu}) \\ 0 & \overline{\phi^*_{n+1}}(\gamma;\mathring{\mu}) & \overline{\phi_{n+1}}(\gamma;\mathring{\mu}) \end{vmatrix},$$

where $c_n(z) = (1-\overline{\beta}_n z)$.

Examples

Chebyshev ORFs on $\mathbb{T}$

- $\mathring{w}(\theta) = 1 \pm \cos\theta$

$$\phi_n(z;\mathring{w}) = c_n \frac{a_n + z(z-b_n)\frac{zB_{n-1}(z)}{1-\overline{\beta}_n z}}{(z\pm 1)^2}$$

- $\mathring{w}(\theta) = \sin^2\theta$

$$\phi_n(z;\mathring{w}) = h_n \frac{d_n + e_n z + z^2(f_n + g_n z + z^2)\frac{zB_{n-1}(z)}{1-\overline{\beta}_n z}}{(z^2-1)^2}$$

Application

OPs and rational modifications of a measure

Alternative way to compute OPs w.r.t. rational modification of a measure

$$d\tilde{\mu} = \frac{d\mathring{\mu}}{|h_m(z)|^2},\ h_m(z) = (1-\overline{\alpha}_1 z)\cdot\ldots\cdot(1-\overline{\alpha}_m z),\ |\alpha_j| < 1$$

(cfr. E. Godoy and F. Marcellán: based on functions of the second kind)

- $\phi_n(z;\tilde{\mu}) \in \mathcal{P}_n$ is related with $\phi_{n+m}(z;\mathring{\mu}) \in \tilde{\mathcal{L}}_m \cdot \mathcal{P}_n$
- if $n > m$ and $\phi_n(z;\mathring{\mu}) = \frac{P_n(z)}{h_m(z)} \in \tilde{\mathcal{L}}_m \cdot \mathcal{P}_{n-m}$, then
  $\phi_n(z;\tilde{\mu}) = P_n(z)$
  if $\phi_n(z;\mathring{\mu})$ is monic, then $\frac{P_n(z)}{\eta_{\alpha_1}\cdot\ldots\cdot\eta_{\alpha_m}}$ is monic

Rational modifications of a measure on I

$$d\tilde{\mu} = r_m(x)d\mu, \quad r_m(x) = \frac{p_m(x)}{(1-x/\alpha_1)\cdot\ldots\cdot(1-x/\alpha_m)},$$
$$r_m(x) \geq 0 \text{ on } I,\ \alpha_i \notin I \text{ and } p_m(\alpha_i)\cdot p_m(\beta_j) \neq 0$$

$$\begin{aligned} r_m(x) &= c\frac{\prod_{k=1}^{p}(x-\gamma_k)\prod_{k=1}^{r}|x-\tilde{\gamma}_k|^2}{\prod_{j=1}^{q}(1-x/\alpha_j)\prod_{j=1}^{s}|1-x/\tilde{\alpha}_j|^2},\ p+2r \leq q+2s = m \\ &= r_S(x)\cdot|r_T(x)|^2,\ S = \max\{p,q\} \text{ and } T = \max\{r,s\} \end{aligned}$$

with

$$\begin{aligned} r_S(x) &= \frac{c_1\prod_{k=1}^{p}(x-\gamma_k)}{(1-x/\epsilon)^{t_1}\prod_{j=1}^{q}(1-x/\alpha_j)},\ \epsilon = \infty \text{ and } t_1 = \max\{0,p-q\} \\ r_T(x) &= \frac{c_2\prod_{k=1}^{r}(x-\tilde{\gamma}_k)}{(1-x/\epsilon)^{t_2}\prod_{j=1}^{s}(1-x/\tilde{\alpha}_j)},\ t_2 = \max\{0,r-s\} \end{aligned}$$

Rational modifications of a measure on I

- relation I and $\mathbb{T}$: $\phi_n(x;\mu) \to \phi_n(z;\mathring{\mu})$ where

$$\mathring{\mu}(E) = \mu\left(\{\cos\theta, \theta \in E \cap [0,\pi)\}\right) + \mu\left(\{\cos\theta, \theta \in E \cap [-\pi,0)\}\right)$$

- Joukowski Transform $x = J(z) = (z + z^{-1})/2$: $r_m(x) \to r_m(z)$

$$r_m(z) = c\prod_{k=1}^{m}\frac{z-\varepsilon_k}{1-\overline{\delta}_k z},\ c \neq 0,\ \gamma_k = J(\varepsilon_k),\ \alpha_k = J(\delta_k)$$

- rational modification $d\mathring{\mu}_* = |r_m(z)|^2 d\mathring{\mu}$
- relation I and $\mathbb{T}$: $\phi_n(z;\mathring{\mu}_*) \to \phi_n(x;\tilde{\mu})$

Rational modifications of a measure on I

Alternative: $\hat{\mathcal{L}}_{n+T+S} \to \{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_T, \beta_1, \ldots, \beta_{n-1}, \alpha_1, \ldots, \alpha_S, \beta_n\}$

$$\phi_{S+T+n}(x;\mu) - c_n^{-1} r_S(x) r_T(x) \phi_n(x;\tilde{\mu}) = \sum_{l=0}^{n-1} 0 \cdot r_T(x)\phi_l(x;\tilde{\mu})$$
$$\sum_{k=0}^{S-1} \Lambda_k \phi_{n+T+k}(x;\mu) + \sum_{j=1}^{T} \lambda_j \hat{k}_{T+S+n-1}(x,\gamma_j;\mu)$$

where

$$c_n = \left[\frac{r_S(x) r_T(x)}{x^{S+T}} \right]_{x=\beta_n}.$$

Evaluate in zeros of $r_S(x)$ and $r_T(c)$ to eliminate the Λ_k's and λ_j's.

Measures on I

Rational modifications of a measure on I

For computational purposes $\longrightarrow$ rational modifications of degree 1:

R_1 $\left|\frac{x-\gamma}{1-x/\alpha}\right|$, $\gamma \in \mathbb{R} \setminus (-1,1)$ and $\alpha \in (\mathbb{R} \cup \{\infty\}) \setminus I$

$\left|\frac{\gamma}{1-x/\alpha}\right|$, $\gamma \in \mathbb{C}_0$ and $\alpha \in (\mathbb{R} \cup \{\infty\}) \setminus I$

R_2 $\left|\frac{x-\gamma}{1-x/\alpha}\right|^2$, $\gamma \in \mathbb{C}$ and $\alpha \in (\mathbb{R} \cup \{\infty\}) \setminus I$

$\left|\frac{\gamma}{1-x/\alpha}\right|^2$, $\gamma \in \mathbb{C}_0$ and $\alpha \in (\mathbb{R} \cup \{\infty\}) \setminus I$

R_3 R_2 with $\alpha \in (\mathbb{C} \setminus \mathbb{R})$

Measures on I

R_2: $S = 0$ and $T = 1$

$$c_n^{-1} r_1(x) \phi_n(x; \tilde{\mu}) = \phi_{n+1}(x; \mu) - \frac{\phi_{n+1}(\gamma; \mu)}{\hat{k}_n(\gamma, \gamma; \mu)} \hat{k}_n(x, \gamma; \mu).$$

and use Christoffel-Darboux relation

$$\hat{k}_n(x, y; \mu) = \frac{\overline{\left(\frac{\phi_{n+1}(y;\mu)}{Z_{\beta_n}(y)}\right)} \left(\frac{\phi_n(x;\mu)}{Z_{\xi_n}(x)}\right) - \left(\frac{\phi_{n+1}(x;\mu)}{Z_{\beta_n}(x)}\right) \overline{\left(\frac{\phi_n(y;\mu)}{Z_{\xi_n}(y)}\right)}}{E_{n+1} H(x, y)}$$

where

$$H(x, y) = \frac{\overline{y} - x}{x\overline{y}}, \quad Z_\beta(x) = \frac{x}{1 - x/\beta}, \text{ with } \beta \in (\mathbb{R} \cup \{\infty\}) \setminus I$$

$$E_{n+1} = \left[\frac{\phi_{n+1}(x; \mu)}{Z_{\beta_n}(x) \phi_n(x; \mu)}\right]_{x=\xi_n}, \ \xi_1 = \alpha, \ \xi_n = \beta_{n-1}, \ n > 1$$

Measures on I

R_1: $S = 1$ and $T = 0$

$$c_n^{-1} r_1(x)\phi_n(x;\tilde{\mu}) = \phi_{n+1}(x;\mu) - \frac{\phi_{n+1}(\gamma;\mu)}{\phi_n(\gamma;\mu)}\phi_n(x;\mu).$$

R_3: $S = 0$ and $T = 1$

$$c_n^{-1}|r_1(x)|^2\phi_n(x;\tilde{\mu}) = \phi_{n+2}(x;\mu) + a_n\phi_{n+1}(x;\mu) + b_n\phi_n(x;\mu)$$

and evaluate in γ and $\overline{\gamma}$ to eliminate a_n and b_n.
? Christoffel-Darboux relation for arbitrary complex poles ?

References

- K. Deckers, J. Van Deun, and A. Bultheel, *"An extended relation between orthogonal rational functions on the unit circle and the interval* $[-1, 1]$*"*, Journal of Mathematical Analysis and Applications, 334(2):1260–1275, 2007.
- E. Godoy and F. Marcellán, *"An analog of the Christoffel formula for polynomial modification of a measure on the unit circle"*, Bollettino della Unione Mathematica Italiana, 7(5–A),1–12, 1991.
- K. Deckers and A. Bultheel, *"Orthogonal rational functions and rational modifications of a measure on the unit circle"*, Journal of Approximation Theory, 2008. To appear.

References

- E. Godoy and F. Marcellán, *"Orthogonal polynomials and rational modifications of measures"*, Canadian Journal of Mathematics, 45(5):930–943, 1993.
- A. Bultheel, R. Cruz-Barroso, K. Deckers, and P. González-Vera, *"Rational Szegő quadratures associated with Chebyshev weight functions"*, Mathematics of Computation, 2008. To appear.