Normal Offsets for digital image decomposition

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Abstract

This paper describes the use of normal offsets with regard to image compression. Given a gray scale image, the decomposition aims at a sparse representation i.e. lots of (nearly) zero coefficients. The class of images we want to represent have smoothly colored regions separated by edges -abrupt changes - along smooth lines. An image can also be seen as a surface lying in the three dimensional space. This surface can be approximated by a mesh. Usually a mesh is constructed given input data consisting of three coefficients, of which two of them represent the location in the domain and only one contains information about the geometry of the mesh. Normal offsets try to resolve this using only one coefficient is needed in order to compute another point, depending on already known mesh points. In this manner normal offsets are suitable for multiresolution purposes, i.e. getting a more and more accurate approximation when moving to higher resolution levels.

1 Introduction

Tensor product wavelet methods already established themselves in the field of image processing. They exhibit nice theoretical properties and have proven their practical usefullness by replacing the cosine transform in image compression standards. Comming in many different flavours, one can choose among a whole set of wavelets and pick the right one for the job. One can show an equivalence between a sequence norm applied to the wavelet coefficients and Besov norms [7]. If the images to be approximated belong to some smoothness spaces like Besov or Sobolev spaces, it can be proven that nonlinear wavelet approximation have an optimal approximation rate [2]. However only a small part of the images belonging to certain smoothness spaces represent lifelike images. From the connection between wavelet coefficients and Besov norms it can be seen that the order in which the coefficients in each resolution level appear does not matter. Swapping the position of the wavelet coefficients -and thus destroying artifacts like edges- does not change the smoothness characteritics of the image after reconstruction. However this image is unlikely to resemble a lifelike image. When dealing with images with smooth contours, wavelets do not seem to take advantage of this redundance in the domain. Since tensor product wavelets are only capable of detecting point singularities they perform well on texture-like regions , whereas line singularities are not well approximated. Where in one dimension only point singularities are to be dealt with, univariate wavelets where extremely suited. Therefore tensor product wavelets are a poor generalisation of univariate wavelets for one dimensional wavelet applications. This can be seen from the fact that the number of wavelet functions that are intersected by a line singularity rize exponentially. This is why we propose a nonlinear method of which its nonlinear approximation behaviour outperforms that of wavelet methods on images consisting of smoothly colored regions separated by smooth contours.

2 Beyond wavelets

There have been proposed several decompositions in order to compress two-dimensional piecewise smooth data. For instance, the use of curvelets [1] has led to a supprizing approximation rate of $\mathcal{O}(M^{-2}(\log_2 M)^3)$, when the discontinuities are lying along a C^2 curve. We can also approximate by means of nonlinear transforms combined with a nonlinear approximation criterium. In this view bandelets have been proposed in [6]. Bandelets make use of a quadtree representation of the domain. Over each subdomain flow vectors are defined, depending on the function values, on which warped wavelet bases are constructed.

Also normal offsets belong to the class of nonlinear transforms. It can be proven [4] that normal offset based methods have an optimal approximation rate for piecewise smooth data under some mild assumptions.

3 Normal Offsets

Normal offsets have already been studied in case of one dimensional non discretized data (see [5]). The concept of normal offsets can be seen in the light of the lifting scheme [8] used to construct wavelets. Lifting comprises three steps. In the first step, incoming data is being partitioned into two disjunct sets. Even when the partioning rule is not a trivial *even* versus *odd* splitting, one still refers to even and odd sets. So, these terms have to be seen in a more general context. In the second step, each of the *odd* set points is being predicted out of surrouding *even* set points. Only the differce between the *odd* value and the predicted value is being kept. The last step consists of an update step, adding a value to the *even* values in order to maintain several order moments. It are those updated *even* values that represent the signal on a coarser resolution level. Now we return to normal offsets. For the sake of clearity, we give a breef explanation for the one dimensional case. Consider the following. We use a simple linear interpolation for the prediction. But instead of measuring the vertical distance between the prediction and the incoming function values, we now take the normal distance (see Figure 1).

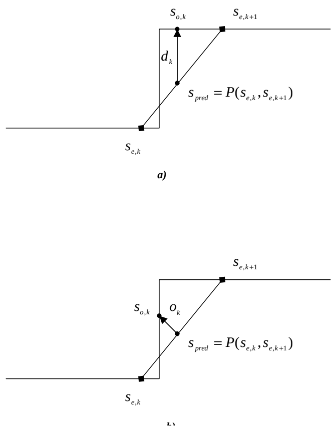

Figure 1: ***Prediction step****: a) The detail coefficient d_k represents the vertical difference between the predicted value and odd value, while in b) o_k represents difference lying in the normal direction.*

We see the resemblance and also the main differences between normal offsets and the lifting scheme explained earlier. The normal offset can be seen as the detailcoefficient coming from the prediction step. Being more specific, the offset represents the difference between the predicted value - which is the linear interpolation- and the real function value lying in the normal direction. Notice however, that the update step does not have a counterpart.

Earlier work already examined the properties of normal offsets on 1D and 2D piecewise smooth functions [4]. Also for representation of $3D$ surfaces normal offsets prove there usefullness as exposed in [3], where a given $3D$ mesh is converted from a 3 parameter representation into a purely scalar representation.

Contrary to the wavelet method, this method is a refinement method decomposing an image bottom up, i.e. starting from a coarse grid producing finer grids. The distance between the prediction and the piercing point with the function, called the normal offset, is kept for later reconstruction. This way, a piercing point only needs one coefficient to be completely defined.

2D normal offsets In the prospect of image compression we mainly focus on the discretized 2D setting. For smooth images belonging to some Besov or Sobolev spaces, there already exist wavelet transforms possessing optimal nonlinear approximation rates. So, no further improvement can be expected from a nonlinear transform. However, for 2D piecewise $\mathcal{C}^2$ smooth functions with discontinuities lying along $\mathcal{C}^2$ smooth contours, it has been proven that wavelets have a suboptimal approximation rate of $\mathcal{O}(M^{-1})$, compared to a theoretical optimal rate of $\mathcal{O}(M^{-2})$ [2] *. This is because tensor product wavelets have a blocky domain partitioning, which is not suited to approximate the smooth curves in the domain (see Figure 2).

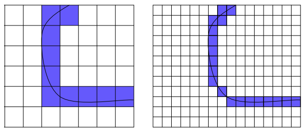

Figure 2: *As the grid dyadically refines, the number of wavelet bases that come into contact with the curve of discontinuity rizes exponential.*

There are several considerations to be made when expanding the concept of normal offsets to higher dimensions. We want to approximate the target images by piecewise linear trianglar meshes which have a conform triangulation in the domain. In order to obtain a local method, i.e. a coefficient only has its influence on a local area of the domain, we choose for an edge refining method as depicted in figure 3. —Note that we can no longer speak of base functions connected to coefficients.— This allows us to locally subdivide each triangular piece into four subtriangles, still leaving some degree of freedom considering there topology as depicted in figure 3. This way we obtain a multiresolution method with nested triangulations. The decomposition takes on a quadtree like structure; each triangle having four children. For compression this allows cutting off a branch when a certain treshold has been reached which only influences a local part of the image without interfering with the rest of the reconstruction.

The main advantage of normal offsets lies in the fact that the piercing point are likely to be attracted towards discontinuities. By carefully chosing the right subdivision of a triangle, we can approximate the contours by linear piecewise curves in the domain. This adaptivity in domain partitioning allows a better approximation rate with respect to the target images.

For the discrete wavelet transform (DWT) elegant and fast implementations are known and used in digital image processing. If we want to apply the normal offset method on a discrete setting, mostly more then one coefficient is needed to represent the piercing point. An extra vertical offset is needed in addition to the piercing point to represent the original function value. So, discretisation already cuts away a major advantage of the method. Fortunatly these vertical offsets are likely to be small. We can also reduce the number of offsets needed per pixel to $\frac{3}{2}$ by letting two pixels define a piercing point, demanding for only one extra vertical offset (see figure 4).

Looking in the normal direction has some extra cost when implementing the algorithm. For each piercing point to be found, one has to search among pixel values which are located on the edges of the domain triangulation.

The paper considers theorethical aspects as well as implementation issues. Several tests are being performed on artificial images and real images.

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*In the remainder of the text the approximation rates are given with respect to square of the L_2 norm, $\|\cdot\|_{L_2}^2$

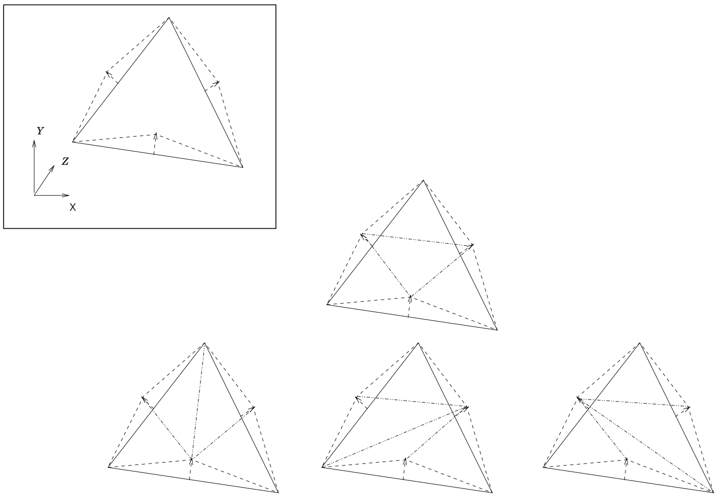

Figure 3: ***Once the piercing points have been found we have left several posibilities of subdividing the coarse triangle. The criterium on which we decide to take on particular refinement should not depend on the original image, because if it does we have to store additional information because we do not have the original image at hand during the reconstruction.***

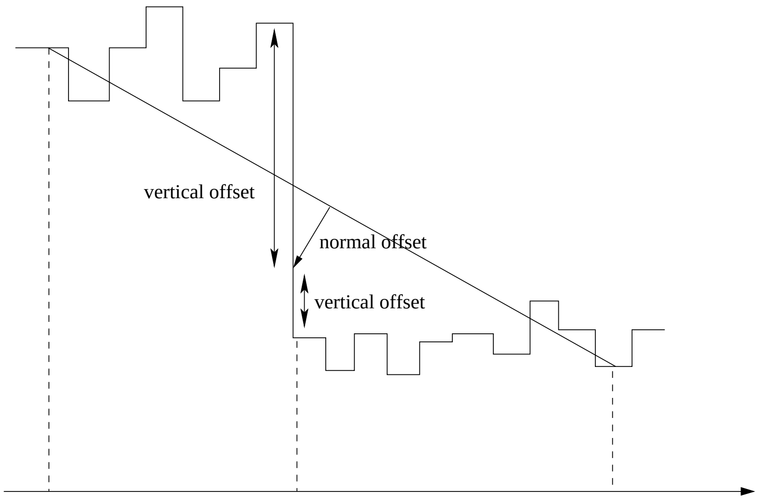

Figure 4: ***In order to reduce the information needed per pixel, we represent two pixels by three offsets.***

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