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A note on Thiele n-fractions

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In this paper we construct an n-fraction which is a generalization of a Thiele continued fraction. We prove that, under certain conditions, the mth approximant of this n-fraction solves the vector case of the rational interpolation problem.

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1. Introduction

Thiele continued fractions are used to solve the following rational interpolation problem: let $\{z_m\}$ be a sequence of distinct complex numbers, and let $f(z)$ be a given function. The problem is to find a rational function $f_m(z)$ which interpolates $f(z)$ at the given points z_i (which are assumed to be distinct):

$$f_m(z_i) = f(z_i), \quad i = 0, \ldots, m. \tag{1}$$

If we define the sequence $\{v_m(z)\}$ recursively by

$$v_0(z) = f(z), \quad v_{k+1}(z) = \frac{z - z_k}{v_k(z) - v_k(z_k)}, \quad k = 0, 1, \ldots,$$

then the Thiele continued fraction is given by

$$v_0(z_0) + \frac{z - z_0|}{|v_1(z_1)} + \frac{z - z_1|}{|v_2(z_2)} + \frac{z - z_2|}{|v_3(z_3)} + \cdots$$

(if $v_k(z_k) \neq 0, \infty$). If we denote by $f_m(z)$ the mth approximant of this continued fraction:

$$f_m(z) = v_0(z_0) + \frac{z - z_0|}{|v_1(z_1)} + \ldots + \frac{z - z_{m-1}|}{|\ v_m(z_m)},$$

then it follows immediately from the definition of the $v_m(z)$ that the mth approximant satisfies (1) if it exists. Another and equivalent way to introduce Thiele fractions is by using the so-called reciprocal differences defined by

$$\rho_{-1} = 0, \rho_0(z_0) = f(z_0),$$

$$\rho_m(z_0 z_1 \ldots z_m) = \frac{z_0 - z_m}{\rho_{m-1}(z_0 z_1 \ldots z_{m-1}) - \rho_{m-1}(z_1 z_2 \ldots z_m)} + \rho_{m-2}(z_1 z_2 \ldots z_{m-1}),$$
$$m = 1, 2, \ldots$$

(if they exist). These are related to the $v_m(z)$ in the following manner:

$$v_m(z) = \rho_m(z_0 z_1 \ldots z_{m-1} z) - \rho_{m-2}(z_0 z_1 \ldots z_{m-2}).$$

One of the advantages of using reciprocal differences is that they are symmetric functions of their arguments (see [9–11]).

The rational interpolation problem has received considerable attention in the literature, see for instance [2,5,8,13], and, more recently, [7].

In this paper we define a generalization of Thiele continued fractions in the form of an n-fraction or generalized continued fraction (see de Bruin [1]), and we prove that it solves the following generalized rational interpolation problem: let $\{z_m\}$ be a sequence of distinct complex numbers, and let $f^{(1)}(z), f^{(2)}(z), \ldots, f^{(n)}(z)$ be given functions. Find n rational functions $f_m^{(1)}(z), f_m^{(2)}(z), \ldots, f_m^{(n)}(z)$ with a common denominator which interpolate $f^{(j)}(z)$ $(1 \leqslant j \leqslant n)$ at the given points z_i:

$$f_m^{(j)}(z_i) = f^{(j)}(z_i), \quad i = 0, \ldots, m,$$

$(1 \leqslant j \leqslant n)$.

This problem has been considered before, most recently by Graves-Morris [3,4,6], who uses another generalization of Thiele fractions based on the Samelson inverse for vectors, and by Van Barel and Bultheel [12] using an altogether different approach, which, however, is basically also of continued fraction nature. The relationship between these methods will be discussed in a forthcoming paper.

In the second section of the paper we define a generalization of the reciprocal differences, which leads in a natural way to the construction of a generalized continued fraction. We show that the mth approximant of this generalized continued fraction interpolates the given data if certain conditions are satisfied. In the next section we look at the degrees of the numerator and denominator polynomials of the interpolating n-tuple, and we prove that the interpolants $f_m^{(j)}(z)$ do not depend on the ordering of the interpolation points. In the fourth and final section we give explicit expressions for the numerator and denominator polynomials for the special case $n = 2$ and we see what happens if all the interpolation points coincide (confluency).

2. Generalized Thiele continued fractions or Thiele n-fractions

Let $n \geqslant 1$ be a fixed integer. Let us remind you (see [1]) that the n-fraction associated with the following set of data:

$$\begin{pmatrix} & a_1^{(1)} & a_2^{(1)} & \dots & a_k^{(1)} & \dots \\ b_0^{(1)} & a_1^{(2)} & a_2^{(2)} & \dots & a_k^{(2)} & \dots \\ \vdots & \vdots & \vdots & & \vdots & \\ b_0^{(n-1)} & a_1^{(n)} & a_2^{(n)} & \dots & a_k^{(n)} & \dots \\ b_0^{(n)} & b_1 & b_2 & \dots & b_k & \dots \end{pmatrix}$$

is defined by its sequence of n-tuples of approximants $\{A_k^{(j)}/B_k, j=1,\dots,n\}$, $k \geqslant 0$. The sequences $A_k^{(1)}, \dots, A_k^{(n)}, B_k$ all satisfy the recurrence relation

$$X_k = b_k X_{k-1} + a_k^{(n)} X_{k-2} + \dots + a_k^{(2)} X_{k-n} + a_k^{(1)} X_{k-n-1}, \quad (k \geqslant 1)$$

with initial values:

$$B_{-i} = 0 \qquad \text{for } i = 1,\dots,n; \quad B_0 = 1;$$

$$A_{-i}^{(j)} = \begin{cases} 0, & j+i \neq n+1 \\ 1, & j+i = n+1 \end{cases} \qquad \text{for } i = 1,\dots,n; \quad A_0^{(j)} = b_0^{(j)} \quad (1 \leqslant j \leqslant n).$$

Now let $z_0, z_1, \dots, z_k, \dots$ be a sequence of different complex numbers, and let us define quantities $v_k^{(j)}(z)$, $(1 \leqslant j \leqslant n)$ formally by

$$\begin{aligned} v_0^{(j)}(z) &= f^{(j)}(z), & j &= 1,\dots,n, \\ v_{k+1}^{(j)}(z) &= \frac{v_k^{(j+1)}(z) - v_k^{(j+1)}(z_k)}{v_k^{(1)}(z) - v_k^{(1)}(z_k)}, & j &= 1,\dots,n-1, \\ v_{k+1}^{(n)}(z) &= \frac{z - z_k}{v_k^{(1)}(z) - v_k^{(1)}(z_k)}, \end{aligned} \tag{2}$$

for $k \geqslant 0$ and where the $f^{(j)}(z)$ are given functions (as we shall see later, only the function values of $f^{(j)}(z)$ at the points z_i, $i \geqslant 0$, have to be known).

We consider the n-fraction denoted by

$$\begin{pmatrix} & (z-z_0) & (z-z_1)(z-z_0) & \dots & \prod_{j=1}^{n}(z-z_{k-j}) & \dots \\ v_0^{(1)}(z_0) & v_1^{(1)}(z_1)(z-z_0) & v_2^{(1)}(z_2)(z-z_1)(z-z_0) & \dots & v_k^{(1)}(z_k)\prod_{j=1}^{n-1}(z-z_{k-j}) & \dots \\ \vdots & \vdots & \vdots & \vdots & & \\ v_0^{(n-1)}(z_0) & v_1^{(n-1)}(z_1)(z-z_0) & v_2^{(n-1)}(z_2)(z-z_1) & \dots & v_k^{(n-1)}(z_k)(z-z_{k-1}) & \dots \\ v_0^{(n)}(z_0) & v_1^{(n)}(z_1) & v_2^{(n)}(z_2) & \dots & v_k^{(n)}(z_k) & \dots \end{pmatrix}. \tag{3}$$

Here we assume that a factor $z - z_i$ $(i = -1, -2, \dots)$ in one of the coefficients is taken to be equal to 1.

The kth approximant $\{f_k^{(j)}(z)\}$, $(k \geqslant 0)$, of the n-fraction (2) is then given by

$$f_k^{(j)}(z) = \frac{A_k^{(j)}(z)}{B_k(z)}, \quad j = 1, \ldots, n, \tag{4}$$

where $A_k^{(j)}(z)$ $(1 \leqslant j \leqslant n)$ and $B_k(z)$ are polynomials in z that satisfy the $(n+1)$st order recurrence relation:

$$\begin{aligned} X_k =& v_k^{(n)}(z_k)X_{k-1} + v_k^{(n-1)}(z_k)(z - z_{k-1})X_{k-2} + \ldots \\ &+ v_k^{(1)}(z_k)\prod_{j=1}^{n-1}(z - z_{k-j})X_{k-n} + \prod_{j=1}^{n}(z - z_{k-j})X_{k-n-1}, \quad (k = 1, 2, \ldots) \end{aligned} \tag{5}$$

with initial values:

$$\begin{aligned} &B_{-i}(z) = 0 && \text{for } i = 1, \ldots, n; \quad B_0(z) = 1; \\ &A_{-i}^{(j)}(z) = \begin{cases} 0, & j + i \neq n + 1 \\ 1, & j + i = n + 1 \end{cases} && \text{for } i = 1, \ldots, n; \quad A_0^{(j)}(z) = v_0^{(j)}(z_0) \quad (1 \leqslant j \leqslant n). \end{aligned}$$

We note that the kth approximant of the n-fraction can also be calculated using the nonlinear recurrences that generate (4): if we calculate

$$\begin{aligned} \xi_{k,k}^{(1)}(z) &= v_k^{(1)}(z_k)\prod_{j=1}^{n-1}(z - z_{k-j}), \\ \xi_{k,k}^{(2)}(z) &= v_k^{(2)}(z_k)\prod_{j=1}^{n-2}(z - z_{k-j}), \\ &\vdots \\ \xi_{k,k}^{(n-1)}(z) &= v_k^{(n-1)}(z_k)(z - z_{k-1}), \\ \xi_{k,k}^{(n)}(z) &= v_k^{(n)}(z_k), \\ \xi_{m,k}^{(1)}(z) &\doteq v_m^{(1)}(z_m)\prod_{j=1}^{n-1}(z - z_{m-j}) + \frac{\prod_{j=0}^{n-1}(z - z_{m-j})}{\xi_{m+1,k}^{(n)}(z)}, \\ \xi_{m,k}^{(2)}(z) &= v_m^{(2)}(z_m)\prod_{j=1}^{n-2}(z - z_{m-j}) + \frac{\xi_{m+1,k}^{(1)}(z)}{\xi_{m+1,k}^{(n)}(z)}, \\ &\vdots \\ \xi_{m,k}^{(n-1)}(z) &= v_m^{(n-1)}(z_m)(z - z_{m-1}) + \frac{\xi_{m+1,k}^{(n-2)}(z)}{\xi_{m+1,k}^{(n)}(z)}, \\ \xi_{m,k}^{(n)}(z) &= v_m^{(n)}(z_m) + \frac{\xi_{m+1,k}^{(n-1)}(z)}{\xi_{m+1,k}^{(n)}(z)}, \end{aligned} \tag{6}$$

for $m = k-1, \ldots, 1, 0$, then $f_k^{(j)}(z) = \xi_{0,k}^{(j)}(z)$ $(1 \leqslant j \leqslant n)$.

From now on we assume that the kth approximant of (3) as defined in (4) or (6) exists. The kth approximant n-tuple then interpolates the n-tuple $\{f^{(j)}(z), j = 1, \ldots, n\}$ at the points $z_0, z_1, \ldots, z_k$. We have the following theorem:

THEOREM 1

If $\xi_{t+1,k}^{(n)}(z_t) \neq 0$, then $f_k^{(j)}(z_t) = f^{(j)}(z_t)$, for $t = 0, 1, \ldots, k$ $(1 \leqslant j \leqslant n)$.

Proof

Let us define

$$r_{m,k}^{(j)}(z) = \frac{\xi_{m,k}^{(j)}(z)}{\prod_{i=1}^{n-j}(z - z_{m-i})}, \quad m = 0, 1, \ldots, k, (1 \leqslant j \leqslant n).$$

Then we can rewrite eqs. (6):

$$r_{m,k}^{(1)}(z) = v_m^{(1)}(z_m) + (z - z_m)\frac{1}{r_{m+1,k}^{(n)}(z)},$$

$$r_{m,k}^{(2)}(z) = v_m^{(2)}(z_m) + (z - z_m)\frac{r_{m+1,k}^{(1)}(z)}{r_{m+1,k}^{(n)}(z)},$$

$$\vdots$$

$$r_{m,k}^{(n)}(z) = v_m^{(n)}(z_m) + (z - z_m)\frac{r_{m+1,k}^{(n-1)}(z)}{r_{m+1,k}^{(n)}(z)}, \tag{7}$$

and furthermore we have $\xi_{0,k}^{(j)}(z) = r_{0,k}^{(j)}(z)$.

Hence, it is easy to see that by taking $m = t = 0$ and $z = z_0$ in (7), we get

$$f_k^{(j)}(z_t) = r_{0,k}^{(j)}(z_t) = v_0^{(j)}(z_0) = f^{(j)}(z_0), \quad (1 \leqslant j \leqslant n),$$

since $\xi_{1,k}^{(n)}(z_0) \neq 0$. This proves the theorem for $t = 0$. If $t > 0$, we use (7) with $m = t-1$ and $z = z_t$:

$$r_{t-1,k}^{(1)}(z_t) = v_{t-1}^{(1)}(z_{t-1}) + (z_t - z_{t-1})\frac{1}{v_t^{(n)}(z_t)},$$

$$r_{t-1,k}^{(2)}(z_t) = v_{t-1}^{(2)}(z_{t-1}) + (z_t - z_{t-1})\frac{v_t^{(1)}(z_t)}{v_t^{(n)}(z_t)},$$

$$\vdots$$

$$r_{t-1,k}^{(n)}(z_t) = v_{t-1}^{(n)}(z_{t-1}) + (z_t - z_{t-1})\frac{v_t^{(n-1)}(z_t)}{v_t^{(n)}(z_t)}. \tag{8}$$

From (2) we have

$$v_{t-1}^{(1)}(z_t) - v_{t-1}^{(1)}(z_{t-1}) = \frac{z_t - z_{t-1}}{v_t^{(n)}(z_t)},$$
$$v_{t-1}^{(j+1)}(z_t) - v_{t-1}^{(j+1)}(z_{t-1}) = v_t^{(j)}(z_t)(v_{t-1}^{(1)}(z_t) - v_{t-1}^{(1)}(z_{t-1})), \quad (1 \leqslant j \leqslant n-1). \quad (9)$$

Note that $v_t^{(n)}(z_t) \neq 0$ as a consequence of its definition (2). If we use (9) in (8) we get immediately that

$$r_{t-1,k}^{(j)}(z_t) = v_{t-1}^{(j)}(z_t), \quad (1 \leqslant j \leqslant n).$$

Using induction we then find

$$r_{0,k}^{(j)}(z_t) = v_0^{(j)}(z_t) = f^{(j)}(z_t), \quad (1 \leqslant j \leqslant n).$$

This completes the proof of theorem 1. □

EXAMPLE 1

For the situation described in table 1 we find, using our method, the following third approximant triple:

$$\left(f_3^{(1)}(z) = \frac{(z+1)(15z+6)}{13z^2 - z + 18}, \quad f_3^{(2)}(z) = \frac{3(z+1)(z-2)}{13z^2 - z + 18}, \right.$$
$$\left. f_3^{(3)}(z) = \frac{(z+1)(z-1)(z+6)}{13z^2 - z + 18}\right).$$

This example was taken from Graves-Morris [3]. In this case Graves-Morris' method gives the following interpolating triple:

$$\left(\frac{0.5(z+1)(3z^2+2z+2)}{5z^2 - 3z + 3}, \quad \frac{0.5(z+1)(z-2)}{5z^2 - 3z + 3}, \quad \frac{0.5(z+1)(z-1)(z+2)}{5z^2 - 3z + 3}\right).$$

3. Ordering of the interpolation points

It is easy to see from (2) and (7) that the kth approximant n-tuple of the n-fraction does not only depend upon the function values $f^{(j)}(z_i)$, $(1 \leqslant j \leqslant n, 0 \leqslant i \leqslant k)$, but also on the interpolation points $z_0, z_1, \ldots, z_k$. What we want to prove next is that

Table 1

i	0	1	2	3
z_i	-1	0	1	2
$(f^{(j)}(z_i))_{j=1}^3$	$(0, 0, 0)$	$(1/3, -1/3, -1/3)$	$(7/5, -1/5, 0)$	$(27/17, 0, 6/17)$
$(v_1^{(j)}(z_i))_{j=1}^3$		$(-1, -1, 3)$	$(-1/7, 0, 10/7)$	$(0, 2/9, 17/9)$
$(v_2^{(j)}(z_i))_{j=1}^3$			$(7/6, -11/6, 7/6)$	$(11/9, -10/9, 2)$
$(v_3^{(j)}(z_i))_{j=1}^3$				$(13, 15, 18)$

this approximant n-tuple is independent of the ordering of the interpolation points. To do this we need to know more about the degrees of the numerator and denominator polynomials of the approximant n-tuples. This information is contained in the following theorem:

THEOREM 2

The solutions $A_k^{(j)}(z)$ $(1 \leqslant j \leqslant n)$ and $B_k(z)$ are polynomials in z with leading terms given by the following expressions:

$$B_{(n+1)\nu}(z) = z^{n\nu} + \ldots ,$$
$$B_{(n+1)\nu+i}(z) = \mu^{(i)}_{(n+1)\nu+i} z^{n\nu+i-1} + \ldots \quad (i = 1, \ldots, n) ,$$

and for $j = 1, \ldots, n$:

$$A^{(j)}_{-n-1+j+i}(z) = \mu^{(i)}_{-n-1+j+i} z^{-n-1+j+i} + \ldots \quad (i = n-j+1, \ldots, n) ,$$
$$A^{(j)}_{(n+1)\nu+j}(z) = z^{n\nu+j} + \ldots ,$$
$$A^{(j)}_{(n+1)\nu+j+i}(z) = \mu^{(i)}_{(n+1)\nu+j+i} z^{n\nu+j+i-1} + \ldots \quad (i = 1, \ldots, n) ,$$

for $\nu = 0, 1, 2, \ldots$, where the coefficients of the leading terms of $A_k^{(j)}(z)$ and $B_k(z)$ are defined by the following recurrence relations:

$$\mu_i^{(1)} = \mu_{i-n-1}^{(1)} + v_i^{(1)}(z_i)\mu_{i-n}^{(2)} + \ldots + v_i^{(n-2)}(z_i)\mu_{i-3}^{(n-1)} + v_i^{(n-1)}(z_i)\mu_{i-2}^{(n)} + v_i^{(n)}(z_i) ,$$
$$\mu_i^{(2)} = \mu_{i-n-1}^{(2)} + v_i^{(1)}(z_i)\mu_{i-n}^{(3)} + \ldots + v_i^{(n-2)}(z_i)\mu_{i-3}^{(n)} + v_i^{(n-1)}(z_i) ,$$
$$\mu_i^{(3)} = \mu_{i-n-1}^{(3)} + v_i^{(1)}(z_i)\mu_{i-n}^{(4)} + \ldots + v_i^{(n-2)}(z_i) ,$$
$$\vdots$$
$$\mu_i^{(n)} = \mu_{i-n-1}^{(n)} + v_i^{(1)}(z_i) ,$$

for $i = 0, 1, \ldots$. Here $\mu_i^{(j)}$ is assumed to be zero if $i \leqslant 0$ and for all $j = 1, \ldots, n$.

Proof

We prove the expressions for $B_i(z)$. It is easy to see from (5) and the initial values that

$$B_i(z) = v_i^{(n-i+1)}(z_i)z^{i-1} + \text{(lower order terms)} , \quad (1 \leqslant i \leqslant n) ,$$
$$B_{n+1}(z) = z^n + \text{(l.o.)} .$$

For the rest we use induction.

Let us assume the theorem holds for $B_j(z)$ for $j = 0, 1, \ldots, (n+1)(\nu+1) + i - 1$. Then we have from (5) for $i = 1, \ldots, n$:

$$
\begin{aligned}
B_{(n+1)(\nu+1)+i}(z) =& v^{(n)}_{(n+1)(\nu+1)+i} B_{(n+1)(\nu+1)+i-1}(z) + \dots \\
&+ v^{(n-i+2)}_{(n+1)(\nu+1)+i} z^{i-2} B_{(n+1)(\nu+1)+1}(z) + v^{(n-i+1)}_{(n+1)(\nu+1)+i} z^{i-1} B_{(n+1)(\nu+1)}(z) \\
&+ v^{(n-i)}_{(n+1)(\nu+1)+i} z^{i} B_{(n+1)\nu+n}(z) + \dots \\
&+ v^{(1)}_{(n+1)(\nu+1)+i} z^{n-1} B_{(n+1)\nu+i+1}(z) + z^{n} B_{(n+1)\nu+i}(z) + (\text{l.o.}) \\
=& v^{(n)}_{(n+1)(\nu+1)+i} \mu^{(i-1)}_{(n+1)(\nu+1)+i-1} z^{n(\nu+1)+i-2} + \dots \\
&+ v^{(n-i+2)}_{(n+1)(\nu+1)+i} z^{i-2} \mu^{(1)}_{(n+1)(\nu+1)+1} z^{n(\nu+1)} + v^{(n-i+1)}_{(n+1)(\nu+1)+i} z^{i-1} z^{n(\nu+1)} \\
&+ v^{(n-i)}_{(n+1)(\nu+1)+i} z^{i} \mu^{(n)}_{(n+1)\nu+n} z^{n\nu+n-1} + \dots \\
&+ v^{(1)}_{(n+1)(\nu+1)+i} z^{n-1} \mu^{(i+1)}_{(n+1)\nu+i+1} z^{n\nu+i} + z^{n} \mu^{(i)}_{(n+1)\nu+i} z^{n\nu+i-1} + (\text{l.o.})
\end{aligned}
$$

and hence

$$
\begin{aligned}
B_{(n+1)(\nu+1)+i}(z) =& (v^{(n-i+1)}_{(n+1)(\nu+1)+i} + v^{(n-i)}_{(n+1)(\nu+1)+i} \mu^{(n)}_{(n+1)\nu+n} + \dots \\
&+ v^{(1)}_{(n+1)(\nu+1)+i} \mu^{(i+1)}_{(n+1)\nu+i+1} + \mu^{(i)}_{(n+1)\nu+i}) z^{n(\nu+1)+i-1} + (\text{l.o.}) \\
=& \mu^{(i)}_{(n+1)(\nu+1)+i} z^{n(\nu+1)+i-1} + (\text{l.o.})\,.
\end{aligned}
$$

(Note that by $v_i^{(j)}$ we mean $v_i^{(j)}(z_i)$.) The proof for $i = n+1$ is similar. □

We now introduce some notations: let $R_i^{(m)}$, $(m, i \geqslant 0)$, be the (Vandermonde) matrix:

$$
R_i^{(m)} = \begin{pmatrix} z_0^{m-1} & \dots & z_0 & 1 \\ \vdots & & \vdots & \vdots \\ z_i^{m-1} & \dots & z_i & 1 \end{pmatrix}
$$

and let

$$
R_i^{(m,j)} = \begin{pmatrix} z_0^{m-1} f^{(j)}(z_0) & \dots & z_0 f^{(j)}(z_0) & f^{(j)}(z_0) \\ \vdots & & \vdots & \vdots \\ z_i^{m-1} f^{(j)}(z_i) & \dots & z_i f^{(j)}(z_i) & f^{(j)}(z_i) \end{pmatrix}
$$

with $1 \leqslant j \leqslant n$. Furthermore let

$$
D_\nu^{(i)} = \begin{pmatrix}
R^{(n\nu+i)}_{(n+1)\nu+i} & \dots & 0 & 0 & \dots & 0 & R^{(n\nu+i,1)}_{(n+1)\nu+i} \\
\vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\
0 & \dots & R^{(n\nu+i)}_{(n+1)\nu+i} & 0 & \dots & 0 & R^{(n\nu+i,i)}_{(n+1)\nu+i} \\
0 & \dots & 0 & R^{(n\nu+i+1)}_{(n+1)\nu+i} & \dots & 0 & R^{(n\nu+i,i+1)}_{(n+1)\nu+i} \\
\vdots & & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & 0 & \dots & R^{(n\nu+i+1)}_{(n+1)\nu+i} & R^{(n\nu+i,n)}_{(n+1)\nu+i}
\end{pmatrix}.
$$

THEOREM 3

If the conditions of theorem 1 are satisfied, and if the matrix $D_\nu^{(i)}$, with ν and i such that $k=(n+1)\nu+i\,(0\leqslant i\leqslant n)$, is non-singular, then $f_k^{(j)}(z)$, $(1\leqslant j\leqslant n)$, is independent of the ordering of the interpolation points.

Proof

Since the conditions of theorem 1 are satisfied, we have that

$$f_k^{(j)}(z_t)=f^{(j)}(z_t)$$

for $t=0,1,\ldots,k$, $(1\leqslant j\leqslant n)$, and hence

$$A_k^{(j)}(z_t)-f^{(j)}(z_t)B_k(z_t)=0\,,\quad t=0,1,\ldots,k,\, j=1,\ldots,n\,. \tag{10}$$

If we use theorem 2 and take the coefficients of the polynomials $A_k^{(j)}(z)$ and $B_k(z)$ as the unknowns in this system of equations, we get a linear system consisting of $n((n+1)\nu+i+1)$ equations with the same number of unknowns. The matrix of this system is $D_\nu^{(i)}$, and the right hand side vector $C_\nu^{(i)}$ is given by

$$(C_\nu^{(0)})_{(j-1)(k+1)+m+1}=z_m^{n\nu}f^{(j)}(z_m)\quad\text{for } m=0,\ldots,k,\, j=1,\ldots,n\,,$$

for $i=0$ and for $i=1,\ldots,n$ we have:

$$\begin{aligned}(C_\nu^{(i)})_{m+i(k+1)-k}&=z_m^{n\nu+i}\ \text{for } m=0,\ldots,k\,,\\ &=0\qquad\text{otherwise}\,.\end{aligned}$$

Since this matrix $D_\nu^{(i)}$ is assumed to be non-singular, the system has a unique solution. If we solve the system using Cramer's rule, then the theorem follows immediately from the fact that interchanging two rows in a determinant can only change the sign of this determinant. □

4. The case $n=2$

In this section we give some explicit determinantal expressions for the numerator and denominator polynomials of the kth approximant n-tuple and we look what happens if all interpolation points coincide. For the sake of simplicity we restrict ourselves to the special case $n=2$. All results can easily be generalized to the case of arbitrary n. The formulas in (2) are then replaced by:

$$v_0^{(1)}(z)=f^{(1)}(z)\,,\quad v_{k+1}^{(1)}(z_0z_1\ldots z_kz)=\frac{v_k^{(2)}(z_0\ldots z_{k-1}z)-v_k^{(2)}(z_0\ldots z_{k-1}z_k)}{v_k^{(1)}(z_0\ldots z_{k-1}z)-v_k^{(1)}(z_0\ldots z_{k-1}z_k)}\,,$$

$$v_0^{(2)}(z)=f^{(2)}(z)\,,\quad v_{k+1}^{(2)}(z_0z_1\ldots z_kz)=\frac{z-z_k}{v_k^{(1)}(z_0\ldots z_{k-1}z)-v_k^{(1)}(z_0\ldots z_{k-1}z_k)}\,,$$

for $k = 0, 1, \ldots$. Here we have changed our notation to indicate the dependence of $v_k^{(1)}$ and $v_k^{(2)}$ on the interpolation points. Theorem 2 gives us for the leading terms in the denominator and numerator polynomials:

$$\begin{aligned}
B_{3i}(z) &= z^{2i} + \ldots, \\
A_{3i}^{(1)}(z) &= \mu_{3i}^{(2)} z^{2i} + \ldots, \\
A_{3i}^{(2)}(z) &= \mu_{3i}^{(1)} z^{2i} + \ldots, \\
B_{3i+1}(z) &= \mu_{3i+1}^{(1)} z^{2i} + \ldots, \\
A_{3i+1}^{(1)}(z) &= z^{2i+1} + \ldots, \\
A_{3i+1}^{(2)}(z) &= \mu_{3i+1}^{(2)} z^{2i+1} + \ldots, \\
B_{3i+2}(z) &= \mu_{3i+2}^{(2)} z^{2i+1} + \ldots, \\
A_{3i+1}^{(1)}(z) &= \mu_{3i+2}^{(1)} z^{2i+1} + \ldots, \\
A_{3i+2}^{(2)}(z) &= z^{2i+2} + \ldots.
\end{aligned} \tag{11}$$

In this case we can give explicit expressions for the denominator and numerator polynomials. For example, using the abbreviation $f_\nu^{(j)}$ for $f^{(j)}(z_\nu)$, we find from (10):

$$B_{3\nu}(z) = \begin{vmatrix}
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & z^{2\nu} & z^{2\nu-1} & \ldots & 1 \\
z_0^{2\nu} & z_0^{2\nu-1} & \ldots & 1 & 0 & 0 & \ldots & 0 & z_0^{2\nu} f_0^{(1)} & z_0^{2\nu-1} f_0^{(1)} & \ldots & f_0^{(1)} \\
\vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\
z_{3\nu}^{2\nu} & z_{3\nu}^{2\nu-1} & \ldots & 1 & 0 & 0 & \ldots & 0 & z_{3\nu}^{2\nu} f_{3\nu}^{(1)} & z_{3\nu}^{2\nu-1} f_{3\nu}^{(1)} & \ldots & f_{3\nu}^{(1)} \\
0 & 0 & \ldots & 0 & z_0^{2\nu} & z_0^{2\nu-1} & \ldots & 1 & z_0^{2\nu} f_0^{(2)} & z_0^{2\nu-1} f_0^{(2)} & \ldots & f_0^{(2)} \\
\vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\
0 & 0 & \ldots & 0 & z_{3\nu}^{2\nu} & z_{3\nu}^{2\nu-1} & \ldots & 1 & z_{3\nu}^{2\nu} f_{3\nu}^{(2)} & z_{3\nu}^{2\nu-1} f_{3\nu}^{(2)} & \ldots & f_{3\nu}^{(2)}
\end{vmatrix} / D_\nu^{(0)},$$

with

$$D_\nu^{(0)} = \begin{vmatrix}
z_0^{2\nu} & z_0^{2\nu-1} & \ldots & 1 & 0 & 0 & \ldots & 0 & z_0^{2\nu-1} f_0^{(1)} & \ldots & f_0^{(1)} \\
\vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\
z_{3\nu}^{2\nu} & z_{3\nu}^{2\nu-1} & \ldots & 1 & 0 & 0 & \ldots & 0 & z_{3\nu}^{2\nu-1} f_{3\nu}^{(1)} & \ldots & f_{3\nu}^{(1)} \\
0 & 0 & \ldots & 0 & z_0^{2\nu} & z_0^{2\nu-1} & \ldots & 1 & z_0^{2\nu-1} f_0^{(2)} & \ldots & f_0^{(2)} \\
\vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\
0 & 0 & \ldots & 0 & z_{3\nu}^{2\nu} & z_{3\nu}^{2\nu-1} & \ldots & 1 & z_{3\nu}^{2\nu-1} f_{3\nu}^{(2)} & \ldots & f_{3\nu}^{(2)}
\end{vmatrix}$$

and

$$A_{3\nu}^{(1)}(z) = -\begin{vmatrix} z^{2\nu} & z^{2\nu-1} & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ z_0^{2\nu} & z_0^{2\nu-1} & \dots & 1 & 0 & 0 & \dots & 0 & z_0^{2\nu}f_0^{(1)} & z_0^{2\nu-1}f_0^{(1)} & \dots & f_0^{(1)} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ z_{3\nu}^{2\nu} & z_{3\nu}^{2\nu-1} & \dots & 1 & 0 & 0 & \dots & 0 & z_{3\nu}^{2\nu}f_{3\nu}^{(1)} & z_{3\nu}^{2\nu-1}f_{3\nu}^{(1)} & \dots & f_{3\nu}^{(1)} \\ 0 & 0 & \dots & 0 & z_0^{2\nu} & z_0^{2\nu-1} & \dots & 1 & z_0^{2\nu}f_0^{(2)} & z_0^{2\nu-1}f_0^{(2)} & \dots & f_0^{(2)} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 & z_{3\nu}^{2\nu} & z_{3\nu}^{2\nu-1} & \dots & 1 & z_{3\nu}^{2\nu}f_{3\nu}^{(2)} & z_{3\nu}^{2\nu-1}f_{3\nu}^{(2)} & \dots & f_{3\nu}^{(2)} \end{vmatrix} / D_\nu^{(0)},$$

$$A_{3\nu}^{(2)}(z) = -\begin{vmatrix} 0 & 0 & \dots & 0 & z^{2\nu} & z^{2\nu-1} & \dots & 1 & 0 & 0 & \dots & 0 \\ z_0^{2\nu} & z_0^{2\nu-1} & \dots & 1 & 0 & 0 & \dots & 0 & z_0^{2\nu}f_0^{(1)} & z_0^{2\nu-1}f_0^{(1)} & \dots & f_0^{(1)} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ z_{3\nu}^{2\nu} & z_{3\nu}^{2\nu-1} & \dots & 1 & 0 & 0 & \dots & 0 & z_{3\nu}^{2\nu}f_{3\nu}^{(1)} & z_{3\nu}^{2\nu-1}f_{3\nu}^{(1)} & \dots & f_{3\nu}^{(1)} \\ 0 & 0 & \dots & 0 & z_0^{2\nu} & z_0^{2\nu-1} & \dots & 1 & z_0^{2\nu}f_0^{(2)} & z_0^{2\nu-1}f_0^{(2)} & \dots & f_0^{(2)} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 & z_{3\nu}^{2\nu} & z_{3\nu}^{2\nu-1} & \dots & 1 & z_{3\nu}^{2\nu}f_{3\nu}^{(2)} & z_{3\nu}^{2\nu-1}f_{3\nu}^{(2)} & \dots & f_{3\nu}^{(2)} \end{vmatrix} / D_\nu^{(0)},$$

and similar expressions hold for A's and B's with indices $3\nu + 1$, respectively $3\nu + 2$. The coefficients of the highest-order terms satisfy the following relations:

$$\mu_i^{(2)}(z_0 \dots z_i) = v_i^{(1)}(z_0 \dots z_i) + \mu_{i-3}^{(2)}(z_0 \dots z_{i-3}),$$
$$\mu_i^{(1)}(z_0 \dots z_i) = v_i^{(1)}(z_0 \dots z_i)\mu_{i-2}^{(2)}(z_0 \dots z_{i-2}) + v_i^{(2)}(z_0 \dots z_i) + \mu_{i-3}^{(1)}(z_0 \dots z_{i-3}), \quad (12)$$

for $i = 0, 1, 2, \dots$, with $\mu_{-3}^{(j)} = \mu_{-2}^{(j)} = \mu_{-1}^{(j)} = 0$ $(j = 1, 2)$. This is a consequence of theorem 2. These coefficients are functions of $z_0, \dots, z_i$ but they are symmetric in the z_m (theorem 3). They form a generalization of the reciprocal differences of the introduction.

Using the definition of $v_i^{(1)}$ and $v_i^{(2)}$ it is possible to eliminate them from eqs. (12), and we get

$$\mu_0^{(1)}(z_0) = f^{(1)}(z_0), \quad \mu_0^{(2)}(z_0) = f^{(2)}(z_0),$$

$$\mu_i^{(2)}(z_0 \dots z_i) = \frac{\mu_{i-1}^{(1)}(z_0 \dots z_{i-2}z_i) - \mu_{i-1}^{(1)}(z_0 \dots z_{i-1})}{\mu_{i-1}^{(2)}(z_0 \dots z_{i-2}z_i) - \mu_{i-1}^{(2)}(z_0 \dots z_{i-1})},$$

$$\mu_i^{(1)}(z_0 \dots z_i) = (\mu_i^{(2)}(z_0 \dots z_i) - \mu_{i-3}^{(2)}(z_0 \dots z_{i-3}))\mu_{i-2}^{(2)}(z_0 \dots z_{i-2}) + \mu_{i-3}^{(1)}(z_0 \dots z_{i-3})$$
$$+ \frac{z_i - z_{i-1}}{\mu_{i-1}^{(2)}(z_0 \dots z_{i-2}z_i) - \mu_{i-1}^{(2)}(z_0 \dots z_{i-1})}. \quad (13)$$

Here the last two equations can be replaced by

$$\mu_i^{(2)}(z_0 \dots z_i) = \frac{\mu_{i-1}^{(1)}(z_0 \dots z_{i-1}) - \mu_{i-1}^{(1)}(z_1 \dots z_i)}{\mu_{i-1}^{(2)}(z_0 \dots z_{i-1}) - \mu_{i-1}^{(2)}(z_1 \dots z_i)},$$

$$\begin{aligned}\mu_i^{(1)}(z_0\dots z_i) =&(\mu_i^{(2)}(z_0\dots z_i)-\mu_{i-3}^{(2)}(z_1\dots z_{i-2}))\mu_{i-2}^{(2)}(z_1\dots z_{i-1})\\ &+\mu_{i-3}^{(1)}(z_1\dots z_{i-2})+\frac{z_0-z_i}{\mu_{i-1}^{(2)}(z_0\dots z_{i-1})-\mu_{i-1}^{(2)}(z_1\dots z_i)}\,. \qquad (14)\end{aligned}$$

This is done by ordering the interpolation points as $z_1, z_2, \dots, z_i, z_0$. Explicit formulas for these coefficients can be obtained from the determinantal expressions given above: for instance

$$\mu_{3\nu}^{(2)}(z_0\dots z_{3\nu})$$

$$= -\begin{vmatrix} z_0^{2\nu-1} & \dots & 1 & 0 & 0 & \dots & 0 & z_0^{2\nu}f_0^{(1)} & z_0^{2\nu-1}f_0^{(1)} & \dots & f_0^{(1)} \\ \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ z_{3\nu}^{2\nu-1} & \dots & 1 & 0 & 0 & \dots & 0 & z_{3\nu}^{2\nu}f_{3\nu}^{(1)} & z_{3\nu}^{2\nu-1}f_{3\nu}^{(1)} & \dots & f_{3\nu}^{(1)} \\ 0 & \dots & 0 & z_0^{2\nu} & z_0^{2\nu-1} & \dots & 1 & z_0^{2\nu}f_0^{(2)} & z_0^{2\nu-1}f_0^{(2)} & \dots & f_0^{(2)} \\ \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & z_{3\nu}^{2\nu} & z_{3\nu}^{2\nu-1} & \dots & 1 & z_{3\nu}^{2\nu}f_{3\nu}^{(2)} & z_{3\nu}^{2\nu-1}f_{3\nu}^{(2)} & \dots & f_{3\nu}^{(2)} \end{vmatrix} \Big/ D_\nu^{(0)},$$

$$\mu_{3\nu}^{(1)}(z_0\dots z_{3\nu}) = \begin{vmatrix} z_0^{2\nu} & z_0^{2\nu-1} & \dots & 1 & 0 & \dots & 0 & z_0^{2\nu}f_0^{(1)} & z_0^{2\nu-1}f_0^{(1)} & \dots & f_0^{(1)} \\ \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ z_{3\nu}^{2\nu} & z_{3\nu}^{2\nu-1} & \dots & 1 & 0 & \dots & 0 & z_{3\nu}^{2\nu}f_{3\nu}^{(1)} & z_{3\nu}^{2\nu-1}f_{3\nu}^{(1)} & \dots & f_{3\nu}^{(1)} \\ 0 & 0 & \dots & 0 & z_0^{2\nu-1} & \dots & 1 & z_0^{2\nu}f_0^{(2)} & z_0^{2\nu-1}f_0^{(2)} & \dots & f_0^{(2)} \\ \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 & z_{3\nu}^{2\nu-1} & \dots & 1 & z_{3\nu}^{2\nu}f_{3\nu}^{(2)} & z_{3\nu}^{2\nu-1}f_{3\nu}^{(2)} & \dots & f_{3\nu}^{(2)} \end{vmatrix} \Big/ D_\nu^{(0)}.$$

As in the case of a Thiele continued fraction it is possible to let two or more arguments of the μ's coincide by taking a limit. We will treat the case where all interpolation points are confluent:

$$\mu_i^{(1)}(t) = \mu_i^{(1)}(\overbrace{t\dots t}^{(i+1)\times}) = \lim_{z_0,\dots,z_i\to t}\mu_i^{(1)}(z_0\dots z_i)\,,$$

$$\mu_i^{(2)}(t) = \mu_i^{(2)}(\overbrace{t\dots t}^{(i+1)\times}) = \lim_{z_0,\dots,z_i\to t}\mu_i^{(2)}(z_0\dots z_i)\,.$$

These expressions can be calculated from the following recurrence relations:

$$\mu_0^{(1)}(t) = f^{(1)}(t)\,,$$

$$\mu_0^{(2)}(t) = f^{(2)}(t)\,,$$

$$\mu_i^{(2)}(t) = \frac{D\mu_{i-1}^{(1)}(t)}{D\mu_{i-1}^{(2)}(t)},$$

$$\mu_i^{(1)}(t) = (\mu_i^{(2)}(t) - \mu_{i-3}^{(2)}(t))\mu_{i-2}^{(2)}(t) + \mu_{i-3}^{(1)}(t) + \frac{i}{D\mu_{i-1}^{(2)}(t)}. \tag{15}$$

(Here $D = \mathrm{d}/\mathrm{d}t$.) The proof of this proceeds in the same way as the proof in [9]: from (14) we get

$$\frac{\mu_{i-1}^{(2)}(zzz\ldots z) - \mu_{i-1}^{(2)}(tzz\ldots z)}{z-t}$$
$$= \frac{-1}{(\mu_i^{(2)}(tzz\ldots z) - \mu_{i-3}^{(2)}(zzz\ldots z))\mu_{i-2}^{(2)}(zzz\ldots z) - \mu_i^{(1)}(tzz\ldots z) + \mu_{i-3}^{(1)}(zzz\ldots z)},$$

$$\frac{\mu_{i-1}^{(2)}(tzz\ldots z) - \mu_{i-1}^{(2)}(ttz\ldots z)}{z-t}$$
$$= \frac{-1}{(\mu_i^{(2)}(ttz\ldots z) - \mu_{i-3}^{(2)}(tzz\ldots z))\mu_{i-2}^{(2)}(tzz\ldots z) - \mu_i^{(1)}(ttz\ldots z) + \mu_{i-3}^{(1)}(tzz\ldots z)},$$

$$\vdots$$

$$\frac{\mu_{i-1}^{(2)}(tt\ldots tz) - \mu_{i-1}^{(2)}(tt\ldots tt)}{z-t}$$
$$= \frac{-1}{(\mu_i^{(2)}(tt\ldots tz) - \mu_{i-3}^{(2)}(tt\ldots tt))\mu_{i-2}^{(2)}(tt\ldots tt) - \mu_i^{(1)}(tt\ldots tz) + \mu_{i-3}^{(1)}(ttt\ldots t)}.$$

Adding these equations and letting $z \to t$, we obtain

$$D\mu_{i-1}^{(2)}(t) = \frac{-i}{(\mu_i^{(2)}(t) - \mu_{i-3}^{(2)}(t))\mu_{i-2}^{(2)}(t) - \mu_i^{(1)}(t) + \mu_{i-3}^{(1)}(t)}$$

and similarly

$$D\mu_{i-1}^{(1)}(t) = \mu_i^{(2)}(t) D\mu_{i-1}^{(2)}(t).$$

Solving for $\mu_i^{(1)}(t)$ and $\mu_i^{(2)}(t)$ gives us (15).

If we now define (using (12))

$$v_i^{(1)}(t) = \mu_i^{(2)}(t) - \mu_{i-3}^{(2)}(t),$$
$$v_i^{(2)}(t) = \mu_i^{(1)}(t) - \mu_{i-3}^{(1)}(t) - (\mu_i^{(2)}(t) - \mu_{i-3}^{(2)}(t))\mu_{i-2}^{(2)}(t),$$

then the 2-fraction associated with the recurrence relation

$$y_1 = v_1^{(2)}(t)y_0 + v_1^{(1)}(t)(z-t)y_{-1} + (z-t)y_{-2},$$
$$y_i = v_i^{(2)}(t)y_{i-1} + v_i^{(1)}(t)(z-t)y_{i-2} + (z-t)^2 y_{i-3}, \quad i = 2, 3, \ldots \tag{16}$$

generalizes the Thiele interpolation formula (see [9]) and can be seen as the C-2-fraction on the main stepline in the table of simultaneous Padé approximants of type II for $f^{(1)}, f^{(2)}$.

EXAMPLE 2

Let us take $f^{(1)}(z) = e^{w_2 z}$ and $f^{(2)}(z) = e^{w_1 z}$. Then we find for $\mu_i^{(1)}(t)$ and $\mu_i^{(2)}(t)$:

$$\mu_{3i}^{(1)}(t) = \left(\frac{w_2 - w_1}{w_1}\right)^i e^{w_2 t},$$

$$\mu_{3i+1}^{(1)}(t) = \left(\frac{w_2}{w_1 - w_2}\right)^i \frac{i+1}{w_1} e^{-w_1 t},$$

$$\mu_{3i+2}^{(1)}(t) = \left(\frac{-w_1}{w_2}\right)^{i+1} \frac{i+1}{w_1 - w_2} e^{(w_1 - w_2)t},$$

$$\mu_{3i}^{(2)}(t) = \left(\frac{w_1 - w_2}{w_2}\right)^i e^{w_1 t},$$

$$\mu_{3i+1}^{(2)}(t) = (-1)^i \left(\frac{w_2}{w_1}\right)^{i+1} e^{(w_2 - w_1)t},$$

$$\mu_{3i+2}^{(2)}(t) = -\left(\frac{w_1}{w_2 - w_1}\right)^{i+1} \frac{i+1}{w_2} e^{-w_2 t},$$

for $i = 0, 1, \ldots$, and for $v_i^{(1)}(t)$ and $v_i^{(2)}(t)$:

$$v_{3i+1}^{(1)}(t) = \left(\frac{-w_2}{w_1}\right)^i \frac{w_1 + w_2}{w_1} e^{(w_2 - w_1)t},$$

$$v_{3i+2}^{(1)}(t) = \left(\frac{w_1}{w_2 - w_1}\right)^i \frac{(w_2 - 2w_1)i - w_1}{w_2(w_2 - w_1)} e^{-w_2 t},$$

$$v_{3i+3}^{(1)}(t) = \left(\frac{w_1 - w_2}{w_2}\right)^i \frac{w_1 - 2w_2}{w_2} e^{w_1 t},$$

$$v_{3i+1}^{(2)}(t) = \left(\frac{w_2}{w_1 - w_2}\right)^i \frac{3i+1}{w_1} e^{-w_1 t},$$

$$v_{3i+2}^{(2)}(t) = \left(\frac{-w_1}{w_2}\right)^{i+1} \frac{3i+2}{w_1 - w_2} e^{(w_1 - w_2)t},$$

$$v_{3i+3}^{(2)}(t) = \left(\frac{w_2 - w_1}{w_1}\right)^{i+1} 3e^{w_2 t},$$

but with $v_1^{(1)}(t) = w_2/w_1 \ e^{(w_2 - w_1)t}$. In this case the 2-fraction associated with (16) is equivalent with the C-2-fraction for this example given in [1].

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