POSITIVE INTERPOLATORY QUADRATURE FORMULAS AND PARA-ORTHOGONAL POLYNOMIALS

ADHEMAR BULTHEEL, LEYLA DARUIS, AND PABLO GONZÁLEZ-VERA

Dedicated to Olav Njåstad on the occasion of his 70th anniversary.

Abstract. We establish a relation between quadrature formulas on the interval $[-1,1]$, that approximate integrals of the form $J_\mu(F) = \int_{-1}^{1} F(x)\mu(x)\mathrm{d}x$, and Szegő quadrature formulas on the unit circle that approximate integrals of the form $I_\omega(f) = \int_{-\pi}^{\pi} f\left(e^{i\theta}\right)\omega(\theta)\mathrm{d}\theta$. The functions $\mu(x)$ and $\omega(\theta)$ are assumed to be weight functions on $[-1,1]$ and $[-\pi,\pi]$, respectively, and related by $\omega(\theta) = \mu(\cos\theta)|\sin\theta|$. It is well known that the nodes of Szegő formulas are the zeros of the so called para-orthogonal polynomials $B_n(z,\tau) = \Phi_n(z) + \tau\Phi_n^*(z)$, $|\tau| = 1$, $\Phi_n(z)$ and $\Phi_n^*(z)$ being the orthogonal and reciprocal polynomials, respectively, with respect to the weight function $\omega(\theta)$. Furthermore, for $\tau = \pm 1$ we have recently obtained Gauss-type quadrature formulas on $[-1,1]$, (see [1]). In this paper, making use of the para-orthogonal polynomials with $\tau \neq \pm 1$, a one-parameter family of interpolatory quadrature formulas with positive coefficients for $J_\mu(F)$ is obtained along with error expressions for analytic integrands. Finally, some illustrative numerical examples are also included.

1. Introduction

Consider the integral

$$J_\mu(F) = \int_{-1}^{+1} F(x)\mu(x)\mathrm{d}x, \tag{1.1}$$

where μ is a weight function on $[-1,1]$.

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In general, by a weight function $\mu(x)$ on an interval $[a,b]$, $(-\infty \leq a < b \leq \infty)$ we mean a nonnegative function on $[a,b]$ such that $\int_I \mu(x)\mathrm{d}x > 0$, for any interval $I \subset [a,b]$.

To approximate $J_\mu(F)$, Gaussian quadrature formulas are often preferred. These are formulas of the form

$$J_n(F) = \sum_{k=1}^{n} \lambda_k F(x_k). \tag{1.2}$$

They have positive coefficients λ_k and the nodes x_k are chosen as the zeros of the nth orthogonal polynomial with respect to the inner product

$$\langle F, G\rangle = \int_{-1}^{+1} F(x)G(x)\mu(x)\mathrm{d}x. \tag{1.3}$$

These zeros are all distinct in the open interval $(-1,+1)$.

Let $\mathbb{P}_N$ be the set of polynomials of degree at most N. It is known that there cannot exist quadrature formulas like in formula (1.2) to be exact in $\mathbb{P}_{2n}$, i.e., there cannot exist quadrature formulas satisfying $J_n(F) = J_\mu(F)$ for all $F \in \mathbb{P}_{2n}$.

The main advantage of choosing the nodes $\{x_k\}_{k=1}^n$ as the zeros of the nth orthogonal polynomial with respect to the inner product (1.3), is that then the quadrature formula is exact in the space $\mathbb{P}_{2n-1}$.

For any other choice of the nodes, the quadrature will only be exact in a smaller set of polynomials. It is reasonable though to require that we have at least exactness in $\mathbb{P}_{n-1}$. This may be obtained by choosing $J_n(F) = J_\mu(L_n)$ where L_n is the unique polynomial in $\mathbb{P}_{n-1}$ that interpolates F at the nodes $\{x_k\}_{k=1}^n$. We then speak of an interpolatory quadrature formula. It is also desirable to construct interpolatory quadrature formulas with positive coefficients, since the convergence of the quadratures for all Riemann integrable functions F on $[-1,1]$ is guaranteed.

On the other hand, for the computation of an integral

$$I_\omega(f) = \int_{-\pi}^{+\pi} f(e^{i\theta})\omega(\theta)\mathrm{d}\theta, \tag{1.4}$$

where ω is a weight function on $[-\pi,+\pi]$, one could also use interpolatory quadrature formulas. In this case the role played by the set of polynomials is replaced by the set of Laurent polynomials. Let p and q be nonnegative integers. Then we denote by $\Lambda_{-p,q}$ as the following set of Laurent polynomials

$$\Lambda_{-p,q} = \{ \sum_{k=-p}^{q} \alpha_k z^k,\ \alpha_k \in \mathbb{C}\}.$$

Let us denote by $\mathbb{T} = \{z : |z| = 1\}$ the unit circle, by $\mathbb{D} = \{z : |z| < 1\}$, the open unit disc and by $\mathbb{E} = \{z : |z| > 1\}$ the exterior of the unit disc.

Now the interpolatory quadrature formulas are formulas of the form

$$I_n(f) = \sum_{k=1}^{n} A_k f(z_k). \tag{1.5}$$

Starting from n distinct nodes $\{z_k\}_{k=1}^n$ on $\mathbb{T}$, they may be obtained by constructing the unique interpolating Laurent polynomial $l_n \in \Lambda_{-p,q}$, $(p+q = n-1)$ and setting $I_n(f) = I_\omega(l_n)$, which will obviously result in a quadrature formula that is exact in $\Lambda_{-p,q}$, $(p+q = n-1)$.

To increase the dimension of the space of Laurent polynomials for which we have exactness, the nodes should be chosen in a particular way.

Indeed, let Φ_n be the nth monic orthogonal polynomial with respect to the inner product

$$\langle f, g\rangle = \int_{-\pi}^{+\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\omega(\theta)\mathrm{d}\theta,$$

i.e., Φ_n represents the nth monic Szegő polynomial with respect to $\omega(\theta)$.

As in the real case, one can think of choosing its zeros as the nodes for the quadrature formula. However, since the zeros of the polynomials Φ_n are all in $\mathbb{D}$, while we want the nodes to be on $\mathbb{T}$, these zeros are not candidates for the nodes of the quadrature formulas. However, the so called para-orthogonal polynomials: $B_n(z,\tau) = \Phi_n(z) + \tau\Phi_n^*(z)$ with $\tau \in \mathbb{T}$ and $\Phi_n^*(z) = z^n\overline{\Phi(1/\overline{z})}$, are known to be simple and on $\mathbb{T}$ [10]. When taking the appropriate coefficients A_k for these nodes, which turn out to be all positive, more precisely, $A_k = \int_{-\pi}^{\pi} \left|l_k(e^{i\theta})\right|^2 \omega(\theta)\mathrm{d}\theta$, with $l_k \in \mathbb{P}_{n-1}$ such that $l_k(z_j) = \delta_{kj}$, $1 \le k, j \le n$, (see [9] for further details), the quadrature formulas can be made exact in $\Lambda_{-n+1,n-1}$. These are the so called Szegő quadrature formulas. Like the Gaussian formulas, the Szegő formulas are exact in a (Laurent) polynomial space of maximal dimension $2n-1$, i.e., there cannot exist an n-point quadrature formula with nodes on $\mathbb{T}$ to be exact neither in $\Lambda_{-(n-1),n}$ nor in $\Lambda_{-n,n-1}$. The first systematic study of Szegő formulas was undertaken by Jones et al. [10].

Of course, by the Joukowsky transform $e^{i\theta} \in \mathbb{T} \leftrightarrow \cos\theta \in [-1,1]$, or more generally $z \leftrightarrow (z+z^{-1})/2$, the unit circle can be mapped onto the interval $[-1,1]$. So there should be some relation between the Gauss and the Szegő formulas. First note that by this transform, the weight function $\mu(x)$ transforms into

$$\omega(\theta) = \mu(\cos\theta)|\sin\theta|. \tag{1.6}$$

This leads to a relation between the orthogonal polynomials on the unit circle and the interval that has already been studied by Szegő [7, Chapter IX, 5, Section 6.4] and Geronimus [13, Section 11.5]. The investigation of the relation between the quadrature formulas has been undertaken in [1]. There it was noted that when $\tau = 1$, then the nodes of the $2n$-point Szegő quadrature formula come in complex conjugate pairs, so that with the n pairs of nodes for the Szegő formulas, we can associate n real nodes in $(-1, 1)$ and the Szegő formula transform into a Gauss quadrature formula on the interval $[-1, 1]$. When $\tau = -1$, then the $(2n + 2)$-point Szegő formula has n complex pairs of nodes and two real ones, which leads to n real nodes in $(-1, 1)$ and 2 nodes at the end points. Therefore the Szegő formula for the circle transforms into the Gauss-Lobatto formula for the interval. For the $(2n + 1)$-point Szegő formulas and $\tau = \pm 1$, we get again complex conjugate pairs of nodes plus either $+1$ or -1. So the Szegő formulas will transform in this case to Gauss-Radau formulas, including one of the end points of the interval $[-1, 1]$.

In this paper, we want to investigate the relation between the Szegő formulas when τ needs not to be ± 1. The corresponding quadrature formulas on $[-1, 1]$ will then not be optimal. They are only exact in the polynomial space $\mathbb{P}_{n-1}$, except for special choices of τ, in which case they can be made exact in $\mathbb{P}_n$. This will be shown in Section 2, giving some illustrative examples in Section 3. In Section 4 we shall provide error expressions for these quadrature formulas along with convergence results. Finally, in Section 5 we include some numerical results.

2. Positive quadrature formulas on $[-1, 1]$

As we mentioned above, the n-point Gaussian quadrature formula with respect to the weight function μ on $[-1, 1]$ can be obtained by using the $(2n)$-point Szegő formulas with respect to ω on $[-\pi, \pi]$, where μ and ω are related by (1.6) and $\tau = \pm 1$. For this purpose, we have first to compute the $(2n)$th monic Szegő polynomial $\Phi_{2n}(z)$. Usually, it is done recursively by using the so-called Levinson algorithm. Therefore, in general, to obtain $\Phi_{2n}(z)$ we need to compute first $\{\Phi_0, \Phi_1, \ldots, \Phi_{2n-1}(z)\}$. Thus, how can we take advantage of these calculations? I.e., can we use Φ_k, $0 \leq k \leq 2n - 1$ to construct quadrature formulas on $[-1, 1]$ when $\tau \neq \pm 1$?

Let us start our investigation with the following Proposition.

Proposition 2.1. *The zeros of the nth para-orthogonal polynomial $B_n(z, \tau)$ with respect to $\omega(\theta) = \mu(\cos\theta)|\sin\theta|$ appear in conjugate pairs if and only if $\tau = \pm 1$.*

Proof. Suppose that ξ and $\overline{\xi}$ are zeros of $B_n(z,\tau) = \Phi_n(z) + \tau\Phi_n^*(z)$. Then

$$\Phi_n(\xi) + \tau\Phi_n^*(\xi) = 0, \tag{2.1}$$

and

$$\Phi_n(\overline{\xi}) + \tau\Phi_n^*(\overline{\xi}) = 0.$$

Since all the coefficients of the Szegő polynomial $\Phi_n(z)$ are real, one has

$$\overline{\Phi_n(\overline{\xi}) + \tau\Phi_n^*(\overline{\xi})} = \Phi_n(\xi) + \overline{\tau}\Phi_n^*(\xi) \tag{2.2}$$

Thus from (2.1) $\tau = -\Phi_n(\xi)/\Phi_n^*(\xi)$ while by (2.2) this equals $\overline{\tau}$. Therefore $\tau = \overline{\tau}$ and the proof follows. □

Our next step is to prove that when $\tau \neq \pm 1$, then the n-point Szegő formulas transform into $n-$point quadrature formulas on $[-1,1]$ that are only exact in $\mathbb{P}_{n-1}$, unless some particular choice of τ is made.

Theorem 2.2. *Let $I_n(f) = \sum_{j=1}^n \lambda_j f(z_j)$ be the nth Szegő quadrature formula with respect to $\omega(\theta) = \mu(\cos\theta)|\sin\theta|$ where the nodes $\{z_j = e^{i\theta_j}\}_{j=1}^n$ are the zeros of the para-orthogonal polynomial $B_n(z,\tau)$ with $\tau \neq \pm 1$. Set*

$$x_j = \cos\theta_j,\ j = 1,\dots,n,$$

and consider the n-point interpolatory quadrature formula based upon these nodes:

$$J_n^\tau(F) = \sum_{j=1}^n A_j F(x_j) \tag{2.3}$$

for $J_\mu(F) = \int_{-1}^1 F(x)\mu(x)\mathrm{d}x$. Then

$$A_j = \frac{\lambda_j}{2} > 0,\ j = 1,\dots,n.$$

Furthermore, (2.3) is exact in $\mathbb{P}_n$ if and only if

$$\tau = -\Phi_n(0) \pm i\sqrt{1 - \Phi_n^2(0)}. \tag{2.4}$$

Proof. Let $0 \leq j \leq n-1$. First note that

$$J_n^\tau(x^j) = \sum_{k=1}^n A_k x_k^j = \sum_{k=1}^n A_k(\cos\theta_k)^j = \sum_{k=1}^n \frac{\lambda_k}{2}(x_k)^j.$$

Since $x = \frac{z+z^{-1}}{2}$, the monomial x^j transforms into a Laurent polynomial in z:

$$x^j = \left(\frac{z+z^{-1}}{2}\right)^j = L(z) \in \Lambda_{-j,j}, \quad (z \in \mathbb{T}).$$

Because the nth Szegő quadrature formula is exact in $\Lambda_{-n+1,n-1}$, i.e., $I_n(f) = I_\omega(f)$ for all $f \in \Lambda_{-n+1,n-1}$, we have

$$J_n^\tau(x^j) = \sum_{k=1}^{n} \frac{\lambda_k}{2} L(e^{i\theta_k}) = \frac{1}{2}\int_{-\pi}^{\pi} L(e^{i\theta})\omega(\theta)\mathrm{d}\theta.$$

Since,

$$J_\mu(x^j) = \int_{-1}^{1} x^j \mu(x)\mathrm{d}x = \frac{1}{2}\int_{-\pi}^{\pi} (\cos\theta)^j\, \omega(\theta)\mathrm{d}\theta = \frac{1}{2}\int_{-\pi}^{\pi} L(e^{i\theta})\omega(\theta)\mathrm{d}\theta,$$

this implies that $J_n^\tau(F)$ is of interpolatory type. Let us see for what values of τ, this is also exact for polynomials with exact degree n, i.e., for what values of τ we have $J_n^\tau(x^n) = J_\mu(x^n) = \int_{-1}^{1} x^n\mu(x)\mathrm{d}x$. By considering the nodal polynomial $W_n(x) = \prod_{j=1}^{n}(x - x_j)$ which has exact degree n, the latter equality is equivalent with $J_n^\tau(W_n) = J_\mu(W_n)$. Since $J_n^\tau(W_n) = 0$, we must check when $J_\mu(W_n) = 0$.

We have,

$$J_\mu(W_n) = \int_{-1}^{1} W_n(x)\mu(x)\mathrm{d}x = \frac{1}{2}\int_{-\pi}^{\pi} W_n\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)\omega(\theta)\mathrm{d}\theta.$$

Consider $Q_n(z) \in \Lambda_{-n,n}$ given by

$$W_n\left(\frac{z+z^{-1}}{2}\right) = \prod_{j=1}^{n}\frac{1}{2}(z+z^{-1}-2x_j^\tau) = \frac{1}{2^n}\prod_{j=1}^{n}(z+z^{-1}-2\cos\theta_j) = \frac{1}{2^n}Q_n(z).$$

Thus, $J_\mu(W_n) = 0$, if and only if, $\int_{-\pi}^{\pi} Q_n(e^{i\theta})\omega(\theta)\mathrm{d}\theta = 0$. We can write $Q_n(z) = \frac{P_{2n}(z)}{z^n}$, $P_{2n} \in \mathbb{P}_{2n}$ such that $P_{2n}(z) = z^{2n} + \cdots + 1$. Furthermore, since the zeros of $B_n(z,\tau)$ are zeros of $Q_n(z)$, there must be some polynomial S such that

$$Q_n(z) = \frac{B_n(z,\tau)}{z^n}S(z), \quad S \in \mathbb{P}_n.$$

Assume that $B_n(z,\tau) = \Phi_n(z) + \tau\Phi_n^*(z)$, with $\Phi_n(z)$ the nth monic Szegő polynomial, i.e.,

$$\Phi_n(z) = z^n + \cdots + \Phi_n(0), \quad \Phi_n(0) \in \mathbb{R}, \quad |\Phi_n(0)| < 1.$$

Then $B_n(z,\tau) = (1+\tau\Phi_n(0))\, z^n + \cdots + (\tau + \Phi_n(0))$. Set $S(z) = b_n z^n + \cdots + b_1 z + b_0$. Since $B_n(z,\tau)S(z) = P_{2n}(z)$, it follows,

$$b_n\,(1+\tau\Phi_n(0)) = 1, \quad \text{and} \quad b_0\,(\tau+\Phi_n(0)) = 1,$$

or equivalently,

$$b_n = \frac{1}{1+\tau\Phi_n(0)}, \quad b_0 = \frac{1}{\tau+\Phi_n(0)}.$$

On the other hand, taking into account that

$$\langle \Phi_n, z^n \rangle_\omega = \langle \Phi_n^*, 1 \rangle_\omega = \langle \Phi_n, \Phi_n \rangle_\omega = \|\Phi_n\|_\omega^2 ,$$

we have,

$$\begin{aligned} \langle B_n(z,\tau), 1 \rangle_\omega &= \tau \langle \Phi_n^*, 1 \rangle_\omega = \tau \|\Phi_n\|_\omega^2 \\ \langle B_n(z,\tau), z^n \rangle_\omega &= \langle \Phi_n, z^n \rangle_\omega = \|\Phi_n\|_\omega^2 . \end{aligned}$$

Now, we compute

$$\begin{aligned} \int_{-\pi}^{\pi} Q_n(e^{i\theta})\omega(\theta)\mathrm{d}\theta &= \int_{-\pi}^{\pi} e^{-in\theta} B_n\left(e^{i\theta},\tau\right) S(e^{i\theta})\omega(\theta)\mathrm{d}\theta \\ &= \left\langle B_n(z,\tau), z^n \overline{S}\left(1/z\right) \right\rangle_\omega = \left\langle B_n(z,\tau), \textstyle\sum_{j=0}^{n} \overline{b}_{n-j} z^j \right\rangle_\omega \\ &= \textstyle\sum_{j=1}^{n} \left\langle B_n(z,\tau), z^j \right\rangle_\omega b_{n-j}. \end{aligned}$$

Since $\left\langle B_n(z,\tau), z^j \right\rangle_\omega = 0$, $j = 1, \ldots, n-1$, it follows that

$$\int_{-\pi}^{\pi} Q_n(e^{i\theta})\omega(\theta)\mathrm{d}\theta = b_n \langle B_n(z,\tau), 1 \rangle_\omega + b_0 \langle B_n(z,\tau), z^n \rangle_\omega .$$

Thus,

$$\begin{aligned} \int_{-\pi}^{\pi} Q_n(e^{i\theta})\omega(\theta)\mathrm{d}\theta &= b_n\tau \|\Phi_n\|_\omega^2 + b_0 \|\Phi_n\|_\omega^2 \\ &= \|\Phi_n\|_\omega^2 \left(\tfrac{\tau}{1+\tau\Phi_n(0)} + \tfrac{1}{\tau+\Phi_n(0)} \right) \\ &= \|\Phi_n\|_\omega^2 \left(\tfrac{\tau^2+2\tau\Phi_n(0)+1}{(1+\tau\Phi_n(0))(\tau+\Phi_n(0))} \right). \end{aligned}$$

Since $|\Phi_n(0)| < 1$ and $|\tau| = 1$, the denominator of the above expression is different than zero. Thus, $\int_{-\pi}^{\pi} Q_n(e^{i\theta})\omega(\theta)\mathrm{d}\theta = 0$ if and only if

$$\tau^2 + 2\tau\Phi_n(0) + 1 = 0,$$

yielding $\tau = -\Phi_n(0) \pm i\sqrt{1-\Phi_n^2(0)}$. □

Remark 2.3. The value τ given by (2.4) will be called "optimal".

3. Illustrative examples

Example 3.1. Let us first consider the Jacobi-type weight functions given by

$$\mu(x) = (1-x)^\alpha (1+x)^\beta, \quad \alpha, \beta > -1.$$

By formula (1.6), we have

$$\omega(\theta) = (1-\cos x)^{\alpha+1/2}(1+\cos x)^{\beta+1/2}, \quad \alpha, \beta > -1.$$

It is known (see e.g., [12]), that the reflection coefficients, or Schur parameters, are,

$$\Phi_n(0) = \frac{\alpha + \frac{1}{2} + (-1)^n(\beta + \frac{1}{2})}{n + \alpha + \beta + 1}. \tag{3.1}$$

Next we shall deal with the Chebyshev weight functions, i.e., with the values $\alpha, \beta \in \{\pm\frac{1}{2}\}$.

For $\alpha = \beta = -\frac{1}{2}$ one has $\mu(x) = 1/\sqrt{1-x^2}$, $x \in [-1,1]$. Thus $\omega(\theta) = 1$, $\theta \in [-\pi, \pi]$. Therefore $\Phi_n(z) = z^n$, and $B_n(z,\tau) = z^n + \tau$. Setting $\tau = e^{i\varphi}$, its zeros are $z_k = e^{i\theta_k}$, $\theta_k = [\varphi + (2k+1)\pi]/n$.

On the other hand, the weights of the corresponding n-point Szegő formula are given by $\lambda_j = 2\pi/n$, $j = 1, \ldots, n$ ([9]). So, the n-point formula

$$J_n^\tau(F) = \frac{\pi}{n} \sum_{j=1}^{n} F(\cos\theta_j)$$

is exact in $\mathbb{P}_{n-1}$.

Furthermore, for the optimal values of τ, i.e., $\tau = \pm i$, the corresponding formulas are exact in $\mathbb{P}_n$. Both of them have equal coefficients. Since the Gaussian formula is the only n-point quadrature formula with equal coefficients that is exact in $\mathbb{P}_n$ (see [11, p. 183]), both formulas must coincide with the corresponding Gaussian quadrature formula, i.e.,

$$J_n^{\pm i}(F) = \frac{\pi}{n} \sum_{j=1}^{n} F(\cos\left(\frac{2j-1}{2n}\pi\right)).$$

For the other values of α and β, the optimal values of the parameter τ are given by formulas (3.1) and (2.4). They are given in the following table:

$\alpha = \beta = \frac{1}{2}$	$\alpha = -\frac{1}{2},\ \beta = \frac{1}{2}$	$\alpha = \frac{1}{2},\ \beta = -\frac{1}{2}$
$\tau_n = \begin{cases} \pm i, & \text{if } n \text{ is odd}, \\ -\frac{2}{n+2} \pm i\frac{\sqrt{n^2+4n}}{n+2}, & \text{if } n \text{ is even}. \end{cases}$	$\tau_n = -\frac{(-1)^n}{n+1} \pm i\frac{\sqrt{n^2+2n}}{n+1}$	$\tau_n = -\frac{1}{n+1} \pm i\frac{\sqrt{n^2+2n}}{n+1}$

For these particular choices of α and β, we have the following explicit expressions for the para-orthogonal polynomials, (see [3])

$\alpha=\beta=\frac{1}{2}$	$B_{2n}(z,\tau_n)=\frac{1}{n+1}\sum_{k=0}^{n}(k+1+\tau_n(n-k+1))z^{2k}$, $B_{2n+1}(z,\tau_n)=\frac{1}{n+1}\left(\sum_{k=0}^{n}(k+1)z^{2k+1}+\tau_n\sum_{k=0}^{n}(n-k+1)z^{2k}\right)$
$\alpha=-\frac{1}{2},\ \beta=\frac{1}{2}$	$B_n(z,\tau_n)=\frac{(-1)^n}{n+1}\sum_{k=0}^{n}(-1)^k(k+1+(n-k+1)\tau_n)z^k$
$\alpha=\frac{1}{2},\ \beta=-\frac{1}{2}$	$B_n(z,\tau_n)=\frac{1}{n+1}\sum_{k=0}^{n}(k+1+(n-k+1)\tau_n)z^k$

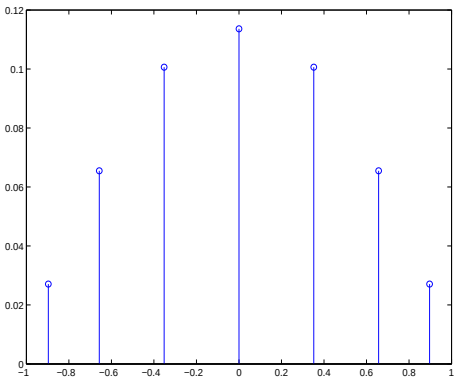
We also know explicit expressions for the corresponding weights in Szegő formulas (see [2]). Here $z_j=e^{i\theta_j}$, $j=1,\ldots,n$ are the zeros of the para-orthogonal polynomial $B_n(z,\tau_n)$.

$\alpha=\beta=\frac{1}{2}$	$\alpha=-\frac{1}{2},\ \beta=\frac{1}{2}$	$\alpha=\frac{1}{2},\ \beta=-\frac{1}{2}$
$\lambda_j=\begin{cases}\frac{1+\tau_n(n+1-z_j^2(n+2))}{4(n+1)z_j^3B_n'(z_j,\tau_n)}, & \text{if } n \text{ is odd},\\ \frac{z_j+\tau_n(n+1-z_j^2(n+2))}{4(n+1)z_j^3B_n'(z_j,\tau_n)}, & \text{if } n \text{ is even}.\end{cases}$	$\lambda_j=\frac{1+\tau_n(n+1-z_j(n+2))}{2(n+1)z_j^2B_n'(z_j,\tau_n)}$	$\lambda_j=(-1)^n\frac{1+\tau_n(n+1+z_j(n+2))}{2(n+1)z_j^2B_n'(z_j,\tau_n)}$

For $\alpha=\beta=\frac{1}{2}$, i.e., $\mu(x)=\sqrt{1-x^2}$, we have calculated the nodes and weights for the corresponding quadratures on $[-1,1]$, choosing optimal values of τ. The results are displayed in the following tables.

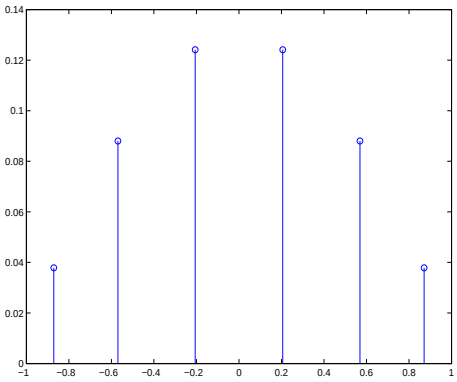
For $n=7$, and $\tau=i$

Nodes	Weights
−0.89659889	0.02709140
−0.65700693	0.06547419
−0.35189103	0.10061622
0	0.11363636
0.35189103	0.10061622
0.65700693	0.06547419
0.89659889	0.02709140



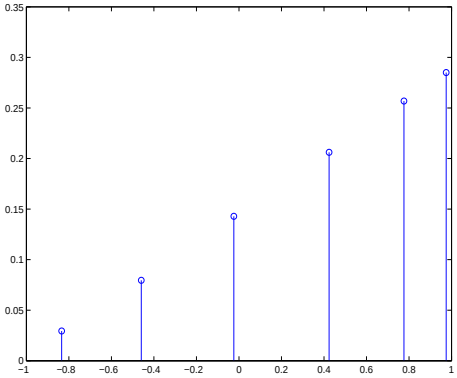
For $n = 6$ and $\tau = \frac{1}{4} + i\frac{\sqrt{15}}{4}$

Nodes	Weights
−0.87099857	0.03785435
−0.56919244	0.08803732
−0.20586750	0.12410831
0.20586750	0.12410831
0.56919244	0.08803732
0.87099857	0.03785435



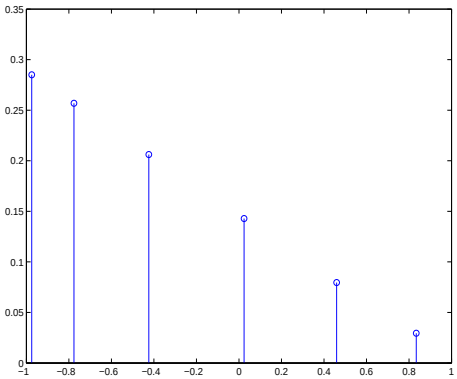
If $\alpha = -\frac{1}{2}$ and $\beta = \frac{1}{2}$, for $n = 6$ and $\tau = -\frac{1}{7} - i\frac{4\sqrt{3}}{7}$ $\left(\mu(x) = \sqrt{\frac{1+x}{1-x}}\right)$

Nodes	Weights
−0.83387852	0.02945035
−0.45959477	0.07960630
−0.02421406	0.14289066
0.42388873	0.20613859
0.77620093	0.25687989
0.97474055	0.28503418



If $\alpha = \frac{1}{2}$ and $\beta = -\frac{1}{2}$, for $n = 6$ and $\tau = -\frac{1}{7} + i\frac{4\sqrt{3}}{7}$ $\left(\mu(x) = \sqrt{\frac{1-x}{1+x}}\right)$

Nodes	Weights
−0.97474055	0.28503418
−0.77620093	0.25687989
−0.42388873	0.20613859
0.02421406	0.14289066
0.45959477	0.07960630
0.83387852	0.02945035



Example 3.2. Now we will consider rational modifications of certain Jacobi weight functions, i.e.,

$$\mu(x) = \frac{(1-x)^\alpha (1+x)^\beta}{P_m(x)}, \quad \alpha, \beta > -1,$$

where $P_m(x) = \prod_{j=1}^{k} (x - \alpha_j)^{p_j}$, $\sum_{j=1}^{k} p_j = m$, $\alpha_j \notin [-1, 1]$ and $P_m(x) > 0$, $\forall x \in [-1, 1]$.

If $\alpha = \beta = -\frac{1}{2}$, then (see e.g., [4]), $\omega(\theta) = \frac{\lambda}{|h(e^{i\theta})|^2}$, where $h(z) = \prod_{j=1}^{k}(z-a_j)^{p_j}$, with $a_j = \alpha_j \pm \sqrt{(\alpha_j)^2 - 1}$ such that $0 < |a_j| < 1$ and $\lambda = 2^m \prod_{j=1}^{k} |a_j|^{p_j}$.

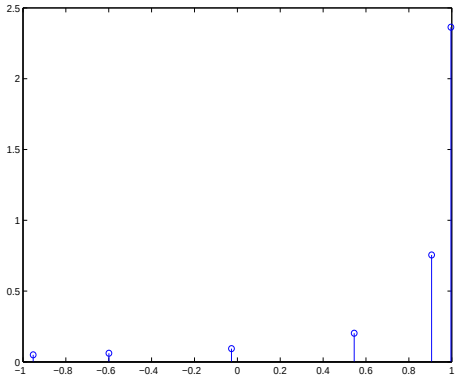
In this case it is known that (see [9]) the para-orthogonal polynomials for $n \geq m$ are given by $B_n(z, \tau_n) = \Phi_n(z) + \tau_n \Phi_n^*(z) = z^{n-m}h(z) + \tau_n h^*(z)$. Thus, the reflection coefficients are $\Phi_n(0) = 0$ and the optimal parameters of τ_n are $\pm i$, provided that $n \geq m$.

Let $\{z_j\}_{j=1}^n$ be the zeros of B_n. In this case we have also an explicit expression for the weights in the Szegő quadrature formula (see [4])

$$\lambda_j = \frac{\lambda}{2\,|h(z_j)|^2 \left(n - m + \sum_{i=1}^{k} p_i \frac{1-|a_j|^2}{|z_j - a_j|^2} \right)}, \quad j = 1, \ldots, n.$$

To fix ideas, take $m = 1$ and $\alpha_1 = 1.01$. Then $a_1 = 0.868226$ and for $n = 6$ and $\tau = i$, we have the following table:

Nodes	Weights
−0.952408	0.0502320
−0.599149	0.0610685
−0.027404	0.0938298
0.545181	0.2027683
0.905962	0.7553279
0.996045	2.3635015



Example 3.3. Let $\mu(x) = 1$. Then $\omega(\theta) = |\sin\theta|$. By formula (3.1) ($\alpha = \beta = 0$), one has

$$\Phi_n(0) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \frac{1}{n+1}, & \text{if } n \text{ is even.} \end{cases}$$

So the corresponding optimal parameters are

$$\tau_n = \begin{cases} \pm i, & \text{if } n \text{ is odd,} \\ -\frac{1}{n+1} \pm i\frac{\sqrt{n^2+2n}}{n+1}, & \text{if } n \text{ is even.} \end{cases}$$

In this case we do not have an explicit expression for Szegő polynomials. For their computation, it seems advisable to use the so-called *split Levinson* algorithm (see [6]), since in this case the trigonometric moments for ω are clearly real.

For $n = 6$ we have computed the Szegő polynomial $\Phi_6(z) = z^6 + \frac{3}{7}z^4 + \frac{9}{35}z^2 + \frac{1}{7}$. Then if we choose the optimal value $\tau = -\frac{1}{7} + \frac{4\sqrt{3}}{7}i$, the nodes in the corresponding Szegő formula are the zeros of $B_6(z, -\frac{1}{7} + \frac{4\sqrt{3}}{7}i)$.

In order to calculate the weights in Szegő formula, we can proceed by computing the so-called *associated polynomials* $\Omega_n(z)$, given by (see [10])

$$\Omega_n(z) = \int_{-\pi}^{\pi} \frac{z+x}{z-x} \left(\Phi_n(x) - \Phi_n(z)\right) \omega(\theta) \mathrm{d}\theta, \; x = e^{i\theta},$$

which satisfies the same recurrence relations as Szegő polynomials Φ_n but with the initial condition $\Omega_0 := -\int_{-\pi}^{\pi} \omega(\theta)\mathrm{d}\theta$.

Thus, we computed them recursively and obtained (for $n = 6$):

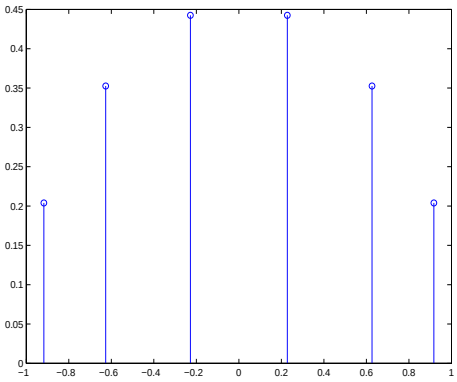
$$\Omega_6(z) = -4z^6 + \frac{20}{21}z^4 + \frac{68}{105}z^2 + \frac{4}{7}.$$

Let $A_n(z,\tau) = \Omega_n(z) - \tau\Omega_n^*(z)$. Then the weights, in general, can be expressed, in terms of $B_n(z,\tau)$ and $A_n(z,\tau)$, as (see [3]):

$$\lambda_j = -\frac{1}{2z_j} \frac{A_n(z_j,\tau)}{B_n'(z_j,\tau)}, \; j = 1, \ldots, n.$$

By virtue of Theorem 2.2, the nodes and weights for $n = 6$ are

Nodes	Weights
−0.917193	0.203921
−0.626545	0.353621
−0.227838	0.442457
0.227838	0.442457
0.626545	0.353621
0.917193	0.203921



4. Error and convergence

In this Section we will obtain an error expression for the n-point interpolatory quadrature formula on $[-1,1]$ i.e., for $J_\mu(F) - J_n^\tau(F)$, $\tau \neq \pm 1$, F being an analytic function.

First we will express this error in terms of the error of a certain interpolating L-polynomial, and secondly, we will establish a relation with the error in the corresponding Szegő quadrature formulas.

Recall that, if μ is a weight function on $[-1,1]$ and $I_n(f) = \sum_{j=1}^n \lambda_j f(z_j)$ denotes the nth Szegő quadrature formula with respect to $\omega(\theta) = \mu(\cos\theta)|\sin\theta|$ for $I_\omega(f) = \int_{-\pi}^{+\pi} f(e^{i\theta})\omega(\theta)\mathrm{d}\theta$, then the nodes $\{z_j = e^{i\theta_j}\}_{j=1}^n$ are the zeros of the para-orthogonal polynomial $B_n(z,\tau)$, and by virtue of Theorem 2.2, for $\tau \neq \pm 1$ the corresponding quadrature formula on $[-1,1]$ has the form: $J_n^\tau(F) = \sum_{j=1}^n \frac{\lambda_j}{2} F(\cos\theta_j)$.

On the other hand, since $J_\mu(P) = J_n^\tau(P)$, $\forall P \in \mathbb{P}_{n-1}$, it follows that $J_n^\tau(F) = J_\mu(P_{n-1}(F,x))$, where $P_{n-1}(F,x) = P_{n-1}(x)$ is the unique polynomial in $\mathbb{P}_{n-1}$ satisfying $P_{n-1}(F,x_j) = F(x_j)$, $j = 1,\dots,n$.

Now, setting $R_n(z) = \frac{1}{2}P_{n-1}\left(\frac{z+z^{-1}}{2}\right)$ then clearly $R_n(z) \in \Lambda_{-(n-1),n-1}$ and it satisfies

$$R_n(z_j) = \frac{1}{2}P_{n-1}\left(\frac{z_j + z_j^{-1}}{2}\right) = \frac{1}{2}P_{n-1}(x_j) = \frac{1}{2}F(x_j) = f(z_j)$$

Similarly, $R_n(\overline{z}_j) = R_n(z_j) = f(z_j) = f(\overline{z}_j)$. (Recall that $f(z) = \frac{1}{2}F(\frac{z+z^{-1}}{2})$).

So, $R_n(z) \in \Lambda_{-(n-1),n-1}$, a space of dimension $2n-1$ and it interpolates $f(z)$ at the $2n$ distinct nodes on $\mathbb{T}$: $z_1,\dots,z_n,\overline{z}_1,\dots,\overline{z}_n$.

Furthermore, the only two subspaces of Λ with dimension $2n$ and containing $\Lambda_{-(n-1),n-1}$ are $\Lambda_{-(n-1),n}$ and $\Lambda_{-n,n-1}$. Thus, given the $2n$ distinct nodes $z_1,\dots,z_n,\overline{z}_1,\dots,\overline{z}_n$, on $\mathbb{T}$, since both $\Lambda_{-(n-1),n}$ and $\Lambda_{-n,n-1}$ are Chebyshev spaces on $\mathbb{T}$ with dimension $2n$, there exist unique L-polynomials $\widetilde{R}_n(z)$ and $\widehat{R}_n(z)$ in $\Lambda_{-n,n-1}$ and $\Lambda_{-(n-1),n}$, respectively, such that

$$\begin{aligned} &\widetilde{R}_n(z_j) = f(z_j), \quad \widetilde{R}_n(\overline{z}_j) = f(\overline{z}_j), \quad j = 1,\dots,n, \\ &\widehat{R}_n(z_j) = f(z_j), \quad \widehat{R}_n(\overline{z}_j) = f(\overline{z}_j), \quad j = 1,\dots,n. \end{aligned} \tag{4.1}$$

Since $R_n(z)$ also satisfies condition (4.1), it holds by uniqueness that

$$\widetilde{R}_n(z) = \widehat{R}_n(z) = R_n(z) = \frac{1}{2}P_{n-1}\left(\frac{z+z^{-1}}{2}\right), \tag{4.2}$$

and consequently

$$\begin{aligned} J_\mu(F) - J_n^\tau(F) &= J_\mu(F - P_{n-1}) = I_\omega\left(\tfrac{1}{2}F\left(\tfrac{z+z^{-1}}{2}\right) - \tfrac{1}{2}P_{n-1}\left(\tfrac{z+z^{-1}}{2}\right)\right) \\ &= I_\omega(f(z) - R_n(z)) = I_\omega(f(z) - \widetilde{R}_n(z)) = I_\omega(f(z) - \widehat{R}_n(z)). \end{aligned} \tag{4.3}$$

Next, we will give an expression for $f(z) - R_n(z)$. Let $\widetilde{R}_n(z)$ be the L-polynomial in $\Lambda_{-n,n-1}$ satisfying the interpolatory conditions given in (4.1). Then we can write

$$\widetilde{R}_n(z) = \frac{\widetilde{P}_{2n-1}(z)}{z^n}, \quad \widetilde{P}_{2n-1} \in \mathbb{P}_{2n-1}.$$

Clearly, $\widetilde{P}_{2n-1}(z_j) = z_j^n f(z_j)$, and if we set $\widetilde{g}(z) = z^n f(z)$ we have

$$f(z) - \widetilde{R}_n(z) = \frac{\widetilde{g}(z) - \widetilde{P}_{2n-1}(z)}{z^n}.$$

Similarly, for $\widehat{R}_n(z)$ in $\Lambda_{-(n-1),n}$, satisfying the corresponding interpolatory conditions given in (4.1), we have

$$\widehat{R}_n(z) = \frac{\widehat{P}_{2n-1}(z)}{z^{n-1}}, \quad \widehat{P}_{2n-1} \in \mathbb{P}_{2n-1},$$

so that $\widehat{P}_{2n-1}(z_j) = z_j^{n-1} f(z_j)$. Thus, if we set $\widehat{g}(z) = z^{n-1} f(z)$ it follows,

$$f(z) - \widehat{R}_n(z) = \frac{\widehat{g}(z) - \widehat{P}_{2n-1}(z)}{z^{n-1}}.$$

Because of (4.2) we can write

$$\frac{\widehat{g}(z) - \widehat{P}_{2n-1}(z)}{z^{n-1}} = \frac{\widetilde{g}(z) - \widetilde{P}_{2n-1}(z)}{z^n},$$

yielding

$$\frac{z\widehat{g}(z) - z\widehat{P}_{2n-1}(z)}{z^n} = \frac{\widetilde{g}(z) - \widetilde{P}_{2n-1}(z)}{z^n},$$

which implies, since $z\widehat{g}(z) = \widetilde{g}(z)$, that

$$z\widehat{P}_{2n-1}(z) = \widetilde{P}_{2n-1}(z).$$

Thus we deduce that

$$\widetilde{P}_{2n-1}(0) = 0 \quad \text{and} \quad \widehat{P}_{2n-1} \in \mathbb{P}_{2n-2}.$$

Hence, we have come to the following

$$f(z) - R_n(z) = \frac{\widetilde{g}(z) - \widetilde{P}_{2n-1}(z)}{z^n} = \frac{\widehat{g}(z) - \widehat{P}_{2n-1}(z)}{z^{n-1}}.$$

Thus, if we assume that $F(x)$ is analytic in a neighbourhood of $[-1, 1]$, then $f(z) = \frac{1}{2} F\left(\frac{z+z^{-1}}{2}\right)$ is analytic in a neighbourhood $\mathbb{B}$ of $\mathbb{T}$, so are $\widetilde{g}(z) = z^n f(z)$ and $\widehat{g}(z) = z^{n-1} f(z)$. Let $C = \partial\mathbb{B}$ denote the boundary of $\mathbb{B}$, then we can use the integral representation of the polynomial interpolation error for analytic functions (see [14, pp. 50]) to obtain

$$\widetilde{g}(z) - \widetilde{P}_{2n-1}(z) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{(z-z_1)\dots(z-z_n)(z-\bar{z}_1)\dots(z-\bar{z}_n)}{(t-z_1)\dots(t-z_n)(t-\bar{z}_1)\dots(t-\bar{z}_n)} \frac{\widetilde{g}(t)}{t-z} \mathrm{d}t, \tag{4.4}$$

and

$$\widehat{g}(z) - \widehat{P}_{2n-1}(z) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{(z-z_1)\dots(z-z_n)(z-\bar{z}_1)\dots(z-\bar{z}_n)}{(t-z_1)\dots(t-z_n)(t-\bar{z}_1)\dots(t-\bar{z}_n)} \frac{\widehat{g}(t)}{t-z} \mathrm{d}t. \tag{4.5}$$

Now, since $(z-z_1)\dots(z-z_n) = \gamma_n B_n(z, \tau)$, with $\gamma_n \neq 0$, and taking into account that the trigonometric moments for $\omega(\theta)$, i.e., $\int_{-\pi}^{\pi} e^{-ik\theta}\omega(\theta)\mathrm{d}\theta$ are real, we can write $(z-\bar{z}_1)\cdots(z-\bar{z}_n) = \tilde{\gamma}_n B_n(z, \bar{\tau})$, with $\tilde{\gamma}_n \neq 0$.

Hence, from (4.3) and (4.5), we have proved the following

Theorem 4.1. *Assume that $F(x)$ is analytic in a neighborhood of $[-1, 1]$ and $f(z) = \frac{1}{2} F\left(\frac{z+z^{-1}}{2}\right)$ analytic in a neighborhood $\mathbb{B}$ of $\mathbb{T}$, and let $\mathcal{C} = \partial\mathbb{B}$ be the boundary of $\mathbb{B}$, then*

$$J_\mu(F) - J_n^\tau(F) = I_\omega(f - R_n) = \frac{1}{2\pi i} \int_{\mathcal{C}} f(z) H_n(z) \mathrm{d}z, \tag{4.6}$$

where

$$H_n(z) = \frac{z^{n-1}}{B_n(z,\tau)B_n(z,\bar{\tau})}\int_{-\pi}^{\pi}\frac{x^{-(n-1)}B_n(x,\tau)B_n(x,\bar{\tau})}{z-x}\omega(\theta)\mathrm{d}\theta,\ x=e^{i\theta}. \tag{4.7}$$

Remark 4.2. Similarly, using (4.3) and (4.4), we can also obtain,

$$J_\mu(F) - J_n^\tau(F) = I_\omega(f-R_n) = \frac{1}{2\pi i}\int_{\mathcal{C}} f(z)H_n^*(z)\mathrm{d}z, \tag{4.8}$$

where

$$H_n^*(z) = \frac{z^{n}}{B_n(z,\tau)B_n(z,\bar{\tau})}\int_{-\pi}^{\pi}\frac{x^{-n}B_n(x,\tau)B_n(x,\bar{\tau})}{z-x}\omega(\theta)\mathrm{d}\theta,\ x=e^{i\theta}. \tag{4.9}$$

For our puposes, it is enough to limit ourselves to compute $H_n(z)$ for $z\in\mathbb{E}$. This is due to the error expression (4.6) given in tems of $H_n^*(z)$ knowing that

$$\forall z\in\mathbb{C}\setminus\mathbb{T}: \overline{H(1/\overline{z})} = -z^2H_n^2(z). \tag{4.10}$$

Furthermore, in some cases both functions $H_n(z)$ and $H_n^*(z)$ can be exactly computed. Indeed, let us again consider the Jacobi weight functions $\mu(x)=(1-x)^\alpha(1+x)^\beta,\ \alpha,\beta>-1$, so that one has

$$\omega(\theta)=h(x)=\left(1-\frac{1}{2}\left(x+\frac{1}{x}\right)\right)^{\alpha+1/2}\left(1+\frac{1}{2}\left(x+\frac{1}{x}\right)\right)^{\beta+1/2},$$

with $x=e^{i\theta}$.

Hence, when $\alpha+\frac{1}{2}$ and $\beta+\frac{1}{2}$ are both integers, the integrals in (4.7) and (4.9) can be calculated by means of the Residue Theorem.

For instance, for $\alpha=\beta=-\frac{1}{2}$, i.e., $\mu(x)=\frac{1}{\sqrt{1-x^2}}$ and $\omega(\theta)=1$, one can obtain the following representation for $H_n(z)$ in the case of the Lebesgue measure:

$$H_n(z)=\frac{2\pi}{z(z^n+\tau)(z^n+\bar{\tau})},\quad \text{as}\quad |z|>1. \tag{4.11}$$

As it is well known, rational modifications of the Chebyshev weight function give rise to the so-called Bernstein-Szegő weight function $\mu(x)$ of the form,

$$\mu(x)=\frac{1}{P_k(x)\sqrt{1-x^2}},\quad x\in(-1,1),$$

where $P_k(x)$ is a polynomial of degree k, k arbitrary, which is positive on $[-1,1]$. Then,

$$\omega(\theta)=\frac{1}{|h(z)|^2},\quad z=\mathrm{e}^{i\theta},$$

where h is a polynomial of degree k with all its zeros in the open unit disk.

Thus suppose that $h(z) = \prod_{j=1}^{k}(z-\alpha_j)$ with $0 < |\alpha_j| < 1$, and $\alpha_i \neq \alpha_j$, $\forall i \neq j$. Then, the corresponding para-orthogonal polynomial is given by $B_n(z,\tau) = z^{n-k}h(z) + \tau h^*(z)$, (see [9]) and one can also obtain the following representation for $H_n(z)$ when $|z| > 1$:

$$H_n(z) = \frac{z^{n-1}}{B_n(z,\tau)B_n(z,\bar{\tau})}\left(\sum_{j=1}^{k}\frac{h^*(\alpha_j)}{\alpha_j^{n-k}h'(\alpha_j)(z-\alpha_j)} - \frac{h^*(z)}{z^{n-k}h(z)}\right).$$

Now, when taking again the error, by Theorem 2.2 we have

$$\begin{aligned} J_\mu(F) - J_n^\tau(F) &= \int_{-1}^{1} F(x)\mu(x)\mathrm{d}x - \sum_{j=1}^{n} A_j F(x_j) \\ &= \frac{1}{2}\int_{-\pi}^{\pi} F\left(\frac{z+z^{-1}}{2}\right)\omega(\theta)\mathrm{d}\theta - \frac{1}{2}\sum_{j=1}^{n}\lambda_j F\left(\frac{z_j+z_j^{-1}}{2}\right) \\ &= \int_{-\pi}^{\pi} f(z)\omega(\theta)\mathrm{d}\theta - \sum_{j=1}^{n}\lambda_j f(z_j) \\ &= I_\omega(f) - I_n(f). \end{aligned}$$

Therefore, the error of the positive interpolatory quadrature formula on $[-1,1]$ can be written in terms of the error in the corresponding Szegő quadrature on $\mathbb{T}$.

Hence, by using the results given in [9] concerning the error for the Szegő quadrature formula for analytic integrands, we can write

$$J_\mu(F) - J_n^\tau(F) = I_\omega(f) - I_n(f) = \frac{1}{2\pi i}\int_{\mathcal{C}} f(z)\tilde{H}_n(z)\mathrm{d}z, \tag{4.12}$$

where

$$\tilde{H}_n(z) = \frac{z^{n-1}}{B_n(z,\tau)}\int_{-\pi}^{\pi}\frac{x^{-(n-1)}B_n(x,\tau)}{z-x}\omega(\theta)\mathrm{d}\theta, \quad x = e^{i\theta}. \tag{4.13}$$

Let us express $\tilde{H}_n(z)$ in a different way proceeding as in [9]. For every $Q_n(z) \in \mathbb{P}_n$ we have

$$\frac{Q_n(z) - Q_n(x)}{z-x} = x^{n-1} + R_{n-2}(x),$$

where $R_{n-2} \in \mathbb{P}_{n-2}$ with coefficients depending on z.

Setting $x = e^{i\theta}$,

$$\begin{aligned} \int_{-\pi}^{\pi} B_n(x,\tau)\frac{Q_n(z)-Q_n(x)}{z-x}x^{-(n-1)}\omega(\theta)\mathrm{d}\theta &= \int_{-\pi}^{\pi} B_n(x,\tau)\left(x^{n-1}+R_{n-2}(x)\right)x^{-(n-1)}\omega(\theta)\mathrm{d}\theta \\ &= \int_{-\pi}^{\pi} B_n(x,\tau)\omega(\theta)\mathrm{d}\theta = \delta_n \neq 0, \end{aligned}$$

so that

$$\int_{-\pi}^{\pi}\frac{x^{-(n-1)}B_n(x,\tau)}{z-x}\omega(\theta)\mathrm{d}\theta = \frac{1}{Q_n(z)}\left(\int_{-\pi}^{\pi}\frac{B_n(x,\tau)Q_n(x)}{z-x}x^{-(n-1)}\omega(\theta)\mathrm{d}\theta + \delta_n\right),$$

yielding

$$\tilde{H}_n(z) = \frac{z^{n-1}}{B_n(z,\tau)Q_n(z)} \left(\int_{-\pi}^{\pi} \frac{B_n(x,\tau)Q_n(x)}{z-x} x^{-(n-1)} \omega(\theta) \mathrm{d}\theta + \delta_n \right). \tag{4.14}$$

When $Q_n(z) = B_n(z,\tau)$ we obtain the same expression as in [9], i.e.,

$$\tilde{H}_n(z) = \frac{z^{n-1}}{B_n^2(z,\tau)} \left(\int_{-\pi}^{\pi} \frac{B_n^2(x,\tau)}{z-x} x^{-(n-1)} \omega(\theta) \mathrm{d}\theta + \delta_n \right).$$

When $Q_n(z) = B_n(z,\bar{\tau})$ we initially obtain an expression yet different from formula (4.7), namely

$$\tilde{H}_n(z) = \frac{z^{n-1}}{B_n(z,\tau)B_n(z,\bar{\tau})} \left(\int_{-\pi}^{\pi} \frac{x^{-(n-1)} B_n(x,\tau)B_n(x,\bar{\tau})}{z-x} \omega(\theta) \mathrm{d}\theta + \delta_n \right). \tag{4.15}$$

Thus, from the two error expressions given by (4.6) and (4.12), we have

$$\frac{1}{2\pi i} \int_{\mathcal{C}} f(z) H_n(z) \mathrm{d}z = \frac{1}{2\pi i} \int_{\mathcal{C}} f(z) \tilde{H}_n(z) \mathrm{d}z$$

i.e., it should hold

$$\frac{1}{2\pi i} \int_{\mathcal{C}} f(z) \left(H_n(z) - \tilde{H}_n(z) \right) \mathrm{d}z = 0.$$

Let us check this. By comparing (4.7) and (4.15), it follows that

$$H_n(z) - \tilde{H}_n(z) = -\frac{\delta_n z^{n-1}}{B_n(z,\tau)B_n(z,\bar{\tau})}, \quad \delta_n \neq 0.$$

Hence, we need to prove that

$$I_n = \int_{\mathcal{C}} g(z) \mathrm{d}z = 0, \quad \text{with} \quad g(z) = \frac{z^{n-1} f(z)}{B_n(z,\tau)B_n(z,\bar{\tau})}.$$

For this purpose, recall that $B_n(z,\tau)$ (which we can assume to be monic here) has n distinct zeros on $\mathbb{T}$, namely: $z_1, \ldots, z_n$ and that $B_n(z,\bar{\tau})$ has also n distinct zeros on $\mathbb{T}$: $\bar{z}_1, \ldots, \bar{z}_n$. Thus,

$$I_n = 2\pi i \left(\sum_{j=1}^{n} \Big(\mathrm{Res}(g, z = z_j) + \mathrm{Res}(g, z = \bar{z}_j) \Big) \right). \tag{4.16}$$

Now,

$$\mathrm{Res}(g, z = z_j) = \frac{f(z_j) z_j^{n-1}}{(z_j - \bar{z}_j) \prod_{k=1, k\neq j}^{n} (z_j - z_k) \prod_{k=1, k\neq j}^{n} (z_j - \bar{z}_k)},$$

and

$$\mathrm{Res}(g, z = \bar{z}_j) = \frac{f(\bar{z}_j) \bar{z}_j^{n-1}}{(\bar{z}_j - z_j) \prod_{k=1, k\neq j}^{n} (\bar{z}_j - \bar{z}_k) \prod_{k=1, k\neq j}^{n} (\bar{z}_j - z_k)}.$$

Since $f(z) = \frac{1}{2}F\left(\frac{z+z^{-1}}{2}\right)$, we have $f(z) = f(\bar{z})$, $\forall z \in \mathbb{T}$:

$$\begin{aligned}
\mathrm{Res}(g, z = \bar{z}_j) &= \frac{f(z_j) z_j^{-(n-1)}}{(\bar{z}_j - z_j) \prod_{k=1,k\neq j}^{n} \left(\frac{1}{z_j} - \frac{1}{z_k}\right) \prod_{k=1,k\neq j}^{n} \left(\frac{1}{z_j} - \frac{1}{\bar{z}_k}\right)} \\
&= \frac{f(z_j) z_j^{-(n-1)}}{(\bar{z}_j - z_j) \frac{1}{z_j^{2(n-1)}} \prod_{k=1,k\neq j}^{n} \frac{\bar{z}_k - z_j}{\bar{z}_k} \prod_{k=1,k\neq j}^{n} \frac{z_k - z_j}{z_k}} \\
&= \frac{f(z_j) z_j^{n-1}}{(\bar{z}_j - z_j) \prod_{k=1,k\neq j}^{n} \frac{1}{z_k \bar{z}_k} \prod_{k=1,k\neq j}^{n} (\bar{z}_k - z_j) \prod_{k=1,k\neq j}^{n} (z_k - z_j)} \\
&= -\frac{f(z_j) z_j^{n-1}}{(z_j - \bar{z}_j) \prod_{k=1,k\neq j}^{n} (z_j - z_k) \prod_{k=1,k\neq j}^{n} (z_j - \bar{z}_k)} \\
&= -\mathrm{Res}(g, z = z_j).
\end{aligned}$$

By (4.16), we obtain that $I_n = 0$.

As we mentioned before, if $F(x)$ is analytic in a neighbourhood V of $[-1, 1]$, then $f(z) = \frac{1}{2}F\left(\frac{z+z^{-1}}{2}\right)$ is analytic in a neighbourhood $\mathbb{B}$ of $\mathbb{T}$. Let ∂V denote the boundary of V and $\mathcal{C} = \partial\mathbb{B}$ denote the boundary of $\mathbb{B}$. Then, by (4.6), one can obtain the following upper bound for the error in our quadrature formula

$$\begin{aligned}
\left|J_\mu(F) - J_n^{(\tau_n)}(F)\right| &= \left|\frac{1}{2\pi i}\int_{\mathcal{C}} f(z) H_n(z) dz\right| \leq \frac{1}{2\pi} \max_{x\in\partial V} |F(x)| \int_{\mathcal{C}} |H_n(z)|\, |dz| \\
&= \frac{1}{2\pi} M(F, \mathcal{C}) \max_{z\in\mathcal{C}} |H_n(z)|,
\end{aligned} \tag{4.17}$$

where $M(F, \mathcal{C}) = \max_{x\in\partial V} |F(x)| \mathrm{lenght}(\mathcal{C})$.

Now, assuming that $\mathcal{C}$ is the boundary of the annulus $\{z \in \mathbb{C} : \ 0 < r < |z| < R\}$, $0 < r < 1 < R$, from (4.11) and (4.10), the above upper bound takes the following form for the Lebesgue measure

$$\left|J_\mu(F) - J_n^{(\tau_n)}(F)\right| \leq M(F, \mathcal{C}) E_n(R, r),$$

where $E_n(R,r) = \max\left\{\frac{1}{R(R^n-1)^2}, \frac{r^{2n-1}}{(1-r^n)^2}\right\}$.

In order to study the rate of convergence of $J_n^{\tau_n}(F)$, ($|\tau_n| = 1$, $\tau_n \neq \pm 1$) we will give an estimation for $\limsup_{n\to\infty} |H_n(z)|^{1/n}$. For this purpose, we will use the following result (see [9])

Theorem 4.3. *Let $\{\varphi_n\}$ be the sequence of Szegő orthonormal polynomials and consider the corresponding para-orthonormal polynomials*

$$\chi_n(z) = \chi_n(z,\tau_n) = \varphi_n(z) + \tau_n \varphi_n^*(z), \quad (|\tau_n| = 1).$$

Then, it holds:

(1) $\lim_{n\to\infty} |\chi_n(z)|^{1/n} = |z|$ *uniformly on compact subset of* $\mathbb{E}$.
(2) $\lim_{n\to\infty} |\chi_n(z)|^{1/n} = 1$ *uniformly on compact subset of* $\mathbb{D}$.
(3) Let $M_n = \max_{z\in\mathbb{T}} |\chi_n(z)|$. *Then,* $\lim_{n\to\infty} M_n^{1/n} = 1$.

Thus, from the above Theorem one can prove the following,

Proposition 4.4. *Let $F(x)$ be an analytic function in a neighbourdhood of $[-1,1]$ and let $\{\tau_n\}_n$ be a sequence of complex numbers such that $|\tau_n| = 1$, $\tau_n \neq \pm 1$. Then,*

$$\lim_{n\to\infty} \left| J_\mu(F) - J_n^{(\tau_n)}(F) \right|^{1/n} \leq \gamma(F) < 1, \tag{4.18}$$

where $\gamma(F) = \inf\{\rho :\ F$ is analytic on the set $|\varphi(x)| \geq \rho\}$ with $\varphi(x) = x - \sqrt{x^2-1}$.

Proof. By (4.17), one can write

$$\begin{aligned} \limsup_{n\to\infty} |J_\mu(F) - J_n^{\tau_n}(F)|^{1/n} &\leq \limsup_{n\to\infty} \left| \max_{z\in\mathcal{C}} |H_n(z)| \right|^{1/n} \\ &\leq \max_{z\in\mathcal{C}} \limsup_{n\to\infty} |H_n(z)|^{1/n}, \end{aligned}$$

with $\mathcal{C}$ as in (4.17).

Now, by (4.7) and Theorem 4.3, it follows

$$\lim_{n\to\infty} \left| J_\mu(F) - J_n^{(\tau_n)}(F) \right|^{1/n} \leq r = \max\{r_1, r_2\} < 1,$$

where $r_1 = \max\{|z| :\ z \in \mathcal{C} \bigcap \mathbb{D}\} < 1$ and $r_2 = \max\{|z|^{-1} :\ z \in \mathcal{C} \bigcap \mathbb{E}\} < 1$.

On the other hand, by the Joukowsky transform, $x = \frac{1}{2}(z + z^{-1})$, yielding, $z = x \pm \sqrt{x^2-1}$. Thus, choosing the root so that $|x + \sqrt{x^2-1}| > 1$, the proof is concluded. □

Remark 4.5. In [8], interpolatory quadrature formulas with nodes the zeros of orthogonal polynomials on $[-1,1]$ were considered. There, the same estimate of the rate of convergence was obtained, however, those quadratures did not have positive weights in general.

Remark 4.6. When dealing with the Chebyshev weight function $\mu(x) = \frac{1}{\sqrt{1-x^2}}$, one sees from (4.11) that $\lim_{n\to\infty} |H_n^*(z)| = |z|^{-2}$, $\forall z \in \mathbb{E}$. Similarly, $\forall z \in \mathbb{D}$: $\lim_{n\to\infty} |H_n^*(z)| = |z|^2$. Thus, for this special case, the estimation $\gamma(F)$ of the rate of convergence (4.18) in Proposition 4.4 can be sharpened. Indeed, under the same condition as in Proposition 4.4 and $\mu(x) = \frac{1}{\sqrt{1-x^2}}$, it holds,

$$\lim_{n\to\infty} \left| J_\mu(F) - J_n^{(\tau_n)}(F) \right|^{1/n} \leq \gamma^2(F).$$

Finally, to conclude this Section, a general result of convergence will be established for non-analytic integrands. Because of the positivity of the weights $\{A_{j,n}\}_{j=1}^n$, as a consequence of Banach-Steinhaus Theorem, see [11, p. 61], we can prove the following

Theorem 4.7. *For $n = 1, 2, \ldots$, let $J_n^{\tau_n}(F) = \sum_{j=1}^n A_{j,n} F(x_{j,n})$ be the nth interpolatory formula for $J_\mu(F) = \int_{-1}^1 F(x)\mu(x)\mathrm{d}x$ based upon the nodes $x_{j,n} = \frac{1}{2}(z_{j,n} + z_{j,n}^{-1})$, $j = 1, \ldots, n$, $\{z_{j,n}\}_{j=1}^n$ being the zeros of $B_n(z, \tau_n) = \phi_n(z) + \tau_n \phi_n^*(z)$, $\tau_n \in \mathbb{T}$ and $\tau_n \neq \pm 1$. Then*

$$\lim_{n\to\infty} J_n^{\tau_n}(F) = J_\mu(F),$$

for any function $F(x)$ Riemann integrable on $[-1, 1]$.

5. Numerical experiments

In this section we first compare, on the one hand, the n-point interpolatory quadrature formula $J_n^\tau(F)$, ($|\tau| = 1$, $\tau \neq \pm 1$) with the n-point Gaussian quadrature formula $G_n(F)$ for $\int_{-1}^1 F(x)\mu(x)\mathrm{d}x$, when choosing $\mu(x) = \sqrt{\frac{1-x}{1+x}}$.

On the other hand, we also compare the n-point interpolatory quadrature formula $J_n^\tau(F)$, ($|\tau| = 1$, $\tau \neq \pm 1$) with another n-point interpolatory quadrature formula $C_n(F)$ for $\int_{-1}^1 F(x)\mu(x)\mathrm{d}x$ based on the Chebyshev nodes. To do that, we will take $\mu(x) = 1$.

In both cases we will choose the integrand $F(x)$ as $F(x) = \sin x/(x - R)$, $-1 \leq x \leq 1$, with $|R| > 1$.

Example 5.1. Let us start with the weight function $\mu(x) = \sqrt{\frac{1-x}{1+x}}$ on $[-1, 1]$. By (1.6), we have that $\omega(\theta) = 1 - \cos\theta$, $\theta \in [-\pi, \pi]$.

In order to compute the n-point Gaussian quadrature $\widetilde{G}_n(F) = \sum_{j=1}^n \widetilde{A}_j F(\widetilde{x}_j)$, we first calculate the zeros $\{z_j\}_{j=1}^{2n}$ of the para-orthogonal polynomial w.r.t. $\omega(\theta) = 1 - \cos\theta$, namely $B_{2n}(z, 1)$. Note that these zeros appear in complex conjugate pairs. Then we compute the corresponding weights $\{\lambda_j\}_{j=1}^{2n}$ in the Szegő formula. The nodes in the n-point Gaussian formula

are $\widetilde{x}_j = \mathrm{Re}(z_j)$, $j = 1, \ldots, n$, and the weights are $\widetilde{A}_j = \lambda_j$, $j = 1, \ldots, n$, where $\mathrm{Re}(z)$ denotes the real part of the complex number z.

On the other hand, to calculate the interpolatory-type formula $J_n^\tau(F) = \sum_{j=1}^n A_j x_j$, we compute the zeros $\{z_j\}_{j=1}^n$ of the polynomial $B_n(z,\tau)$, $\tau \neq \pm 1$ which is para-orthogonal w.r.t. $\omega(\theta) = 1 - \cos\theta$. Note that these zeros do not appear in complex conjugate pairs. Then also the corresponding weights in the Szegő formula $\{\lambda_j\}_{j=1}^n$ are computed. According to Theorem 2.2, the result is $x_j = \mathrm{Re}(z_j)$ and $A_j = \frac{\lambda_j}{2}$, $j = 1, \ldots, n$.

We choose $n = 9$ and the "optimal" value of τ is $\tau_9 = -\frac{1}{10} + i\frac{3\sqrt{11}}{10}$. Observe that the interpolatory formula are exact in $\mathbb{P}_9$ so we will first compute the Gaussian formula for $n = 5$, in order to make both quadratures exactly integrate polynomials up to the same degree.

In the next table a comparison between the absolute error in the Interpolatory and Gaussian quadratures for the function $F(x) = \sin x/(x - R)$, $-1 \leq x \leq 1$, $|R| > 1$, is displayed

R	Error (Interpolatory-type)	Error (Gaussian)
$R = 5$	$1.2E-13$	$1.2E-11$
$R = 2$	$7.5E-08$	$1.7E-06$
$R = 1.3$	$4.9E-05$	$5.3E-04$

From this table we can see that the interpolatory formulas give better results than the Gaussian formulas and that both quadratures depend on the location of the singularities of the integrand $F(x)$ with respect to the interval $[-1, 1]$.

If we make a comparison between these two formulas with the same number of nodes, say $n = 9$, in order to compute the 9th Gaussian formula we have to calculate the zeros of $B_{18}(z, 1)$ and the quadrature are exact in $\mathbb{P}_{17}$.

R	Error (Interpolatory-type)	Error (Gaussian)
$R = 5$	$1.2E-13$	$6.0E-19$
$R = 2$	$7.5E-08$	$4.5E-11$
$R = 1.3$	$4.9E-05$	$1.3E-06$

As it could be expected, the Gaussian formula gives better results when the number of nodes is the same.

Let us compare both formulas but now computing the interpolatory quadrature with another value of the parameter τ which is not "optimal". By choosing $\tau = i$, the results are displayed in the following table,

R	Error (Interpolatory-type)	Error (Gaussian)
$R = 5$	$3.1E-12$	$1.2E-11$
$R = 2$	$8.0E-07$	$1.7E-06$
$R = 1.3$	$1.9E-04$	$5.3E-04$

Here, we have used an interpolatory formula with nine nodes versus a Gaussian one with five nodes.

From the above table we can say that the interpolatory quadrature formula are slightly better than the Gaussian formula. But, if we compute our interpolatory formula with ten nodes and, again, the Gaussian formula with five nodes in order that both formulas are exact in the same subspace $\mathbb{P}_9$, we obtain the following results,

R	Error (Interpolatory-type)	Error (Gaussian)
$R = 5$	$8.9E-13$	$1.2E-11$
$R = 2$	$1.9E-07$	$1.7E-06$
$R = 1.3$	$7.7E-05$	$5.3E-04$

Example 5.2. Let us now consider $\mu(x) = 1$. Then by (1.6), we have $\omega(\theta) = |\sin\theta|$.

In this case we will approximate the integral $\int_{-1}^{1} F(x)\mathrm{d}x$ by our interpolatory quadratures $J_n^\tau(F)$, $(\tau \neq \pm 1)$ and also by means of another interpolatory formulas based upon the Chebyshev nodes, namely, $C_n(F) = \sum_{j=1}^{n} w_j F(y_j)$, where $y_j = \cos\left(\frac{2j-1}{2n}\pi\right)$, $j = 1, \ldots, n$. We also have an explicit expression for the weights $\{w_j\}_{j=1}^n$, (see [5, pp. 85]) which are given by

$$w_j = \frac{2}{n}\left(1 - 2\sum_{m=1}^{E[n/2]} \frac{\cos\left(m\frac{2j-1}{n}\pi\right)}{4m^2-1}\right),\ j = 1, \ldots, n,$$

where $E[n/2]$ denotes the entire part of $n/2$. Recall that this formula is exact in $\mathbb{P}_{n-1}$. To distinguish this formulas from our interpolatory ones, we will call them *Chebyshev interpolatory quadrature formulas*.

To compute $J_n^\tau(F)$, $(\tau \neq \pm 1)$ we have chosen $n = 6$ and the optimal value of τ as $-\frac{1}{7} + \frac{4\sqrt{3}}{7}i$. Since such formula is exact in $\mathbb{P}_6$, in order that both formulas exactly integrate polynomials up to the same degree, we have to compute $C_n(F)$ for $n = 7$.

In the next table we compare the error in both formulas:

R	Error (Interpolatory)	Error (Chebyshev Interpolatory)
$R = 5$	$3.1E-09$	$1.9E-09$
$R = 2$	$4.8E-06$	$2.9E-06$
$R = 1.3$	$1.2E-03$	$7.4E-04$

As it can be observed, both formulas have a similar behaviour. Now, if we compare both formulas with the same number of nodes $n = 6$, as we can see in the next table, the interpolatory formula gives better results than the interpolatory Chebyshev formula:

R	Error (Interpolatory)	Error (Chebyshev Interpolatory)
$R = 5$	$3.1E-09$	$2.2E-07$
$R = 2$	$4.8E-06$	$2.8E-05$
$R = 1.3$	$1.2E-03$	$2.7E-03$

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