Quasi-paraorthogonal polynomials, their zeros and quadrature

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Abstract

Real polynomials have zeros that are either real or appear in complex conjugate pairs, i.e., they are symmetric with respect to the real line $\mathbb{R}$. Complex polynomials whose zeros have similar properties with respect to the unit circle $\mathbb{T}$, i.e., their zeros are either on the circle or are symmetric with respect to it, are invariant polynomials. Invariant polynomials of degree n satisfy by definition $P_n(z) = \tau P_n^*(z)$ for some invariance parameter $\tau \in \mathbb{T}$ and where $P_n^*(z) = z^n\overline{P_n(1/\overline{z})}$.

In view of this spectral property, it should be clear that the natural counterpart of orthogonal polynomials on $\mathbb{R}$ are not the orthogonal, but the paraorthogonal polynomials on $\mathbb{T}$. These paraorthogonal polynomials are orthogonal to $\operatorname{span}\{z, z^2, \ldots, z^{n-1}, z^n - \hat{\tau}\}$ for some $\hat{\tau} \in \mathbb{T}$, and their zeros are important for the construction of (positive) quadrature formulas with maximal domain of validity.

By removing some of the orthogonality conditions for the polynomials, we gain extra free parameters, which can be used to place some of the zeros at preselected locations. This resulted in a number of papers on quasi-orthogonal polynomials on $\mathbb{R}$. In this talk we shall consider the analogue concept on $\mathbb{T}$ which are the quasi-paraorthogonal polynomials. We analyze the possibilities of preselecting some of the zeros, in order to build positive quadrature formulas with prefixed nodes and maximal domain of validity. We illustrate with a numerical example how the zeros behave as a function of an invariance parameter τ, when they are all simple and on $\mathbb{T}$, and when they are allowed as nodes in positive quadrature formulas.

References

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