Powell–Sabin Spline Prewavelets on the Hexagonal Lattice

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Abstract

In this paper we give an explicit construction of compactly supported prewavelets on differentiable, twodimensional, piecewise polynomial quadratic finite element spaces of $L_2(\mathbb{R}^2)$, sampled on the hexagonal grid. The obtained prewavelet basis is stable in the Sobolev spaces $\mathcal{H}^s$ for $|s| < \frac{5}{2}$. In particular, the prewavelet basis is generated by one single function vector $\boldsymbol{\psi}$ consisting of three generating functions ψ_1,ψ_2,ψ_3 that are globally invariant by a rotation of $2\pi/3$.

Keywords: Powell–Sabin splines, prewavelets, local support

1 Introduction

Spline multiresolution analysis has become a standard mathematical tool in a large number of areas including signal processing and image compression, computer graphics, and numerical solutions of differential and integral equations. The construction of multivariate wavelets on arbitrary triangulations is very challenging. In fact, even the case of continuous piecewise linear wavelets construction is unexpectedly complicated, see [6, 7, 8, 9, 10, 11, 17]. In this paper, we present an explicit construction of C^1 continuous piecewise quadratic prewavelets with compact support on the hexagonal lattice in $\mathbb{R}^2$. Our main motivation comes from the numerical treatment of elliptic variational problems.

Let $\mathcal{H}^s$, $s \in \mathbb{R}$ denote the usual Sobolev space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}^s}$, and let $a(\cdot, \cdot)$ be a symmetric bilinear form such that $a(v, v) \sim \|v\|^2_{\mathcal{H}^s}$. We always mean by $A \sim B$ that A can be bounded above and below by constant multiples of B. Let $\mathcal{H}^{-s}$ denote the dual of $\mathcal{H}^s$. Consider the elliptic variational problem:

$$\begin{array}{c} \text{Given } f \in \mathcal{H}^{-s}, \text{ find } u \in \mathcal{H}^s \text{ such that} \\ a(u, v) = f(v), \quad \forall v \in \mathcal{H}^s. \end{array} \tag{1}$$

Suppose V is a finite dimensional subspace of $\mathcal{H}^s$ with basis $\{B_j : j = 1, \ldots, N\}$. The Galerkin approximate solution $u_V = \sum_{j=1}^N c_j B_j$ is the unique solution of (1) with $\mathcal{H}^s$ replaced by V and it can be found by solving the linear system

$$(a(B_i, B_j))_{i,j=1}^N (c_j)_{j=1}^N = (f(B_i))_{i=1}^N. \tag{2}$$

The stiffness matrix $A = (a(B_i, B_j))_{i,j=1}^N$ for a typical nodal basis is ill-conditioned. Suppose that there exist constants $0 < \underline{\alpha}, \overline{\alpha} < \infty$ (Riesz bounds) such that

$$\underline{\alpha} \sum_{j=1}^N |c_j|^2 \leq \| \sum_{j=1}^N c_j B_j \|^2_{\mathcal{H}^s} \leq \overline{\alpha} \sum_{j=1}^N |c_j|^2, \tag{3}$$

then it is well-known that $\kappa(A) = \mathcal{O}(\overline{\alpha}/\underline{\alpha})$, [2]. So let $\{B'_j : j = 1, \ldots, N\}$ be another basis for V. Then we can solve the linear system (2) with the basis functions B_j replaced by B'_j. The new system can be considered to arise from (2) by preconditioning. Hence, we are interested in basis functions B'_j for which $\kappa((a(B'_i, B'_j))_{i,j=1}^N) = \mathcal{O}(1)$.

Several (pre-)wavelet constructions on polygonal domains generate Riesz bases on $\mathcal{H}^s$ for s in a range around zero, see e.g. [4, 18]. However, for most constructions at least a subset of the wavelets is only continuous, so that the range of stability is restricted to $s < \frac{3}{2}$. The only available wavelet Riesz basis on general polygons for $s \geq \frac{3}{2}$ is the basis constructed in [3], but, from a practical point of view, this basis is very hard to construct. The wavelet basis that we present in this paper is easy to construct, and it forms a Riesz basis for $\mathcal{H}^s$, $|s| < \frac{5}{2}$, which makes it a useful basis for preconditioning fourth order elliptic problems. It

is however restricted to uniform triangulations of the hexagonal grid. Another construction that provides a Riesz basis for $\mathcal{H}^s$, $|s| < \frac{5}{2}$, is the cubic spline prewavelet basis on the uniform four-directional mesh [1]. The dimension of the cubic spline space in [1] is larger than the Powell–Sabin spline space that we use here, which gives more degrees of freedom in constructing prewavelets with certain properties, but at the cost of the construction complexity. The extra degrees of freedom are used to obtain symmetry properties. In comparison, the Powell–Sabin spline prewavelets have hexagonal symmetry for free.

Section 2 gives a short overview of Powell–Sabin splines on the hexagonal lattice and introduces a stable basis. In Section 3 we construct Powell–Sabin spline prewavelets using the two-step construction method from [17]. Finally we visualize the prewavelets in Section 4.

2 Preliminaries

Consider the hexagonal lattice Δ in $\mathbb{R}^2$ which is defined as the image of $\mathbb{Z}^2$ by a linear transformation corresponding to the matrix

$$\Gamma = \begin{bmatrix} 1 & -1/2 \\ 0 & \sqrt{3}/2 \end{bmatrix},$$

and let Δ^* be its refinement by drawing in the additional grid lines $y = l$, $y = \frac{\sqrt{3}}{3}(x + m)$, and $y = -\frac{\sqrt{3}}{3}(x + n)$, $l, m, n \in \mathbb{Z}$. In fact, Δ^* is the Powell–Sabin 6-split of Δ, see Figure 1. Define $S_2^1(\Delta^*)$ as the space of real-valued functions in $C^1(\mathbb{R}^2)$ whose restrictions on each triangle of the triangulation Δ^* are bivariate quadratic polynomials. Then each function $\phi \in S_2^1(\Delta^*)$ is called a uniform Powell–Sabin (PS) spline.

Let $k \in \Delta$, then the interpolation problem

$$\left[\phi(l), \frac{\partial}{\partial x}\phi(l), \frac{\partial}{\partial y}\phi(l)\right] = \delta_{k,l}\left[\alpha, \beta, \gamma\right], \qquad l \in \Delta, \tag{4}$$

has a unique solution $\phi \in S_2^1(\Delta^*)$, see [16]. This allows to define a function vector $\boldsymbol{\phi} = [\phi_1, \phi_2, \phi_3]^T$ that generates a multiresolution analysis (MRA). Let each ϕ_i, $i = 1, 2, 3$, be the unique solution of (4) with

$$\alpha = \frac{1}{3},\ \beta = \frac{8\sqrt{3}}{9} - \delta_{i1}\frac{24\sqrt{3}}{9},\ \gamma = \left((-1)^{i-1} - \delta_{i1}\right)\frac{8}{3},$$

then the integer translates under Γ of the basis functions ϕ_i form a basis for $S_2^1(\Delta^*)$. Furthermore they

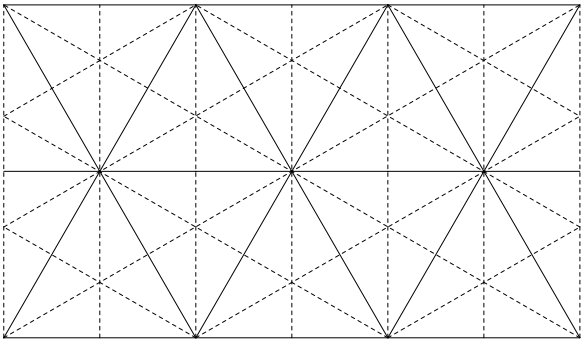

Figure 1: Hexagonal lattice Δ (black lines) with Powell–Sabin 6-split Δ^* (black and dotted lines)

form a partition of unity, see [5]. It can be checked that the three basis functions ϕ_1, ϕ_2, ϕ_3 are invariant by a rotation of $2\pi/3$.

Define the 2×2 dilation matrix D given by

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

then we consider the refinement $\Delta_j := D^{-j}\Delta$, which can be obtained by midedge subdivision, and the corresponding PS 6-split $\Delta_j^* := D^{-j}\Delta^*$. This yields nested subspaces $V_j = S_2^1(\Delta_j^*) \subset L_2(\mathbb{R}^2)$, $j \in \mathbb{Z}$, such that

$$V_j \subset V_{j+1}, \quad j \in \mathbb{Z}, \tag{5}$$

the closure of their union is $L_2(\mathbb{R}^2)$, and their intersection contains only the zero function. In general, the basis functions on all standard refinements Δ_j of Δ can be written as

$$\boldsymbol{\phi}_{j,k}(u) = \boldsymbol{\phi}(D^j(u - k)), \qquad k \in \Delta_j, \quad u \in \mathbb{R}^2,$$

and the set $\left\{2^j\boldsymbol{\phi}_{j,k} \mid k \in \Delta_j\right\}$ forms a Riesz of V_j, i.e., for $\mathbf{c} = \{\mathbf{c}_k\}_{k\in\Delta_j} \in l_2^{3\times 1}(\Delta_j)$, $\mathbf{c}_k := (c_{1,k}, c_{2,k}, c_{3,k})^T$,

$$\left\|\sum_{k\in\Delta_j} \mathbf{c}_k^T 2^j \boldsymbol{\phi}_{j,k}\right\|_{L_2(\mathbb{R}^2)} \sim \|\mathbf{c}\|_{l_2^{3\times 1}(\Delta_j)}, \tag{6}$$

see [13, 14]. With $l_2^{3\times 1}(\Delta_j)$ we denote the Banach space of all sequences of 3×1 vectors $\mathbf{c}_k$ for which $\sqrt{\sum_{k\in\Delta_j} \|\mathbf{c}_k\|_2^2} < \infty$.

Because of the properties listed above we say that the sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R}^2)$ forms a MRA of multiplicity 3, and the function vector $\boldsymbol{\phi}$ is called scaling vector. The nestedness (5) of the MRA

implies that $\boldsymbol{\phi}$ needs to satisfy a matrix refinement equation of the form

$$\boldsymbol{\phi}(u) = \sum_{k\in\mathbb{Z}^2} \mathbf{A}_k \boldsymbol{\phi}(Du - \Gamma k), \quad u \in \mathbb{R}^2, \tag{7}$$

where $\mathbf{A}_k$ are 3×3 mask coefficient matrices. Moreover, $\mathbf{A}_{(-1,-1)}$ and $\mathbf{A}_{(0,-1)}$ are given by

$$\frac{1}{4}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix},$$

$\mathbf{A}_{(-1,0)}$, $\mathbf{A}_{(0,0)}$ and $\mathbf{A}_{(0,1)}$ are given by

$$\frac{1}{4}\begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \frac{1}{6}\begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}, \quad \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix},$$

and $\mathbf{A}_{(0,1)}$ and $\mathbf{A}_{(1,1)}$ are given by

$$\frac{1}{4}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \quad \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix},$$

see e.g. [21].

3 Prewavelets

We are interested in complement spaces W_j of V_j such that

$$V_{j+1} = V_j \oplus^{\perp_{L_2}} W_j,$$

hence

$$L_2(\mathbb{R}^2) = \bigoplus_{j\in\mathbb{Z}}^{\perp_{L_2}} W_j.$$

We will refer to the basis functions of W_j as prewavelets. Note that the prewavelets have three vanishing moments since constant, linear and quadratic polynomials are contained in the space V_j.

The construction of PS prewavelets consists of two steps, cfr. [17]. First we construct a dual basis

$$\{\widetilde{\boldsymbol{\phi}}_{j,k} \mid k \in \Delta_j\} \subset V_{j+1} \tag{8}$$

for the basis $\left\{\boldsymbol{\phi}_{j,k}\ \left(\|\boldsymbol{\phi}_{j,k}\|_{L_2}^2\right)^{-1} \mid k \in \Delta_j\right\}$ of V_j, i.e.

$$\left\langle \widetilde{\boldsymbol{\phi}}_{j,k}, \boldsymbol{\phi}_{j,l}^T \right\rangle_{L_2} = \begin{cases} \|\boldsymbol{\phi}_{j,k}\|_{L_2}^2 & k = l, \\ 0 & k \neq l, \end{cases} \tag{9}$$

with $\|\boldsymbol{\phi}_{j,k}\|_{L_2}^2 := \operatorname{diag}(\langle \boldsymbol{\phi}_{j,k}, \boldsymbol{\phi}_{j,k}^T \rangle_{L_2}) \cdot \mathbf{I}_3$. Since we search for a dual basis in the space V_{j+1} which is much larger than V_j, this basis is not unique. We will use this freedom to select a dual basis that is locally supported. Then, as the second step, for $l \in \Delta_{j+1} \setminus \Delta_j$ we define

$$\boldsymbol{\psi}_{j,l} := \boldsymbol{\phi}_{j+1,l} - \sum_{k\in\Delta_j} \left\langle \boldsymbol{\phi}_{j+1,l}, \boldsymbol{\phi}_{j,k}^T \right\rangle_{L_2} \left(\|\boldsymbol{\phi}_{j,k}\|_{L_2}^2\right)^{-1} \widetilde{\boldsymbol{\phi}}_{j,k}. \tag{10}$$

Note that, because of (8) and (9), $\boldsymbol{\psi}_{j,l} \in V_{j+1} \bigcap V_j^{\perp_{L_2}}$.

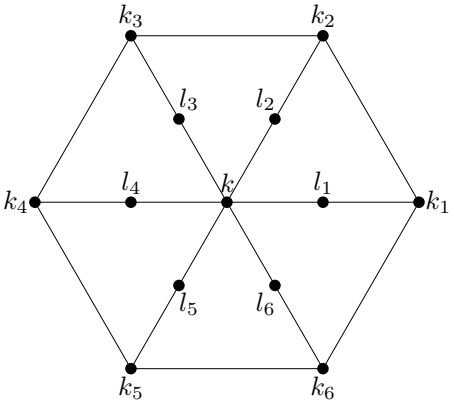

Figure 2: Grid around k

Let $k \in \Delta_j$ and define $k_1, \ldots, k_6 \in \Delta_j$ and $l_1, \ldots, l_6 \in \Delta_{j+1}$ as in Figure 2. We look for a dual function vector $\widetilde{\boldsymbol{\phi}}_{j,k}$ of the form

$$\widetilde{\boldsymbol{\phi}}_{j,k} := \widetilde{\mathbf{A}}_0 \boldsymbol{\phi}_{j+1,k} + \sum_{i=1}^{6} \widetilde{\mathbf{A}}_i \boldsymbol{\phi}_{j+1,l_i}$$

that satisfies (9), where the $\widetilde{\mathbf{A}}_i$ are 3×3 mask coefficient matrices. By construction we know that the support of each scaling vector $\boldsymbol{\phi}_{j,k}$ is the set of all triangles in Δ_j that have $k \in \Delta_j$ as a common vertex. Note that, by construction, the support of $\widetilde{\boldsymbol{\phi}}_{j,k}$ equals the support of $\boldsymbol{\phi}_{j,k}$, see Figure 2. Therefore it is sufficient to solve the system

$$\begin{cases} \langle \widetilde{\boldsymbol{\phi}}_{j,k}, \boldsymbol{\phi}_{j,k}^T \rangle_{L_2} = \|\boldsymbol{\phi}_{j,k}\|_{L_2}^2 & , \\ \langle \widetilde{\boldsymbol{\phi}}_{j,k}, \boldsymbol{\phi}_{j,k_i}^T \rangle_{L_2} = 0 & , \quad i = 1, \ldots, 6, \end{cases} \tag{11}$$

to obtain a dual basis. We have 63 equations and 63 unknown matrix entries. Since the system (11) is non-singular we find a unique solution,

$$\widetilde{\mathbf{A}}_0 = \begin{bmatrix} \frac{382603}{41310} & -\frac{318649}{82620} & -\frac{318649}{82620} \\ -\frac{318649}{82620} & \frac{382603}{41310} & -\frac{318649}{82620} \\ -\frac{318649}{82620} & -\frac{318649}{82620} & \frac{382603}{41310} \end{bmatrix},$$

$$\widetilde{\mathbf{A}}_1 = \begin{bmatrix} \frac{1691}{2754} & -\frac{551}{18360} & -\frac{551}{18360} \\ -\frac{779}{1836} & -\frac{608}{20655} & \frac{11761}{165240} \\ -\frac{779}{1836} & \frac{11761}{165240} & -\frac{608}{20655} \end{bmatrix},$$

$$\widetilde{\mathbf{A}}_2 = \begin{bmatrix} \frac{10241}{82620} & \frac{4123}{18360} & -\frac{2204}{20655} \\ \frac{4123}{18360} & \frac{10241}{82620} & -\frac{2204}{20655} \\ -\frac{82783}{165240} & -\frac{82783}{165240} & \frac{6346}{20655} \end{bmatrix},$$

$$\widetilde{\mathbf{A}}_3 = \begin{bmatrix} -\frac{608}{20655} & -\frac{779}{1836} & \frac{11761}{165240} \\ -\frac{551}{18360} & \frac{1691}{2754} & -\frac{551}{18360} \\ \frac{11761}{165240} & -\frac{779}{1836} & -\frac{608}{20655} \end{bmatrix},$$

$$\widetilde{\mathbf{A}}_4 = \begin{bmatrix} \frac{6346}{20655} & -\frac{82783}{165240} & -\frac{82783}{165240} \\ -\frac{2204}{20655} & \frac{10241}{82620} & \frac{4123}{18360} \\ -\frac{2204}{20655} & \frac{4123}{18360} & \frac{10241}{82620} \end{bmatrix},$$

$$\widetilde{\mathbf{A}}_5 = \begin{bmatrix} -\frac{608}{20655} & \frac{11761}{165240} & -\frac{779}{1836} \\ \frac{11761}{165240} & -\frac{608}{20655} & -\frac{779}{1836} \\ -\frac{551}{18360} & -\frac{551}{18360} & \frac{1691}{2754} \end{bmatrix},$$

$$\widetilde{\mathbf{A}}_6 = \begin{bmatrix} \frac{10241}{82620} & -\frac{2204}{20655} & \frac{4123}{18360} \\ -\frac{82783}{165240} & \frac{6346}{20655} & -\frac{82783}{165240} \\ \frac{4123}{18360} & -\frac{2204}{20655} & \frac{10241}{82620} \end{bmatrix}.$$

Theorem 3.1. *The set*

$$\{2^{j+1}\widetilde{\boldsymbol{\phi}}_{j,k} \mid k \in \Delta_j\} \cup \{2^{j+1}\boldsymbol{\phi}_{j+1,l} \mid l \in \Delta_{j+1} \setminus \Delta_j\} \tag{12}$$

is a Riesz basis for the space V_{j+1}.

Proof. We know that $\{2^{j+1}\boldsymbol{\phi}_{j+1,k} \mid k \in \Delta_{j+1}\}$ is a Riesz basis for V_{j+1}, see (6). Furthermore we have that the basis $\{2^{j+1}\boldsymbol{\phi}_{j+1,k} \mid k \in \Delta_{j+1}\}$ can be transformed into the set (12) by a matrix operation of the form

$$\mathbf{M} = \begin{bmatrix} \mathbf{K} & \mathbf{L} \\ \mathbf{0} & \mathbf{I}_n \end{bmatrix}, \tag{13}$$

where $\mathbf{I}_n$ denotes the $n \times n$ identity matrix with $n = \#\{l \in \Delta_{j+1} \setminus \Delta_j\}$ and $\mathbf{0}$ is a zero matrix. The matrix $\mathbf{K}$ has a block-structure of the form

$$\mathbf{K} = \begin{bmatrix} \widetilde{\mathbf{A}}_0 & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \widetilde{\mathbf{A}}_0 & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \widetilde{\mathbf{A}}_0 & \\ \vdots & \vdots & & \ddots \end{bmatrix},$$

and the matrix $\mathbf{L}$ is a sparse block-structured matrix with blocks $\widetilde{\mathbf{A}}_i$, $i = 1, \ldots, 6$. Note that

$$\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{K}^{-1} & -\mathbf{K}^{-1}\mathbf{L} \\ \mathbf{0} & \mathbf{I}_n \end{bmatrix}.$$

Clearly $\mathbf{M}$ is uniformly bounded and has full rank which proves our claim. □

For all j, for all $l \in \Delta_{j+1} \setminus \Delta_j$, we define the prewavelets $\boldsymbol{\psi}_{j,l}$ as in (10). Note that these prewavelets are compactly supported. It is easily checked that $W_j = \operatorname{span}\{\boldsymbol{\psi}_{j,l} \mid l \in \Delta_{j+1} \setminus \Delta_j\}$. Indeed, suppose $w \in W_j$. Then by Theorem 3.1 we have

$$w = \sum_{l \in \Delta_{j+1} \setminus \Delta_j} \mathbf{c}_l^T \boldsymbol{\phi}_{j+1,l} + \sum_{k \in \Delta_j} \mathbf{c}_k^T \widetilde{\boldsymbol{\phi}}_{j,k},$$

or

$$\begin{aligned} [0 \quad 0 \quad 0] &= \langle w, \boldsymbol{\phi}_{j,k}^T \rangle_{L_2} \\ &= \sum_{l \in \Delta_{j+1} \setminus \Delta_j} \mathbf{c}_l^T \langle \boldsymbol{\phi}_{j+1,l}, \boldsymbol{\phi}_{j,k}^T \rangle_{L_2} + \mathbf{c}_k^T \|\boldsymbol{\phi}_{j,k}\|_{L_2}^2, \end{aligned}$$

which yields

$$\mathbf{c}_k^T = -\sum_{l \in \Delta_{j+1} \setminus \Delta_j} \mathbf{c}_l^T \langle \boldsymbol{\phi}_{j+1,l}, \boldsymbol{\phi}_{j,k}^T \rangle_{L_2} \left(\|\boldsymbol{\phi}_{j,k}\|_{L_2}^2 \right)^{-1}$$

and we get

$$w = \sum_{l \in \Delta_{j+1} \setminus \Delta_j} \mathbf{c}_l^T \boldsymbol{\psi}_{j,l}.$$

Theorem 3.2. *The set*

$$\{2^{j+1}\boldsymbol{\psi}_{j,l} \mid l \in \Delta_{j+1} \setminus \Delta_j\} \tag{14}$$

is a Riesz basis for the space W_j.

Proof. It is sufficient to show that

$$\left\| \sum_{l \in \Delta_{j+1} \setminus \Delta_j} \mathbf{c}_l^T 2^{j+1} \boldsymbol{\psi}_{j,l} \right\|_{L_2} \sim \|\mathbf{c}\|_{l_2(\Delta_{j+1} \setminus \Delta_j)}^{3 \times 1}.$$

By Theorem 3.1 and (10) it holds that

$$\begin{aligned} \left\| \sum_{l \in \Delta_{j+1} \setminus \Delta_j} \mathbf{c}_l^T 2^{j+1} \boldsymbol{\psi}_{j,l} \right\|_{L_2}^2 &\sim \sum_{l \in \Delta_{j+1} \setminus \Delta_j} \mathbf{c}_l^T \mathbf{c}_l \\ + \sum_{k \in \Delta_l} \left(\sum_{l \in \Delta_{j+1} \setminus \Delta_j} \mathbf{c}_l^T \langle \boldsymbol{\phi}_{j+1,l}, \boldsymbol{\phi}_{j,k}^T \rangle_{L_2} \left(\|\boldsymbol{\phi}_{j,k}\|_{L_2}^2 \right)^{-1} \right)^T & \\ \cdot \left(\sum_{l \in \Delta_{j+1} \setminus \Delta_j} \mathbf{c}_l^T \langle \boldsymbol{\phi}_{j+1,l}, \boldsymbol{\phi}_{j,k}^T \rangle_{L_2} \left(\|\boldsymbol{\phi}_{j,k}\|_{L_2}^2 \right)^{-1} \right). & \end{aligned}$$

Both for fixed $k \in \Delta_j$ or for fixed $l \in \Delta_{j+1} \setminus \Delta_j$ the number of nonzero inner products $\langle \boldsymbol{\phi}_{j+1,l}, \boldsymbol{\phi}_{j,k}^T \rangle_{L_2}$ is bounded above by an absolute constant. Furthermore it is straightforward to check that

$$0 \leq \langle \boldsymbol{\phi}_{j+1,l}, \boldsymbol{\phi}_{j,k}^T \rangle_{L_2} \left(\|\boldsymbol{\phi}_{j,k}\|_{L_2}^2 \right)^{-1} \leq C$$

with C an absolute constant. This concludes the proof. □

Remark 3.3. Note that the prewavelets $\boldsymbol{\psi}_{j,l}$ and the dual functions $\widetilde{\boldsymbol{\phi}}_{j,k}$ can be obtained as scaled translates of a finite subset of them. Indeed, we have that

$$\begin{aligned} \boldsymbol{\psi}_{j,l}(u) &= \boldsymbol{\psi}(D^j(u-l)), \quad l \in \Delta_{j+1} \setminus \Delta_j,\ u \in \mathbb{R}^2, \\ \widetilde{\boldsymbol{\phi}}_{j,k}(u) &= \widetilde{\boldsymbol{\phi}}(D^j(u-k)), \quad k \in \Delta_j,\ u \in \mathbb{R}^2, \end{aligned}$$

with $\boldsymbol{\psi}$ and $\widetilde{\boldsymbol{\phi}}$ a generating wavelet vector resp. a generating dual scaling vector. Therefore the inner products appearing in the construction above have to be computed only once. As a consequence of the hexagonal symmetry of $\boldsymbol{\phi}$, both $\widetilde{\boldsymbol{\phi}}$ and $\boldsymbol{\psi}$ have also hexagonal symmetry, i.e. the corresponding generating functions are invariant by a rotation of $2\pi/3$.

Remark 3.4. We are interested in determining the exact range of Sobolev exponents s for which the prewavelet basis forms a Riesz basis for $\mathcal{H}^s$. It turns out that the range of such s is determined by the Sobolev regularity $s(\boldsymbol{\phi})$ of the scaling vector $\boldsymbol{\phi}$, i.e. the wavelet system is a Riesz basis for $\mathcal{H}^s$ for all s with $-s(\boldsymbol{\phi}) < s < s(\boldsymbol{\phi})$ and that this interval is sharp, cfr. [15]. It is well-known that PS splines are in the Sobolev spaces $\mathcal{H}^s$ for any $s < 5/2$, see [13] for a detailed proof.

Remark 3.5. The above construction can be used to construct prewavelets on bounded domains in $\mathbb{R}^2$ due to the explicit construction that we use.

Remark 3.6. It is also possible to extend this construction to arbitrary polygonal domains in $\mathbb{R}^2$. In that case, one should use a triadic refinement scheme instead of the dyadic scheme that we use here, see [20]. Unfortunately it remains an open problem whether the system (11) is non singular for arbitrary settings. However, using perturbation arguments, one can argue that a non-trivial solution to (11) should always exist in some sense.

Remark 3.7. A PS spline wavelet basis on arbitrary polygonal domains in $\mathbb{R}^2$ does exist, see [19]. In [12] it is shown that this wavelet basis is not a Riesz basis for L_2. Fortunately it does form a Riesz basis for $\mathcal{H}^s$ with s in a range around 2 which makes it suitable for preconditioning fourth order elliptic equations.

4 Visualization of the functions

Finally we visualize a scaling function, a dual scaling function and a prewavelet, see Figures 3, 4, and 5.

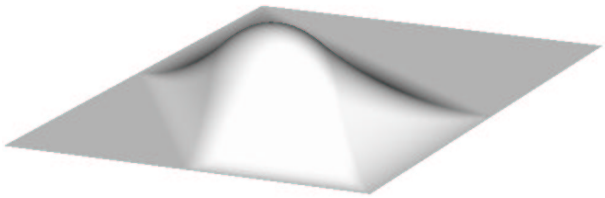

Figure 3: Scaling function

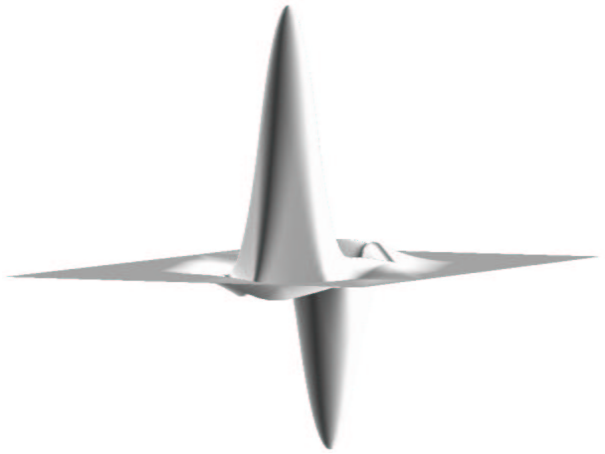

Figure 4: Dual scaling function

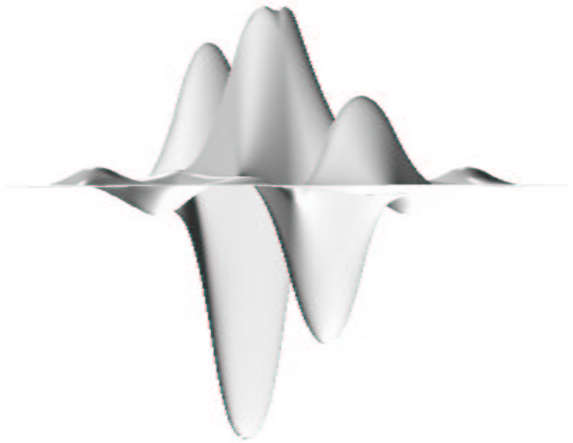

Figure 5: Prewavelet

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