BPX-TYPE PRECONDITIONERS FOR 2ND AND 4TH ORDER ELLIPTIC PROBLEMS ON THE SPHERE*

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Abstract. We develop two Bramble–Pasciak–Xu-type preconditioners for second resp. fourth order elliptic problems on the surface of the two-sphere. To discretize the second order problem we construct C^0 linear elements on the sphere, and for the fourth order problem we construct C^1 finite elements of Powell–Sabin type on the sphere. The main idea why these BPX preconditioners work depends on this particular choice of basis. We prove optimality and provide numerical examples. Furthermore we numerically compare the BPX preconditioners with the suboptimal hierarchical basis preconditioners.

Key words. BPX preconditioner; C^0 and C^1 finite elements; elliptic equations on surfaces.

AMS subject classifications. 65F10, 65F35, 65N30, 35J20, 35J35

1. Introduction. The aim of the present paper is the development of two Bramble–Pasciak–Xu (BPX) [7] preconditioners for second resp. fourth order elliptic problems on the two-dimensional sphere. Such problems arise from several applications in physical geodesy, oceanography and meteorology, [8], and they are even of interest for the graphics community, since surface meshes are often parameterized by using so-called harmonic weights, which correspond to a finite element discretization of the Laplace–Beltrami operator, see, e.g., [1] and references therein.

The geometry of the sphere is a major obstacle in constructing suitable approximation spaces for solving partial differential equations. Often a transformation into spherical coordinates is used which gives rise to singularities at the “poles” of the sphere. This complication is induced by the spherical coordinate system itself. Therefore, an important point in our method is the use of homogeneous polynomials in $\mathbb{R}^3$ which allows us to stick with Cartesian coordinates, hence the “pole problem” is avoided. In order to develop the theory we shall restrict ourselves to the following two most simple equations

$$-\Delta_S u = f \quad \text{on } S, \tag{1.1}$$

and

$$\Delta_S^2 u = f \quad \text{on } S, \tag{1.2}$$

where Δ_S is the Laplace–Beltrami operator on the two-sphere S . In order to work with Cartesian coordinates we write down the Laplace–Beltrami operator in terms of the tangential gradient

$$\nabla_S u := \nabla u - (n \cdot \nabla u)n,$$

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with n the outward normal to S . The Laplace–Beltrami operator on S can now be defined as

$$\Delta_S := \nabla_S \cdot \nabla_S.$$

We use C^0 continuous piecewise linear spherical polynomials to discretize the variational problem

$$\int_S \nabla_S u \nabla_S v \, d\omega = \int_S f v \, d\omega \quad \text{for all } v \in H^1(S) \quad (1.3)$$

corresponding to (1.1), and C^1 continuous piecewise quadratic spherical polynomials to discretize the variational problem

$$\int_S \Delta_S u \Delta_S v \, d\omega = \int_S f v \, d\omega \quad \text{for all } v \in H^2(S) \quad (1.4)$$

corresponding to (1.2). For every $f \in L_2(S)$ with $\int_S f \, d\omega = 0$ there exists a weak solution $u \in H^1(S)$ of (1.3) and a weak solution $u \in H^2(S)$ of (1.4). In both cases u is unique up to a constant, see, e.g., [5, 16].

So let $m \in \{1, 2\}$, and suppose $V \subset H^m(S)$ is a space of conforming C^{m-1} finite elements defined on a spherical triangulation of S with mesh size h . Define $a(u, v)$ as the bilinear form induced by (1.3) resp. (1.4) given $m = 1$ resp. $m = 2$, and let A denote the positive definite selfadjoint operator on V defined by

$$a(u, v) = (Au, v), \quad v \in V, \quad (1.5)$$

where $(\cdot, \cdot)$ denotes the inner product of $L_2(S)$. Then we have to solve the linear operator equation

$$Au = b \quad (1.6)$$

for some $u \in V$, where $b \in V$ is defined by $(b, v) = (f, v)$, $v \in V$. The conjugate gradient method is a very efficient solver for large linear systems arising from problems such as (1.6). However, because of stability reasons, it is necessary that these systems have been suitably preconditioned. It is a known fact (see, e.g., [12]) that if for some constants $0 < \gamma, \Gamma < \infty$ and some invertible operator C

$$\gamma(C^{-1}u, u) \leq a(u, u) \leq \Gamma(C^{-1}u, u), \quad u \in V, \quad (1.7)$$

then the spectral condition number $\kappa(C^{1/2}AC^{1/2})$ is bounded by Γ/γ .

Let us represent the operator A by the stiffness matrix $A_\Phi := (a(\phi_i, \phi_j))_{i,j \in I}$ with respect to some typical nodal basis $\Phi := \{\phi_i : i \in I\}$ of V . Then it is known that $\kappa(A_\Phi) = \mathcal{O}(h^{-2})$ for the problem (1.3) and $\kappa(A_\Phi) = \mathcal{O}(h^{-4})$ for the problem (1.4). In order to precondition the system

$$A_\Phi y = b_\Phi, \quad (b_\Phi)_i := (f, \phi_i), \quad i \in I, \quad (1.8)$$

one can perform a change of basis. So let $\Psi = \{\psi_i : i \in I\}$ be another basis of V , and L be the transfer matrix between the two bases. Then

$$A_\Psi = L^T A_\Phi L,$$

which suggests the use of $C = LL^T$ as preconditioner for the nodal basis discretization.

Several approaches exist to construct a suitable preconditioner, such as the hierarchical basis preconditioner [31] and the closely related BPX preconditioner [7]. The growth rate of the condition numbers was shown to be logarithmic in the size of the problem for the hierarchical basis preconditioner ([31]) and uniformly bounded for the BPX preconditioner in [12, 27]. Originally, these results were formulated for second order problems on two-dimensional planar domains, but they could also be established for fourth order problems on the plane, [14, 20, 26]. Recently, we constructed a hierarchical basis preconditioner for fourth order elliptic problems on the surface of the sphere in [23]. The growth rate of the condition number was shown to be logarithmic which is, as expected, similar to the planar case. It is the aim of the present paper to prove optimality of a BPX preconditioner for the problems (1.3) and (1.4), independent of the discretization, and to give numerical evidence of this optimality. We emphasize that the crucial steps in the optimality proof depend on the particular choice of basis, and, thus, are not valid for arbitrary C^0 or C^1 finite element constructions on the sphere. For both problems we explicitly construct a suitable basis that is easy to implement.

The outline of the remaining sections is as follows. In Section 2, we introduce the C^0 continuous piecewise linear and C^1 continuous piecewise quadratic spherical polynomials that will be used to discretize the problem (1.3) resp. (1.4). The corresponding BPX preconditioners are constructed in Section 3 and we prove their optimality. Finally, in Section 4 we conclude with some numerical experiments that confirm the theory with small absolute condition and iteration numbers.

We finish this introduction with a note about notation. We always mean by $a \sim b$ that $a \lesssim b$ and $a \gtrsim b$ hold, where $a \lesssim b$ means that a can be bounded by a constant multiple of b uniformly in any parameters on which a, b may depend, and $a \gtrsim b$ means $b \lesssim a$.

2. Suitable elements on the sphere. In a series of papers [2, 3, 4], Alföld *et al.* develop spline spaces on triangulations on the sphere analogous to the classical spline spaces on planar triangulations. The idea is to work with homogeneous Bernstein–Bézier polynomials in $\mathbb{R}^3$ which are then restricted to the sphere. A function f defined on $\mathbb{R}^3$ is *homogeneous of degree d* provided that $f(\alpha v) = \alpha^d f(v)$ for all real α and all $v \in \mathbb{R}^3$. The space $\mathbb{H}_d$ of *trivariate polynomials of degree d that are homogeneous of degree d* is a $\binom{d+2}{2}$ dimensional subspace of the space of trivariate polynomials of degree d . Let $\{v_1, v_2, v_3\}$ be a set of linearly independent unit vectors in $\mathbb{R}^3$. We call

$$\mathcal{T} := \{v \in \mathbb{R}^3 \mid v = b_1(v)v_1 + b_2(v)v_2 + b_3(v)v_3 \quad \text{with} \quad b_i(v) \geq 0\}$$

the *trihedron* generated by $\{v_1, v_2, v_3\}$. Each $v \in \mathbb{R}^3$ can be written in the form

$$v = b_1(v)v_1 + b_2(v)v_2 + b_3(v)v_3, \quad (2.1)$$

and we call $b_1(v), b_2(v), b_3(v)$ the *trihedral coordinates of v with respect to $\mathcal{T}$* . Given an integer $d \geq 0$, the *homogeneous Bernstein basis polynomials of degree d on $\mathcal{T}$* are the polynomials

$$B_{ijk}^d(v) := \frac{d!}{i!j!k!} b_1(v)^i b_2(v)^j b_3(v)^k, \quad i + j + k = d,$$

and they form a basis for $\mathbb{H}_d$. We define a *spherical triangle* as the restriction of a trihedron $\mathcal{T}$ to the unit sphere S . The restrictions of the trihedral coordinates (2.1) to a spherical triangle with vertices v_1, v_2 and v_3 are called *spherical barycentric*

coordinates. Any homogeneous polynomial p of degree d and its restriction to a spherical triangle τ has a *Bernstein–Bézier representation* with respect to τ

$$p(v) := \sum_{i+j+k=d} c_{ijk} B_{ijk}^d(v), \quad (2.2)$$

and the coefficients c_{ijk} are the *Bézier ordinates*.

Homogeneous polynomials in their Bernstein–Bézier representation can be evaluated efficiently using the classical de Casteljau algorithm:

$$p(v) = c_{000}^d(v)$$

where for $1 \leq l \leq d$

$$\begin{aligned} c_{ijk}^0(v) &:= c_{ijk}, \\ c_{ijk}^l(v) &:= b_1(v) c_{i+1,j,k}^{l-1} + b_2(v) c_{i,j+1,k}^{l-1} + b_3(v) c_{i,j,k+1}^{l-1}, \quad i+j+k = d-l. \end{aligned}$$

Also continuity conditions can be expressed analogous to the classical bivariate case. Let $\mathcal{T}$ and $\tilde{\mathcal{T}}$ be trihedra with vertices $\{v_1, v_2, v_3\}$ and $\{v_4, v_2, v_3\}$. A necessary and sufficient condition for p and $\tilde{p}$ to be C^r continuous across the common boundary is

$$\tilde{c}_{ijk} = c_{0jk}^i(v_4), \quad i = 0, 1, \dots, r, \quad i+j+k = d. \quad (2.3)$$

We write $\mathbb{H}_d(\Omega)$ for the restriction of $\mathbb{H}_d$ to any subset Ω of the unit sphere S , and refer to $\mathbb{H}_d(\Omega)$ as the *space of spherical polynomials of degree d* . Similarly, we write $\mathbb{H}_d(H)$ for the restriction of $\mathbb{H}_d$ to any hyperplane H in $\mathbb{R}^3 \setminus \{0\}$. This is just the well-known space of bivariate polynomials. All these spaces have the same dimension $\binom{d+2}{2}$. Let Δ be a conforming spherical triangulation of $\Omega \subset S$. Then we define the *space of spherical splines of degree d and smoothness r associated with Δ* to be

$$S_d^r(\Delta) := \{s \in C^r(S) : s|_\tau \in \mathbb{H}_d(\tau), \tau \in \Delta\},$$

where $s|_\tau$ denotes the restriction of s to the spherical triangle τ .

2.1. C^0 linear elements on the sphere. The C^0 continuous piecewise linear spherical polynomials that we describe here are a natural extension of the well-known linear elements introduced by Courant [10]. However, our approach differs significantly from previous constructions (e.g., [6, 16]), see Remark 2.4. Suppose that we are given an initial triangulation Δ_0 of S and that

$$\Delta_0 \subset \Delta_1 \subset \dots \subset \Delta_j \subset \dots, \quad j = 0, 1, \dots,$$

is a sequence of dyadically refined triangulations obtained by subdividing the triangles at level j (i.e. the triangles of Δ_j) into 4 congruent subtriangles of level $j+1$. This refinement is regular, i.e. the minimum angle condition is satisfied and

$$\text{diam } \tau \sim 2^{-j}, \quad \tau \in \Delta_j, \quad j = 0, 1, \dots$$

For each $j = 0, 1, \dots$ we define $v_{i,j}$, $i = 1, \dots, N_j$, as the vertices of the triangulation Δ_j . We create suitable basis functions for the nested spherical spline spaces

$$S_1^0(\Delta_0) \subset S_1^0(\Delta_1) \subset \dots \subset S_1^0(\Delta_j) \subset \dots, \quad j = 0, 1, \dots,$$

and this approach allows us to point out a strong connection with the classical Courant elements on the plane.

So let us define a nodal basis for $S_1^0(\Delta_j)$ by solving the following interpolation problem: find functions $\phi_{i,j} \in S_1^0(\Delta_j)$, $i = 1, \dots, N_j$, such that $\phi_{i,j}(v_{k,j}) = \delta_{i,k}$. Obviously this interpolation problem has a unique solution. If we restrict the spline $\phi_{i,j}$ to any spherical triangle τ in Δ_j we get a spherical Bernstein–Bézier polynomial (2.2) of degree $d = 1$. The interpolation problem determines the three Bézier ordinates c_{ijk} in (2.2) (with $d = 1$) in a unique way. If the spherical triangle τ does not contain vertex $v_{i,j}$ then the three Bézier ordinates equal zero, hence $\phi_{i,j}$ has local support. If τ contains vertex $v_{i,j}$ then the Bézier ordinate that is associated with this vertex takes the value 1, the other two Bézier ordinates take the value 0. It is easily checked that the continuity conditions (2.3) for $r = 0$ are satisfied. Fig. 2.1 shows the spherical Courant element.

FIG. 2.1. Graph of $(\phi_{i,j}(v) + 1)v$ with $v \in S$ and $\phi_{i,j}$ the spherical Courant element.

We can look at each spherical basis functions $\phi_{i,j}$ as the restriction of a trivariate homogeneous function to the sphere S . In particular, let f be any spherical function and $d \in \mathbb{N}$, then we define $(f)_d$ as its *homogeneous extension of degree d* , i.e.

$$(f)_d(v) := |v|^d f\left(\frac{v}{|v|}\right), \quad v \in \mathbb{R}^3 \setminus \{0\}. \quad (2.4)$$

If we restrict the homogeneous extension of degree 1 of $\phi_{i,j}$ to the sphere S we recover $\phi_{i,j}$, i.e. $\phi_{i,j} \equiv (\phi_{i,j})_1|_S$. Moreover, we even have the following theorem.

THEOREM 2.1. *The restriction of $(\phi_{i,j})_1$ to the tangent plane touching S at $v_{i,j}$ is a classical bivariate Courant element defined on this tangent plane centered around the vertex $v_{i,j}$.*

Proof. First we define the *radial projection* R_T from any plane T , that is tangent to S , onto S by

$$R_T \bar{v} := v := \frac{\bar{v}}{|\bar{v}|} \in S, \quad \bar{v} \in T, \quad (2.5)$$

where $|v|$ denotes the Euclidean norm of v . Let $T_{i,j}$ be the tangent plane touching S at vertex $v_{i,j} \in \Delta_j$. Because the mapping $R_{T_{i,j}}$ is one-to-one, the inverse $R_{T_{i,j}}^{-1}$ is well defined. Define $\Delta_{i,j}$ as the 1-ring of vertex $v_{i,j}$ in Δ_j . Let $\bar{\Delta}_{i,j}$ be the image of $\Delta_{i,j}$ under $R_{T_{i,j}}^{-1}$. Since great circles are mapped onto straight lines under $R_{T_{i,j}}^{-1}$, $\bar{\Delta}_{i,j}$ consists of planar neighbouring triangles with one common vertex $v_{i,j}$. The spline

space $S_1^0(\bar{\Delta}_{i,j})$ is just the well-known bivariate linear spline space on the triangulation $\bar{\Delta}_{i,j}$. Let τ be a spherical triangle in $\Delta_{i,j}$ and denote its vertices by v_1, v_2, v_3 . Let $\mathcal{T}$ be the trihedron generated by $\{v_1, v_2, v_3\}$. Then for some $a \in \{1, 2, 3\}$ we have $v_a = v_{i,j}$ (since $\tau \in \Delta_{i,j}$) and

$$(\phi_{i,j})_1(w) = b_a(w)^i, \quad w \in \mathcal{T},$$

with (b_1, b_2, b_3) the trihedral coordinates of w with respect to $\mathcal{T}$. Consequently $(\phi_{i,j})_1$ equals zero at the vertices of $\bar{\Delta}_{i,j}$, except for vertex $v_{i,j} \in \bar{\Delta}_{i,j}$ we get $(\phi_{i,j})_1(v_{i,j}) = 1$. Since $\phi_{i,j} = (\phi_{i,j})_1|_S$ we have $\phi_{i,j}|_\tau \in \mathbb{H}_1(S)$. Let $\bar{\tau}$ be the image of τ under $R_{T_{i,j}}^{-1}$. We find that $(\phi_{i,j})_1|_{\bar{\tau}} \in \mathbb{H}_1(\bar{\tau})$ and thus $(\phi_{i,j})_1|_{T_{i,j}} \in S_1^0(\bar{\Delta}_{i,j})$. This proves that $(\phi_{i,j})_1|_{T_{i,j}}$ is just the well-known classical bivariate Courant element. $\square$

This idea can be exploited to extend several properties of the classical Courant elements to the spherical elements $\phi_{i,j}$, such as

$$0 \leq \phi_{i,j}(v) \leq 1 \quad , \quad v \in S.$$

The following lemma is obvious.

LEMMA 2.2 (Riesz L_∞ -stability). *The nodal basis functions $\{\phi_{i,j} \mid i = 1, \dots, N_j\}$ satisfy*

$$\left\| \sum_{i=1}^{N_j} c_{i,j} \phi_{i,j} \right\|_{L_\infty} \sim \max_i |c_{i,j}|.$$

Proof. There exists a triangle $\tau \in \Delta_j$ and a point $v \in \tau$ such that

$$\left\| \sum_{i=1}^{N_j} c_{i,j} \phi_{i,j} \right\|_{L_\infty} = \left| \sum_{i=1}^{N_j} c_{i,j} \phi_{i,j}(v) \right| \leq \max_i |c_{i,j}| \sum_{i \mid v_{i,j} \in \tau} \|\phi_{i,j}\|_{L_\infty} \lesssim \max_i |c_{i,j}|.$$

The other inequality follows from $|c_{k,j}| = \left| \sum_{i=1}^{N_j} c_{i,j} \phi_{i,j}(v_{k,j}) \right| \leq \left\| \sum_{i=1}^{N_j} c_{i,j} \phi_{i,j} \right\|_{L_\infty}$. $\square$

To derive the optimality of the BPX preconditioner we will need the following theorem.

THEOREM 2.3 (Riesz L_p -stability). *For any $1 < p < \infty$ we have*

$$\left\| \sum_{i=1}^{N_j} c_{i,j} \phi_{i,j} \right\|_{L_p}^p \sim 2^{-2j} \sum_{i=1}^{N_j} |c_{i,j}|^p.$$

Proof. Since we have already established Riesz L_∞ -stability of the basis (Lemma 2.2), the proof is identical to the corresponding proof for the classical Courant elements on the plane from [10]. $\square$

REMARK 2.4. *There exist other constructions of C^0 spherical finite elements in the literature. In [16] problem (1.1) is discretized by approximating the sphere S by a polyhedron S_h . Then linear elements on the surface S_h are used. In [6] spherical linear elements are created, but another definition for spherical barycentric coordinates is used. In [6] the spherical barycentric coordinates are required to form a partition of unity, and therefore they inevitably fail to have many of the important properties that the spherical barycentric coordinates of [2] have. The optimality proof of the BPX-preconditioner that we give in Section 3 only works for our construction.*

2.2. C^1 Powell–Sabin elements on the sphere. In general, maintaining C^1 continuity conditions (2.3) between neighbouring triangles results in non-trivial relations and is not always possible for arbitrary given triangulations, see, e.g., [17]. Therefore, to overcome this problem, we will focus on the *Powell–Sabin 6-split* of a triangulation. Starting from an arbitrary spherical triangulation Δ , we introduce further substructures by subdividing each triangle of Δ into 6 subtriangles in a prescribed way. Because of the special structure of this refined triangulation one introduces sufficient degrees of freedom to maintain overall C^1 continuity. The Powell–Sabin 6-split is obtained as follows:

1. Define for each triangle τ_k in Δ an interior point z_k such that, if two triangles τ_k and τ_l have a common edge (circle segment), then the arc that joins z_k and z_l intersects this common edge (circle segment) at a point r_{kl} between its vertices. The arc between two points on S is defined as the circle segment connecting these two points obtained as the intersection of S with a plane passing through the two points and the origin.
2. Join the points z_k to the vertices of τ_k .
3. For each edge (circle segment) of τ_k
 - that belongs to the boundary $\partial\Omega$, join z_k to some point of the edge.
 - that is common to a triangle τ_l , join z_k to r_{kl} .

Fig. 2.2 shows the split of one triangle. We will refer to this new triangulation as Δ^{PS} . The spline space $S_2^1(\Delta^{PS})$ of piecewise quadratic C^1 spherical polynomials over Δ^{PS} will be called the space of *spherical Powell–Sabin (PS) splines*. Let g_i and h_i be independent unit vectors lying in the tangent plane of S at the vertices v_i , $i = 1, \dots, N$ of the triangulation Δ . The following interpolation problem can be considered for spherical PS splines. Given any set of values $(\alpha_i, \beta_i, \gamma_i)$, $i = 1, \dots, N$, find $s(v) \in S_2^1(\Delta^{PS})$ such that

$$s(v_i) = \alpha_i, \quad \frac{\partial s(v_i)}{\partial g_i} = \beta_i, \quad \frac{\partial s(v_i)}{\partial h_i} = \gamma_i, \quad (2.6)$$

for all $i = 1, \dots, N$. Maes and Bultheel [23] have shown that this interpolation problem has a unique solution, hence the classical result of [28] can be extended to spherical domains, i.e. the dimension of the spherical spline space $S_2^1(\Delta^{PS})$ equals $3N$.

FIG. 2.2. The spherical Powell–Sabin macro-element.

In order to create nested spherical PS spline spaces

$$S_2^1(\Delta_0^{PS}) \subset S_2^1(\Delta_1^{PS}) \subset S_2^1(\Delta_2^{PS}) \subset \dots$$

it is sufficient that we find a refinement procedure that yields nested sequences

$$\begin{aligned} \Delta_0^{PS} &\subset \Delta_1^{PS} \subset \Delta_2^{PS} \subset \dots, \\ \{v_i \in \Delta_0\} &\subset \{v_i \in \Delta_1\} \subset \{v_i \in \Delta_2\} \subset \dots. \end{aligned} \quad (2.7)$$

It was pointed out by Vanraes *et al.* [30] that applying a $\sqrt{3}$ refinement scheme yields nested PS spline spaces. Applying the $\sqrt{3}$ scheme twice yields a triadic scheme. The $\sqrt{3}$ scheme was first introduced by Kobbelt [19] and Labsik and Greiner [21]. Instead of splitting each edge in Δ_0 and performing a 1-to-4 split for each triangle (dyadic refinement), we compute a new vertex for each triangle and retriangulate the old and new vertices. Fig. 2.3 shows the principle. Remark that the new edges in Δ_1 coincide with the lines of the PS 6-split Δ_0^{PS} . In the new triangles new interior points must be chosen on the one line of the new PS 6-split Δ_1^{PS} that is already fixed, that is, the original edge that crosses the triangle.

REMARK 2.5. *Although the $\sqrt{3}$ refinement is applicable to arbitrary (spherical) triangulations, it is not rigorously proven whether the corresponding sequence (2.7) satisfies the minimum angle condition and whether*

$$\text{diam } \tau \sim \sqrt{3}^{-j}, \quad \tau \in \Delta_j^{PS}, \quad j = 0, 1, \dots$$

In our mathematical analysis we will always assume that we are given a nested sequence (2.7) that is regular. By using a PS 12-split as in [26] instead of the PS 6-split one can obtain a provably regularly refined sequence of PS spline spaces by applying dyadic refinement. However we opt for the PS 6-split because the construction of the corresponding basis functions is less complicated, certainly on the sphere, and because the $\sqrt{3}$ refinement is a topologically slower refinement than the dyadic refinement, thus we have more levels of resolution if a prescribed target complexity of the PS spline space must not be exceeded.

With each vertex $v_{i,j} \in \Delta_j$ we associate two directions $g_{i,j}$ and $h_{i,j}$ such that the set $(v_{i,j}, g_{i,j}, h_{i,j})$ forms an orthonormal basis for $\mathbb{R}^3$. For instance, suppose that $v_{i,j}$ has spherical coordinates $(\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi)$, $\theta \in [0, 2\pi]$, $\phi \in [0, \pi]$, then take $g_{i,j} = (\cos \theta \cos \phi, \sin \theta \cos \phi, -\sin \phi)$ and $h_{i,j} = (-\sin \theta, \cos \theta, 0)$. Let us introduce the functionals

$$\lambda_{i,j}^1(f) := f(v_{i,j}), \quad \lambda_{i,j}^2(f) := \frac{\partial f(v_{i,j})}{\partial g_{i,j}}, \quad \lambda_{i,j}^3(f) := \frac{\partial f(v_{i,j})}{\partial h_{i,j}}, \quad f \in C^1(S).$$

Then we construct a nodal basis for $S_2^1(\Delta_j^{PS})$ by solving the following interpolation problems of the form (2.6): find functions $B_{i,j}^k \in S_2^1(\Delta_j^{PS})$, $k = 1, 2, 3$, $i = 1, \dots, N_j$, such that

$$\begin{aligned} \lambda_{m,j}^1(B_{i,j}^k) &= \sqrt{3}^{-j} \delta_{k,1} \delta_{i,m}, \\ \lambda_{m,j}^2(B_{i,j}^k) &= \delta_{k,2} \delta_{i,m}, \\ \lambda_{m,j}^3(B_{i,j}^k) &= \delta_{k,3} \delta_{i,m}, \end{aligned} \quad (2.8)$$

for all $m = 1, \dots, N_j$. Note that these basis functions satisfy $B_{i,j}^k \equiv (B_{i,j}^k)_2|_S$, i.e. the spherical basis function $B_{i,j}^k$ is equal to the restriction of its homogeneous extension

FIG. 2.3. Principle of $\sqrt{3}$ subdivision. Applying the $\sqrt{3}$ subdivision twice results in triadic subdivision.

(2.4) of degree 2 to the sphere S . If we restrict $(B_{i,j}^k)_2$ to the tangent plane touching S at $v_{i,j}$, we get the corresponding planar Hermite basis function of [29] defined on this tangent plane. The proof is similar to the proof of Theorem 2.1. A detailed proof in a more general setting can be found in [23, Theorem 4.1]. Fig. 2.4 shows the three spherical Hermite PS basis functions that are associated with one vertex.

FIG. 2.4. Graphs of $(B_{i,j}^k(v) + 1)v$ with $v \in S$ for $k = 1, 2, 3$ with $B_{i,j}^k$ the spherical Hermite PS basis function.

We now show some stability properties of the nodal basis (2.8) that will be useful in the optimality proof of the BPX preconditioner.

LEMMA 2.6 (Riesz L_∞ -stability). *The nodal basis defined by (2.8) satisfies*

$$\left\| \sum_{i=1}^{N_j} \sum_{k=1}^3 c_{i,j}^k B_{i,j}^k \right\|_{L_\infty} \sim \sqrt{3}^{-j} \max_{i,k} |c_{i,j}^k|.$$

Proof. First we note that this result is well-known for the classical bivariate Hermite basis of Powell–Sabin type on planar triangulations. Indeed, the inequality

$\gtrsim$ can be shown using the Markov inequality for polynomials ([9]), and the inequality $\lesssim$ can be deduced, for instance, from the work in [29, Section 6.2]. This result for the bivariate planar setting can be extended easily to the spherical setting by exploiting the fact that the restriction of $(B_{i,j}^k)_2$ to the tangent plane touching S at $v_{i,j}$ is a classical bivariate Hermite basis function. For a detailed proof, see [23, Corollary 4.2]. $\square$

THEOREM 2.7 (Riesz L_p -stability). *If s is in $S_2^1(\Delta_j^{PS})$, then for any $1 < p < \infty$ we have*

$$\|s\|_{L_p}^p \sim \sqrt{3}^{-2j} \left(\sum_{i=1}^{N_j} |\lambda_{i,j}^1(s)|^p + \sqrt{3}^{-jp} \sum_{i=1}^{N_j} \sum_{k=2}^3 |\lambda_{i,j}^k(s)|^p \right).$$

Proof. Using the Markov inequality for spherical polynomials ([25, Prop. 4.3]), we infer that $|\lambda_{i,j}^k(s)| \lesssim \sqrt{3}^j \|s\|_{L_\infty(\tau_i)}$ for $k = 2, 3$ with $\tau_i \in \Delta_j^{PS}$ such that $v_{i,j} \in \tau_i$. By mapping τ_i to a standard reference triangle and using the fact that all norms on the finite-dimensional space of polynomials are equivalent, we find that $\|s\|_{L_\infty(\tau_i)} \lesssim \sqrt{3}^{2j/p} \|s\|_{L_p(\tau_i)}$, which implies

$$\sqrt{3}^{-2j} \left(\sum_{i=1}^{N_j} |\lambda_{i,j}^1(s)|^p + \sqrt{3}^{-jp} \sum_{i=1}^{N_j} \sum_{k=2}^3 |\lambda_{i,j}^k(s)|^p \right) \lesssim \sum_{i=1}^{N_j} \sum_{k=1}^3 \|s\|_{L_p(\tau_i)}^p \lesssim \|s\|_{L_p}^p.$$

The other inequality follows from the observation that

$$\begin{aligned} |s(v)|^p &= \left| \sum_{i=1}^{N_j} \left(\sqrt{3}^j \lambda_{i,j}^1(s) B_{i,j}^1(v) + \sum_{k=2}^3 \lambda_{i,j}^k(s) B_{i,j}^k(v) \right) \right|^p \\ &\lesssim \sum_{i=1}^{N_j} \left(\sqrt{3}^{jp} |\lambda_{i,j}^1(s)|^p |B_{i,j}^1(v)|^p + \sum_{k=2}^3 |\lambda_{i,j}^k(s)|^p |B_{i,j}^k(v)|^p \right), \end{aligned}$$

which holds because at any $v \in S$ there are at most nine nonzero basis functions. We find that

$$\begin{aligned} \|s\|_{L_p}^p &\lesssim \sum_{i=1}^{N_j} \left(\sqrt{3}^{jp} |\lambda_{i,j}^1(s)|^p \int_S |B_{i,j}^1(v)|^p dv + \sum_{k=2}^3 |\lambda_{i,j}^k(s)|^p \int_S |B_{i,j}^k(v)|^p dv \right) \\ &\lesssim \sqrt{3}^{-2j} \left(\sum_{i=1}^{N_j} |\lambda_{i,j}^1(s)|^p + \sqrt{3}^{-jp} \sum_{i=1}^{N_j} \sum_{k=2}^3 |\lambda_{i,j}^k(s)|^p \right), \end{aligned}$$

where we have used that $\|B_{i,j}^k\|_{L_\infty} \lesssim \sqrt{3}^{-j}$ which follows from the Riesz L_∞ -stability of the basis (Lemma 2.6). $\square$

3. Construction of the BPX preconditioners. In this section we construct BPX preconditioners for the problems (1.3) and (1.4), and we prove that these preconditioners are optimal. Let $m \in \{1, 2\}$ and let $\rho \in \mathbb{R}$ be a scaling factor. If m equals 1 we define S_j as the spline space $S_1^0(\Delta_j)$, we set the scaling factor ρ equal to 2 and

we solve problem (1.3). If m equals 2 we define S_j as the spline space $S_2^1(\Delta_j^{PS})$ and we set the scaling factor ρ equal to $\sqrt{3}$, leading to problem (1.4).

Let $Q_j, j = 0, 1, \dots$, be a sequence of projectors on S_j which are orthogonal with respect to the inner product $(\cdot, \cdot)$ and let $Q_{-1} \equiv 0$. Let Ω be a subset of the sphere S and let $H^m(\Omega), H^m(S)$ be the spherical Sobolev spaces as defined in [22, 25]. We prove the following theorem.

THEOREM 3.1. *Suppose $s \in S_J$. Then*

$$\|s\|_{H^m(S)}^2 \sim \sum_{j=0}^J \rho^{2mj} \|(Q_j - Q_{j-1})s\|_{L_2(S)}^2. \quad (3.1)$$

Proof. Let Ω be a subset of S such that $\text{diam}(\Omega) \leq 1$. Let T_Ω be the tangent plane touching S at r_Ω , with r_Ω the center of a spherical cap of smallest possible radius containing Ω . Here a spherical cap is defined as the region of a sphere which lies on one side of a given plane that intersects with the sphere. Recall the definition of the radial projection R_{T_Ω} from (2.5). Let $\bar{\Omega}$ be the image of Ω under $R_{T_\Omega}^{-1}$ and define $H^m(\bar{\Omega})$ as the usual Sobolev space on domains in $\mathbb{R}^2$. Let $(s)_m$ be the homogeneous extension (2.4) of degree m of s , and define $\bar{s}$ as the restriction of $(s)_m$ to $\bar{\Omega}$. The norm equivalence $\|s\|_{H^m(\Omega)} \sim \|\bar{s}\|_{H^m(\bar{\Omega})}$ holds, see Lemma 3.2 in [25]. Furthermore, we also have $\|s\|_{L_2(\Omega)} \sim \|\bar{s}\|_{L_2(\bar{\Omega})}$, see Lemma 3.1 in [25]. Now let $s = \sum_{j=0}^J s_j$ with each $s_j \in S_j$. Then it follows from the theory of homogeneous polynomials ([2, 3]) that $(s)_m = \sum_{j=0}^J (s_j)_m$, hence $\bar{s} = \sum_{j=0}^J \bar{s}_j$ with $\bar{s}_j$ the restriction of $(s_j)_m$ to $\bar{\Omega}$. Furthermore, each $\bar{s}_j$ is a member of the planar spline space $\bar{S}_j$ which is defined as $S_1^0(R_{T_\Omega}^{-1}(\Delta_j|_\Omega))$ for $m = 1$ and as $S_2^1(R_{T_\Omega}^{-1}(\Delta_j^{PS}|_\Omega))$ for $m = 2$. Proposition 2 in [26] claims that

$$\|\bar{s}\|_{H^m(\bar{\Omega})} \sim \inf \sum_{j=0}^J \rho^{2mj} \|\bar{s}_j\|_{L_2(\bar{\Omega})}^2$$

where the infimum must be taken with respect to all admissible representations $\sum_{j=0}^J \bar{s}_j$ of $\bar{s}$. From the norm equivalences above, we get

$$\|s\|_{H^m(\Omega)} \sim \inf \sum_{j=0}^J \rho^{2mj} \|s_j\|_{L_2(\Omega)}^2. \quad (3.2)$$

Now consider a finite collection of domains Ω_k with $\text{diam}(\Omega_k) \leq 1$, covering S . Equation (3.2) is valid for each sub-domain Ω_k . Furthermore, we have the equivalences $\|s_j\|_{L_2(S)}^2 \sim \sum_k \|s_j\|_{L_2(\Omega_k)}^2$ and $\|s\|_{H^m(S)}^2 \sim \sum_k \|s\|_{H^m(\Omega_k)}^2$. Hence,

$$\|s\|_{H^m(S)} \sim \inf \sum_{j=0}^J \rho^{2mj} \|s_j\|_{L_2(S)}^2,$$

which immediately implies (3.1), see [15, 26]. $\square$

REMARK 3.2. *Proposition 2 in [26] is formulated in terms of C^1 finite elements but it is clear that a similar result holds for C^0 finite elements (with obvious modifications).*

In view of (1.7) let us define the selfadjoint positive definite operator C_J^{-1} on S_J by

$$(C_J^{-1}u, v) = \sum_{j=0}^J \rho^{2mj} ((Q_j - Q_{j-1})u, (Q_j - Q_{j-1})v), \quad (3.3)$$

and let A_J be the operator defined by (1.5) for $V = S_J$. By Poincaré's inequality on S , we have

$$a(u, u) \sim \|u\|_{H^m(S)}^2 \quad (3.4)$$

under the constraint $\int_S u \, d\omega = 0$. Then Theorem 3.1 and (1.7) imply that

$$\kappa(C_J^{1/2} A_J C_J^{1/2}) = \mathcal{O}(1) \quad (3.5)$$

under the constraint that we fix the solution u of (1.3), (1.4) such that $\int_S u \, d\omega = 0$.

We now replace C_J by a spectrally equivalent and computationally simpler preconditioner $\widehat{C}_J$ given by

$$\widehat{C}_J := \sum_{j=0}^J \sum_{i=1}^{N_j} (\cdot, \phi_{i,j}) \phi_{i,j} \quad (3.6)$$

for the problem (1.3) and by

$$\widehat{C}_J := \sum_{j=0}^J \sum_{i=1}^{N_j} \sum_{k=1}^3 (\cdot, B_{i,j}^k) B_{i,j}^k \quad (3.7)$$

for the problem (1.4). We say that two operators A and B are *spectrally equivalent* if

$$\frac{(Av, v)}{(v, v)} \sim \frac{(Bv, v)}{(v, v)}.$$

Note that, by (3.3), we have

$$C_J^{-1} := \sum_{j=0}^J \rho^{2mj} (Q_j - Q_{j-1}). \quad (3.8)$$

Indeed, by the L_2 -orthogonality of the operators Q_j we find

$$\begin{aligned} (C_J^{-1}u, v) &= \left(\sum_{j=0}^J \rho^{2mj} (Q_j - Q_{j-1})u, v \right) \\ &= \left(\sum_{j=0}^J \rho^{2mj} (Q_j - Q_{j-1})u, \sum_{j=0}^J (Q_j - Q_{j-1})v \right) \\ &= \sum_{j=0}^J \rho^{2mj} ((Q_j - Q_{j-1})u, (Q_j - Q_{j-1})v). \end{aligned}$$

Let us focus on the biharmonic problem (1.4), i.e. take $m = 2$. By the orthogonality of the projectors Q_j one finds from (3.8) that

$$C_J = \sum_{j=0}^J \sqrt{3}^{-4j} (Q_j - Q_{j-1}).$$

Because of the decaying scaling factors we are allowed to replace C_J by the spectrally equivalent operator

$$\tilde{C}_J := \sum_{j=0}^J \sqrt{3}^{-4j} Q_j.$$

From Theorem 2.7, we have the Riesz L_2 -stability

$$\left\| \sum_{i=1}^{N_j} \sum_{k=1}^3 c_{i,j}^k B_{i,j}^k \right\|_{L_2}^2 \sim \sqrt{3}^{-4j} \sum_{i=1}^{N_j} \sum_{k=1}^3 |c_{i,j}^k|^2, \quad (3.9)$$

and by the Riesz representation theorem this implies the existence of a dual or biorthogonal basis $\{\tilde{B}_{i,j}^k\}$ such that

$$\left\| \sum_{i=1}^{N_j} \sum_{k=1}^3 c_{i,j}^k \tilde{B}_{i,j}^k \right\|_{L_2}^2 \sim \sqrt{3}^{4j} \sum_{i=1}^{N_j} \sum_{k=1}^3 |c_{i,j}^k|^2. \quad (3.10)$$

The orthogonal projector Q_j has the representation

$$Q_j f = \sum_{i=1}^{N_j} \sum_{k=1}^3 (f, B_{i,j}^k) \tilde{B}_{i,j}^k.$$

Hence,

$$(Q_j f, f) = (Q_j f, Q_j f) = \|Q_j f\|_{L_2}^2 \sim \sqrt{3}^{4j} \sum_{i=1}^{N_j} \sum_{k=1}^3 |(f, B_{i,j}^k)|^2 = (\hat{Q}_j f, f),$$

with

$$\hat{Q}_j := \sqrt{3}^{4j} \sum_{i=1}^{N_j} \sum_{k=1}^3 (\cdot, B_{i,j}^k) B_{i,j}^k,$$

which shows that $\hat{C}_J$ defined in (3.7) is spectrally equivalent to C_J such that, by (3.5),

$$\kappa(\hat{C}_J^{1/2} A_J \hat{C}_J^{1/2}) = \mathcal{O}(1).$$

The optimality of the preconditioner defined in (3.6) for problem (1.3) can be derived analogously using Theorem 2.3. We have, thus, proved the main result of this paper.

THEOREM 3.3. *The BPX preconditioners given by*

$$\sum_{j=0}^J \sum_{i=1}^{N_j} (\cdot, \phi_{i,j}) \phi_{i,j} \quad \text{and} \quad \sum_{j=0}^J \sum_{i=1}^{N_j} \sum_{k=1}^3 (\cdot, B_{i,j}^k) B_{i,j}^k$$

yield uniformly bounded condition numbers for the problems (1.3), resp. (1.4).

COROLLARY 3.4. *Any basis of the general form in [23] which is stable in the sense of Theorem 2.7 gives rise to an optimal BPX preconditioner for (1.4).*

REMARK 3.5. *In the paper [18] Griebel shows that the conjugate gradient method for the semidefinite system that arises from the Galerkin scheme using the nodal basis functions of the finest level and of all coarser levels of discretization, is equivalent to the BPX-preconditioned conjugate gradient method for the linear system that arises from the Galerkin scheme using only the nodal basis functions of the finest level. In our numerical experiments we take the approach of the semidefinite system using the nodal basis functions at all resolution levels. For details on efficient implementation we refer to [18].*

REMARK 3.6. *The above derivation depends heavily on the use of a biorthogonal basis. Its existence is guaranteed by the Riesz representation theorem. Note the weight change from $\sqrt{3}^{-4j}$ in (3.9) for the finite element basis to $\sqrt{3}^{4j}$ in (3.10) for the biorthogonal basis. For properties of Riesz bases in connection with biorthogonality and multiresolution we refer to [11].*

4. Numerical results. In this section we provide the results of numerical experiments illustrating the optimality of the BPX preconditioners developed in the earlier sections. We also compare the results of the BPX preconditioners with those obtained using the corresponding hierarchical preconditioners which are suboptimal.

The first problem that we solve is given by

$$-\Delta_S u = 2x \quad \text{on } S, \quad (4.1)$$

and the exact solution u equals x , which can easily be checked since spherical harmonics are eigenfunctions of the Laplace–Beltrami operator on S ([24]). To discretize the problem (4.1) we use the basis functions $\phi_{i,j}$. We start from an almost uniform triangulation Δ_0 by projecting the twelve vertices of the regular icosahedron onto the sphere. These twelve points define a mesh consisting of twenty equal spherical triangles, cfr. [6]. The finer triangulations Δ_j are constructed by subdividing the triangles of the previous coarser triangulation into four equal subtriangles. Hence the dimension of the spline space increases like $2 + 10 \cdot 4^j$ with the refinement level j . Inner products of the form $(\nabla_S \phi_{i_1,j}, \nabla_S \phi_{i_2,j})$ will have to be computed. Hereto, we use a 3th order Gaussian quadrature formula on a triangle, see also [4, Prop. 4.1].

The second problem that we solve is given by

$$\Delta_S^2 u = 36xy \quad \text{on } S, \quad (4.2)$$

and the exact solution u equals xy . In order to discretize (4.2) we have to compute inner products of the form $(\Delta_S B_{i_1,j}^{k_1}, \Delta_S B_{i_2,j}^{k_2})$. Since the basis functions $B_{i,j}^k$ are piecewise quadratic polynomials, we can use the formula

$$\Delta_S B_{i,j}^k(v) = \Delta B_{i,j}^k(v) - 6B_{i,j}^k(v), \quad v \in S,$$

with Δ the usual Laplace operator on $\mathbb{R}^3$, see [24]. Then, to evaluate the inner products, we use again a 3th order Gaussian quadrature formula on a triangle. We show results for the $\sqrt{3}$ refinement procedure where we start from the same quasi-uniform triangulation Δ_0 as in the first problem (4.1). The dimension of the spline space increases like $6 + 30 \cdot 3^j$ with the refinement level j .

Note that the solution u in (4.1) and (4.2) is only unique up to a constant. From [2, Prop. 7.2] we find that constant functions on the sphere are contained in the

spherical Powell–Sabin spline space $S_2^1(\Delta_j^{PS})$ but not in the spherical piecewise linear spline space $S_1^0(\Delta_j)$. Hence, the stiffness matrix corresponding to the nodal basis $\{B_{i,j}^k\}$ will have one zero eigenvalue with an eigenvector corresponding to the constant function. The stiffness matrix corresponding to the nodal basis $\{\phi_{i,j}\}$ will have an eigenvalue of $\mathcal{O}(h^2)$ with an eigenvector that approximates the constant function up to discretization error $\mathcal{O}(h^2)$ w.r.t the L_2 norm. Note that the condition numbers that we compute are given by $\kappa(C^{1/2}AC^{1/2}) = \lambda_{\max}/\lambda_{\min}$ where $\lambda_{\max}$ denotes the largest eigenvalue of $C^{1/2}AC^{1/2}$ and $\lambda_{\min}$ its smallest nonzero eigenvalue. For obvious reasons we also omit the smallest eigenvalue of $\mathcal{O}(h^2)$ for the Poisson equation. Note that, from Theorem 3.3 and Remark 3.5, the BPX preconditioner uses all nodal basis functions on all levels. For each redundant basis function we will get a zero eigenvalue.

Tables 4.1 and 4.2 show the results. We have used a *nested iteration conjugate gradient method* to solve the problem, i.e. by means of an outer iteration loop going from a coarse resolution level to the finest resolution level we compute the solution to (4.1) or (4.2) at each level with the BPX-preconditioned conjugate gradient method and we use the solution obtained at the previous coarser level as an initial guess. At each level we stop the conjugate gradient iteration if the H^m -norm of the residual is proportional to the discretization error which is of $\mathcal{O}(h)$. In [13] arguments are given for the fact that nested iteration is an asymptotically optimal method in the sense that it provides the solution u at the finest resolution level J up to discretization error in an overall amount of $\mathcal{O}(N_J)$ operations, provided that an optimal preconditioner is used.

dim	J	BPX			HB		
		κ	residual	#iter	κ	residual	#iter
42	1	3.1	2.4897e-05	12	7.6	2.4974e-05	17
162	2	3.7	1.6766e-05	9	10.7	1.9546e-05	16
642	3	4.6	4.7350e-06	11	15.2	8.6198e-06	20
2562	4	5.5	4.5474e-06	11	22.2	5.2361e-06	22
10242	5	6.2	1.6705e-06	12	31.9	3.0622e-06	23
40962	6	6.7	1.0193e-06	12	44.9	1.4750e-06	25
163842	7	7.0	6.2720e-07	12	60.9	6.5043e-07	26
655362	8	7.4	1.6451e-07	13	84.2	3.4960e-07	24

TABLE 4.1
Iteration history for problem (4.1).

dim	J	BPX			HB		
		κ	residual	#iter	κ	residual	#iter
96	1	51.0	3.0555e-11	10	60.7	1.5555e-03	5
276	2	65.7	8.7509e-04	7	82.0	4.9572e-04	9
816	3	79.5	3.9398e-04	8	103.7	5.4025e-04	15
2436	4	88.4	2.6345e-04	11	123.6	2.5576e-04	20
7296	5	96.8	1.7065e-04	11	152.0	1.8184e-04	26
21876	6	103.4	8.9656e-05	13	192.1	1.0873e-04	31
65616	7	107.7	5.0634e-05	13	237.0	6.3035e-05	36
196836	8	110.2	3.2635e-05	14	310.7	3.5946e-05	44

TABLE 4.2
Iteration history for problem (4.2).

Each table has the same setup. The first column shows the dimension of the spline space and the second column contains the resolution level J . Then we distinguish between the results for the BPX preconditioner and the results for the hierarchical basis

(HB) preconditioner. For each preconditioner we display the spectral condition number κ of the system matrix for the linear system of equations that is solved. Moreover we show the H^m -norm of the residuals corresponding to the approximate solution, and the number of iterations that are needed on this level to reach discretization error accuracy.

REMARK 4.1. *Computing the H^m -norm of the residual is easy. Let us concentrate on problem (1.2). We have that*

$$\begin{aligned} \|u_J - u\|_{H^2}^2 &\sim \|Au_J - f\|_{(H^2)'}^2 \\ &= \left\| \sum_{j=0}^J \sum_{i=1}^{N_j} \sum_{k=1}^3 \langle B_{i,j}^k, A(u_j - u) \rangle \tilde{B}_{i,j}^k \right\|_{(H^2)'}^2 \\ &\sim \sum_{j=0}^J \sum_{i=1}^{N_j} \sum_{k=1}^3 |\langle B_{i,j}^k, A(u_j - u) \rangle|^2 \\ &= \sum_{j=0}^J \sum_{i=1}^{N_j} \sum_{k=1}^3 |\langle B_{i,j}^k, Au_j \rangle - \langle B_{i,j}^k, f \rangle|^2. \end{aligned}$$

Here $\{\tilde{B}_{i,j}^k\}$ is the dual frame to $\{B_{i,j}^k\}$. The first equivalence is due to the ellipticity of the operator A . The second equivalence is because the dual frame is a Riesz frame for the dual function space $(H^2)'$. The last expression is just the l_2 norm of the residual of the system (1.8) with respect to the frame $\{B_{i,j}^k\}$ (see also Remark 3.5). This trick only works for elliptic PDEs, because the first equivalence above makes use of the ellipticity condition (3.4).

REFERENCES

- [1] B. AKSOYLU, A. KHODAKOVSKY, AND P. SCHRÖDER, *Multilevel solvers for unstructured surface meshes*, SIAM J. Sci. Comput., 26 (2005), pp. 1146–1165.
- [2] P. ALFELD, M. NEAMTU, AND L. L. SCHUMAKER, *Bernstein–Bézier polynomials on spheres and sphere-like surfaces*, Comput. Aided Geom. Design, 13 (1996), pp. 333–349.
- [3] ———, *Dimension and local bases of homogeneous spline spaces*, SIAM J. Math. Anal., 27 (1996), pp. 1482–1501.
- [4] ———, *Fitting scattered data on sphere-like surfaces using spherical splines*, J. Comput. Appl. Math., 73 (1996), pp. 5–43.
- [5] T. AUBIN, *Nonlinear Analysis on Manifolds: Monge–Ampère Equations*, Springer-Verlag, New York, 1982.
- [6] J. R. BAUMGARDNER AND P. O. FREDERICKSON, *Icosahedral discretization of the two-sphere*, SIAM J. Numer. Anal., 22 (1985), pp. 1107–1115.
- [7] J. H. BRAMBLE, J. E. PASCIAK, AND J. XU, *Parallel multilevel preconditioners*, Math. Comp., 55 (1990), pp. 1–22.
- [8] D. CASTAÑO, G. CONSTANTINESCU, A. KUNOTH, AND W. D. SCHUH, *Approximate continuation of harmonic functions*, (2006). Manuscript, in preparation.
- [9] CH. COATMÉLEC, *Approximation et interpolation des fonctions différentiables de plusieurs variables*, Ann. Sci. Ecole Normale Sup., 83 (1966), pp. 271–341.
- [10] R. COURANT, *Variational methods for the solution of problems of equilibrium and vibrations*, Bull. Amer. Math. Soc., 49 (1943), pp. 1–23.
- [11] W. DAHMEN, *Some remarks on multiscale transformations, stability and biorthogonality*, in Curves and surfaces, P. J. Laurent, A. Le Méhauté, and L. L. Schumaker, eds., Academic Press, 1994.
- [12] W. DAHMEN AND A. KUNOTH, *Multilevel preconditioning*, Numer. Math., 63 (1992), pp. 315–344.
- [13] W. DAHMEN, A. KUNOTH, AND R. SCHNEIDER, *Wavelet least squares methods for boundary value problems*, SIAM J. Numer. Anal., 39 (2002), pp. 1985–2013.

- [14] W. DAHMEN, P. OSWALD, AND X.-Q. SHI, C^1 -hierarchical bases, J. Comput. Appl. Math., 51 (1994), pp. 37–56.
- [15] R. A. DEVORE AND V. A. POPOV, *Interpolation of Besov spaces*, Trans. Amer. Math. Soc., 305 (1988), pp. 397–414.
- [16] G. DZIUK, *Finite elements for the Beltrami operator on arbitrary surfaces*, in Partial differential equations and calculus of variations, S. Hildebrandt and R. Leis, eds., Lecture Notes in Mathematics 1357, Springer, Berlin, 1988, pp. 142–155.
- [17] G. FARIN, *Curves And Surfaces For Computer Aided Geometric Design: A Practical Guide*, Academic Press, Boston, 4th ed., 1997.
- [18] M. GRIEBEL, *Multilevel algorithms considered as iterative methods on semidefinite systems*, SIAM J. Sci. Comput., 15 (1994), pp. 547–565.
- [19] L. KOBBELT, $\sqrt{3}$ -subdivision, in Computer Graphics Proceedings, Annual Conference Series, ACM SIGGRAPH, 2000.
- [20] A. KUNOTH, *Multilevel Preconditioning*, PhD thesis, Verlag Shaker, Aachen, 1994.
- [21] U. LABSIK AND G. GREINER, *Interpolatory $\sqrt{3}$ -subdivision*, Comput. Graph. Forum, 19 (2000), pp. 131–138.
- [22] J. L. LIONS AND E. MAGENES, *Non-Homogeneous Boundary Value Problems and Applications, Vol. I*, Springer-Verlag, New York, 1972.
- [23] J. MAES AND A. BULTHEEL, *A hierarchical basis preconditioner for the biharmonic equation on the sphere*, IMA J. Numer. Anal., 26 (2006), pp. 563–583.
- [24] C. MÜLLER, *Spherical harmonics*, Springer Lecture Notes in Mathematics, Vol. 17, (1966).
- [25] M. NEAMTU AND L. L. SCHUMAKER, *On the approximation order of splines on spherical triangulations*, Adv. in Comp. Math., 21 (2004), pp. 3–20.
- [26] P. OSWALD, *Hierarchical conforming finite element methods for the biharmonic equation*, SIAM J. Numer. Anal., 29 (1992), pp. 1610–1625.
- [27] ———, *On discrete norm estimates related to multilevel preconditioners in the finite element method*, in Constructive Theory of Functions, K. G. Ivanov, P. Petrushev, and B. Sendov, eds., Proc. Int. Conf. Varna, 1991, Bulg. Acad. Sci., Sofia, 1992, pp. 203–214.
- [28] M. J. D. POWELL AND M. A. SABIN, *Piecewise quadratic approximations on triangles*, ACM Trans. on Math. Software, 3 (1977), pp. 316–325.
- [29] P. SABLONNIÈRE, *Error bounds for Hermite interpolation by quadratic splines on an α -triangulation*, IMA J. Numer. Anal., 7 (1987), pp. 495–508.
- [30] E. VANRAES, J. WINDMOLDERS, A. BULTHEEL, AND P. DIERCKX, *Automatic construction of control triangles for subdivided Powell–Sabin splines*, Comput. Aided Geom. Design, 21 (2004), pp. 671–682.
- [31] H. YSERENTANT, *On the multi-level splitting of finite element spaces*, Numer. Math., 49 (1986), pp. 379–412.