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Convergence of modified approximants associated with orthogonal rational functions.

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Abstract

Let $\{\alpha_n\}$ be a sequence in the unit disk $\mathbf{D} = \{z \in \mathbf{C} : | z | < 1\}$ consisting of a finite number of points cyclically repeated, and let $\mathcal{L}$ be the linear space generated by the functions $B_n(z) = \prod_{k=0}^{n} -\frac{\alpha_k}{| \alpha_k |} \frac{(z - \alpha_k)}{(1 - \bar{\alpha}_k z)}$. Let $\{\varphi_n(z)\}$ be orthogonal rational functions obtained from the sequence $\{B_n(z)\}$ (orthogonalization with respect to a given functional on $\mathcal{L}$), and let $\{\psi_n(z)\}$ be the corresponding functions of the second kind (with superstar transforms $\varphi_n^*(z)$ and $\psi_n^*(z)$ respectively). Interpolation and convergence properties of the modified approximants $R_n(z, u_n, v_n) = \frac{u_n \psi_n(z) - v_n \psi_n^*(z)}{u_n \varphi_n(z) + v_n \varphi_n^*(z)}$ that satisfy $|u_n| = |v_n|$, are discussed.

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1 Preliminaries

We shall use the notation $\boldsymbol{T} = \{z \in \boldsymbol{C} : |z| = 1\}$, $\boldsymbol{D} = \{z \in \boldsymbol{C} : |z| < 1\}$ for the unit circle and the unit disk. The kernel $D(t, z)$ is defined by

$$D(t, z) = \frac{t + z}{t - z}. \tag{1.1}$$

Let μ be a finite Borel measure on $[-\pi, \pi]$. The integral transform Ω_μ is defined as the Carathéodory function

$$\Omega_\mu(z) = \int_{\boldsymbol{T}} D(t, z) d\mu(t). \tag{1.2}$$

(We use the simplified notation above for $\int_{-\pi}^{\pi} D(e^{i\theta}, z) d\mu(\theta)$, and analogously in similar cases.)

The real part of a Carathéodory function is a positive harmonic function in $\boldsymbol{D}$, and vice versa. (Recall Riesz-Herglotz representation theorem. Note that the real part of the kernel $D(t, z)$ is the Poisson kernel.)

The substar conjugate f_* of a function f is defined as

$$f_*(z) = \overline{f(1/\bar{z})}. \tag{1.3}$$

When f is a rational function or a series expansion, this may also be written as

$$f_*(z) = \bar{f}(1/z) \tag{1.4}$$

where the bar denotes conjugation of the coefficients. The inner product $<,>_\mu$ is defined on $\boldsymbol{C}(\boldsymbol{T}) \times \boldsymbol{C}(\boldsymbol{T})$ by

$$< f, g >_\mu = \int_{\boldsymbol{T}} f(t)\overline{g(t)} d\mu(t) = \int_{\boldsymbol{T}} f(t) g_*(t) d\mu(t). \tag{1.5}$$

Let $\{\alpha_n : n = 1, 2, ...\}$ be an arbitrary sequence of (not necessarily distinct) points (interpolation points) in $\boldsymbol{D}$. We define the Blaschke factor $\zeta_n(z)$ as the function

$$\zeta_n(z) = \frac{\overline{\alpha_n}}{|\alpha_n|} \cdot \frac{(\alpha_n - z)}{(1 - \overline{\alpha}_n z)}, \qquad n = 1, 2, \ldots. \tag{1.6}$$

(Here $\frac{\overline{\alpha_n}}{|\alpha_n|} = -1$ if $\alpha_n = 0$.)
We also define

$$\pi_0(z) = 1, \quad \pi_n(z) = \prod_{k=1}^{n} (1 - \overline{\alpha_k} z), \qquad n = 1, 2, ..., \tag{1.7}$$

$$\omega_0(z) = 1, \ \ \omega_n(z) = \prod_{k=1}^{n}(z - \alpha_k), \ \ n = 1, 2, \ldots. \tag{1.8}$$

The Blaschke products $B_n(z)$ are defined by

$$B_0(z) = 1, \ \ B_n(z) = \prod_{k=1}^{n} \zeta_k(z) = \eta_n \frac{\omega_n(z)}{\pi_n(z)}, \ \ n = 1, 2, \ldots, \tag{1.9}$$

where

$$\eta_n = (-1)^n \prod_{k=1}^{n} \frac{\overline{\alpha_k}}{|\alpha_k|}. \tag{1.10}$$

We shall also make use of the functions $B_{n\backslash k}(z)$ defined by

$$B_{n\backslash n}(z) = 1, \ \ B_{n\backslash k}(z) = B_n(z)/B_k(z) = \prod_{j=k+1}^{n} \zeta_j(z) \ \text{ for } \ 0 \leq k < n, \ \ n = 1, 2, \ldots. \tag{1.11}$$

(The product means the constant 1 when $k = n$.)

We define the spaces $\mathcal{L}_n$ and $\mathcal{L}_{n*}$ by

$$\mathcal{L}_n = Span\{B_k : k = 0, 1, \ldots, n\} \tag{1.12}$$

$$\mathcal{L}_{n*} = \{f_* : f \in \mathcal{L}_n\}, \tag{1.13}$$

and set $\mathcal{L} = \cup_{n=0}^{\infty} \mathcal{L}_n, \ \ \mathcal{L}_* = \cup_{n=0}^{\infty} \mathcal{L}_{n*}$.
We may then write

$$\mathcal{L}_n = \{\frac{p_n(z)}{\pi_n(z)} : \ p_n \in \Pi_n\} \tag{1.14}$$

$$\mathcal{L}_{n*} = \{\frac{q_n(z)}{\omega_n(z)} : \ q_n \in \Pi_n\} \tag{1.15}$$

where Π_n denotes the space of all polynomials of degree at most n.

For $f_n \in \mathcal{L}_n$ we define its superstar conjugate f_n^* by

$$f_n^*(z) = B_n(z) f_{n*}(z). \tag{1.16}$$

Note that this transformation depends on n. It must be clear from the context what n is. Also note that when $f_n \in \mathcal{L}_n$ then $f_n^* \in \mathcal{L}_n$.

The theory of the function spaces described above is connected with the Nevanlinna-Pick interpolation problem with interpolation points $\{\alpha_n\}$ (cf. [16,17]). These function spaces were introduced by Djrbashian in 1969 (see [11]), and independently in [1,2,10]. The theory has recently been further developed in [3,4,6,8]. (Cf. also [14].) For connections between Nevanlinna-Pick interpolation and system theory, see [9].

We shall in this paper mainly be concerned with a special case, which we shall call the cyclic case. In this case the sequence $\{\alpha_n\}$ consists of a finite number p of points cyclically repeated. Thus $\alpha_{qp+k} = \alpha_k$ for $k = 1, ..., p$, $q = 0, 1, 2, ...$. For more details on the cyclic case see [5,7,12].

When all the interpolation points coalesce at the origin, the space $\mathcal{L}$ reduces to the space of polynomials, and the orthogonal rational functions in $\mathcal{L}$ (see Section 2) are orthogonal polynomials, Szegö polynomials. For a survey of this special situation, see e.g. [13].

2 Orthogonal rational functions

Let the sequence $\{\varphi_n : n = 0, 1, 2, ...\}$ be obtained by orthonormalization of the sequence $\{B_n : n = 0, 1, 2, ...\}$ with respect to $<,>_\mu$. These functions are uniquely determined by the requirement that the leading coefficient κ_n in

$$\varphi_n(z) = \sum_{k=0}^{n} \kappa_k B_k(z) \tag{2.1}$$

is positive. We then have $\kappa_n = \varphi_n^*(\alpha_n)$. The following orthogonality properties are valid

$$< f, \varphi_n >_\mu = 0 \quad \text{for} \quad f \in \mathcal{L}_{n-1}, \tag{2.2}$$

$$< g, \varphi_n^* >_\mu = 0 \quad \text{for} \quad g \in \zeta_n \mathcal{L}_{n-1}. \tag{2.3}$$

(See [3,4].) We define the functions $\varphi_n(z, u, v)$ by

$$\varphi_n(z, u, v) = u\varphi_n(z) + v\varphi_n^*(z), \quad u, v \in \boldsymbol{C}, \ (u, v) \neq (0, 0). \tag{2.4}$$

We note that $\varphi_n(z, u, v)$ belongs to $\mathcal{L}_n$ (as function of z). We call these functions paraorthogonal when $|u| = |v|$.

We define the functions ψ_n of the second kind by

$$\psi_0(z) = 1, \quad \psi_n(z) = \int_T D(t, z)[\varphi_n(t) - \varphi_n(z)]d\mu(t), \quad n = 1, 2, ... \tag{2.5}$$

For the functions ψ_n and ψ_n^* various equivalent expressions can be given. Let us recall the following result (see [3,4]):

Theorem 2.1 *For $n = 1, 2, ...$ the following formulas are valid:*

$$\psi_n(z) = \int_{\boldsymbol{T}} D(t,z)[\frac{B_k(z)}{B_k(t)}\varphi_n(t) - \varphi_n(z)]d\mu(t), \quad k = 0, 1, ..., n-1, \tag{2.6}$$

$$\psi_n^*(z) = -\int_{\boldsymbol{T}} D(t,z)[\frac{B_{n\backslash k}(z)}{B_{n\backslash k}(t)}\varphi_n^*(t) - \varphi_n^*(z)], \quad k = 0, 1, ..., n-1. \tag{2.7}$$

We shall next prove a result valid in the cyclic situation.

Theorem 2.2 *In the cyclic case with p points the following formulas are valid for $n = p+1, \ p+2, ...$:*

$$\psi_n(z) = \int_{\boldsymbol{T}} D(t,z)[\frac{B_{n\backslash qp}(z)}{B_{n\backslash qp}(t)}\varphi_n(t) - \varphi_n(z)]d\mu(t) \quad \textit{where} \quad qp < n \tag{2.8}$$

$$\psi_n^*(z) = -\int_{\boldsymbol{T}} D(t,z)[\frac{B_{qp}(z)}{B_{qp}(t)}\varphi_n^*(t) - \varphi_n^*(z)]d\mu(t) \quad \textit{where} \quad qp < n. \tag{2.9}$$

Proof:

We may write $B_{n\backslash qp}(z) = \prod_{j=n-qp+1}^{n} \zeta_j(z) = \prod_{j=1}^{qp} \zeta_j(z) = B_{qp}(z)$. The results now follow by using $k = qp$ in (2.6) - (2.7). □

We define the functions $\psi_n(z,u,v)$ of the second kind by

$$\psi_n(z,u,v) = u\psi_n(z) - v\psi_n^*(z), \quad u,v \in \boldsymbol{C}, \quad (u,v) \neq (0,0). \tag{2.10}$$

Theorem 2.3 *In the cyclic case with p points the following formulas are valid for $n = p+1, p+2, ...$:*

$$\psi_n(z,u,v) = \int_{\boldsymbol{T}} D(t,z)[\frac{B_{qp}(z)}{B_{qp}(t)}\varphi_n(t,u,v) - \varphi_n(z,u,v)]d\mu(t), \quad \textit{where} \quad qp < n, \tag{2.11}$$

$$\psi_n(z,u,v) = \int_{\boldsymbol{T}} D(t,z)[\frac{B_{n\backslash qp}(z)}{B_{n\backslash qp}(t)}\varphi_n(t,u,v) - \varphi_n(z,u,v)]d\mu(t), \quad \textit{where} \quad qp < n. \tag{2.12}$$

Proof:

Follows by combining (2.7) and (2.8) (resp. (2.6) and (2.9)) for the situation $k = qp$. □

3 Interpolation by rational approximants

We shall in this section study interpolation properties of the rational functions

$$R_n(z,u,v) = \frac{\psi_n(z,u,v)}{\varphi_n(z,u,v)} \tag{3.1}$$

given by (2.4) and (2.10) to the function $-\Omega_\mu(z)$ defined in (1.2).
Let us recall the following result (see [8]):

Theorem 3.1 *The function $\Omega_\mu(z)$ has in $\boldsymbol{D}$ the following Newton series expansion*

$$\Omega_\mu(z) = [\mu_0 + 2\sum_{m=1}^{\infty} \mu_m z\omega_{m-1}(z)], \tag{3.2}$$

where the general moments μ_m are given by

$$\mu_m = \int_{\boldsymbol{T}} \frac{d\mu(t)}{\omega_m(t)}, \quad m = 0, 1, 2, \dots . \tag{3.3}$$

In the following we shall use the notation $q(n), r(n)$ as defined below:

$$n = q(n)p + r(n), \quad r(n) \in \{1, ..., p\}. \tag{3.4}$$

Theorem 3.2 *The rational function $R_n(z,u,v)$ interpolates the function $-\Omega_\mu(z)$ in the sense that for $n > p$:*

$$\psi_n(z,u,v) \; + \; \varphi_n(z,u,v)\Omega_\mu(z) = f_n(z) z\omega_{n-1}(z) \tag{3.5}$$

where $f_n(z)$ is analytic in $\boldsymbol{D}$.

Proof:

One can easily establish the identity

$$1 + 2\sum_{m=1}^{n-1} \frac{z\omega_{m-1}(z)}{\omega_m(t)} = \frac{t+z}{t-z}[1 - \frac{z\omega_{n-1}(z)}{t\omega_{n-1}(t)}] - \frac{z\omega_{n-1}(z)}{t\omega_{n-1}(t)}. \tag{3.6}$$

Hence, after integrating (3.6) with measure μ, we get

$$\mu_0 + 2\sum_{m=1}^{n-1} \mu_m z\omega_{m-1}(z) = \int_{\boldsymbol{T}} \{D(t,z)[1 - \frac{z\omega_{n-1}(z)}{t\omega_{n-1}(t)}] - \frac{z\omega_{n-1}(z)}{t\omega_{n-1}(t)}\} d\mu(t). \tag{3.7}$$

By combining (2.11) and (3.7) we then obtain (since $q(n)p < n$)

$$\psi_n(z,u,v) + \varphi_n(z,u,v)[\mu_0 + 2\sum_{m=1}^{n-1} \mu_m z\omega_{m-1}(z)] \tag{3.8}$$
$$= \int_{\boldsymbol{T}} D(t,z)[\frac{B_{q(n)p}(z)}{B_{q(n)p}(t)}\varphi_n(t,u,v) - \frac{z\omega_{n-1}(z)}{t\omega_{n-1}(t)}\varphi_n(z,u,v)]d\mu(t)$$
$$-\varphi_n(z,u,v)z\omega_{n-1}(z)\int_{\boldsymbol{T}} \frac{1}{t\omega_{n-1}(t)}d\mu(t)$$

and hence

$$\psi_n(z,u,v) + \varphi_n(z,u,v)[\mu_0 + 2\sum_{m=1}^{n-1} \mu_m z\omega_{m-1}(z)] \tag{3.9}$$
$$= -\mu_n'\varphi_n(z,u,v)z\omega_{n-1}(z) + \omega_{q(n)p}(z)\sigma_n(z),$$

where

$$\mu_n' = \int_{\boldsymbol{T}} \frac{1}{t\omega_{n-1}(t)}d\mu(t) \tag{3.10}$$

and

$$\sigma_n(z) = \int_{\boldsymbol{T}} D(t,z)[\frac{\pi_{q(n)p}(t)}{\pi_{q(n)p}(z)\omega_{q(n)p}(t)}\varphi_n(t,u,v) \tag{3.11}$$
$$-\frac{z\prod_{k=q(n)p+1}^{n-1}(z-\alpha_k)}{t\omega_{n-1}(t)}\varphi_n(z,u,v)]d\mu(t).$$

(If $q(n)p = n-1$, the product means the constant 1.)

We are going to prove that $\sigma_n(\alpha_k) = 0$ for $q(n)p+1 \le k \le n-1$.
Let $q(n)p+1 \le k \le n-1$, if $n(q) < n-1$. Then

$$\sigma_n(\alpha_k) = \frac{1}{\pi_{q(n)p}(\alpha_k)}\int_{\boldsymbol{T}} D(t,\alpha_k)\frac{\pi_{q(n)p}(t)}{\omega_{q(n)p}(t)}\varphi_n(t,u,v)d\mu(t). \tag{3.12}$$

We note that $D(t,\alpha_k)\left[\frac{\pi_{q(n)p}(t)}{\omega_{q(n)p}(t)}\right]_* = c\frac{1+\overline{\alpha}_k t}{1-\overline{\alpha}_k t}\frac{\omega_{q(n)p}(t)}{\pi_{q(n)p}(t)} = c\zeta_n(t)L(t)$, where $L(t) \in \mathcal{L}_{n-1}$ and c is a constant, while also $D(t,\alpha_k)\frac{\omega_{q(n)p}(t)}{\pi_{q(n)p}(t)} \in \mathcal{L}_{n-1}$. (Note that $\frac{(1+\overline{\alpha}_k t)\omega_{q(n)p}(t)}{(1-\overline{\alpha}_k t)\pi_{q(n)p}(t)} = \frac{(t-\alpha_k)s_{q(n)p}(t)}{(1-\overline{\alpha}_k t)\pi_{q(n)p}(t)}$ where $s_{q(n)p}(t)$ is a polynomial of degree $q(n)p$, that $(1-\overline{\alpha}_k t)\pi_{q(n)p}(t)$ is a factor in $\pi_n(t)$, and that $(t-\alpha_k)$ is a factor in $\omega_{q(n)p}(t)$.) Thus $\left[\frac{\pi_{q(n)p}(t)}{\omega_{q(n)p}(t)}\right]_* \in \mathcal{L}_{n-1}\cap\zeta_n\mathcal{L}_{n-1}$, and hence

$$\sigma_n(\alpha_k) = \frac{1}{\pi_{q(n)p}(\alpha_k)} < \varphi_n(t,u,v), [\frac{\pi_{q(n)p}(t)}{\omega_{q(n)p}(t)}]_* >_\mu = 0. \tag{3.13}$$

Analogously we find $\sigma_n(0) = 0$.

We have now seen that the second term of the right side of (3.9) in addition to having the factor $\omega_{q(n)p}(z)$ also has the extra factor z and the extra factors $(z - \alpha_k)$ for $q(n)p + 1 \leq k \leq n - 1$ (since $\sigma_n(0)$ and $\sigma_n(\alpha_k) = 0$ for the values of k indicated).

It follows that the second term on the right of (3.9) is of the form $A_n(z)z\omega_{n-1}(z)$. Thus

$$\psi_n(z,u,v) + \varphi_n(z,u,v)[\mu_0 + 2\sum_{m=1}^{n-1} \mu_m z\omega_{m-1}(z)] = g_n(z)z\omega_{n-1}(z), g_n(z) \text{ analytic.} \tag{3.14}$$

Since

$$\Omega_\mu(z) + [\mu_0 + 2\sum_{m=1}^{n-1} \mu_m z\omega_{m-1}(z)] = h_n(z)z\omega_{n-1}(z), \quad h_n(z) \text{ analytic,} \tag{3.15}$$

we conclude that (3.5) holds. □

4 Convergence of rational approximants

We recall that we call the function $\varphi_n(z,u,v)$ paraorthogonal when $|u| = |v|$. Paraorthogonal functions give rise to quadrature formulas. Let us recall the following result (see [3,6]):

Theorem 4.1 *The zeros of $\varphi_n(z,u,v)$ for $|u| = |v|$ are all simple and lie on $\boldsymbol{T}$. Let the zeros be denoted by $\xi_k^{(n)}(u,v)$, $k = 1, ..., n$. Then there exist positive constants $\lambda_k^{(n)}(u,v)$ such that the quadrature formula*

$$\int_{\boldsymbol{T}} L(t)d\mu(t) = \sum_{k=1}^{n} \lambda_k^{(n)}(u,v)L(\xi_k^{(n)}(u,v)) \tag{4.1}$$

is valid for $L \in \mathcal{L}_{n-1} + \mathcal{L}_{(n-1)*}$.

We shall in the rest of this section again consider only the cyclic case with p points, and use the same notation as in Section 3 and Theorem 4.1.

Theorem 4.2 *Let $|u| = |v|$, and assume $n > p$. Then $R_n(z,u,v)$ has the partial fraction decomposition*

$$R_n(z,u,v) = -\sum_{m=1}^{n} \lambda_m^{(n)}(u,v)D(\xi_m^{(n)}(u,v), z). \tag{4.2}$$

Proof:

Consider the function $f(t)$ defined by

$$f(t) = D(t,z)[\frac{B_p(z)}{B_p(t)}\varphi_n(t,u,v) - \varphi_n(z,u,v)]. \tag{4.3}$$

The function $\varphi_n(z,u,v)$ can be written as

$$\varphi_n(z,u,v) = \frac{p_n(z,u,v)}{\pi_n(z)}, \tag{4.4}$$

where $p_n(z,u,v) \in \Pi_n$. It follows that

$$f(t) = \frac{(t+z)[\omega_p(z)\pi_p(t)p_n(t,u,v)\pi_n(z) - \omega_p(t)\pi_p(z)\pi_n(t)p_n(z,u,v)]}{(t-z)\omega_p(t)\pi_p(z)\pi_n(t)}, \tag{4.5}$$

hence since $t - z$ is a factor in the numerator:

$$f(t) = \frac{P_{p+n-1}(z,t)(1-\overline{\alpha_n}t)}{\omega_p(t)\pi_n(t)}, \tag{4.6}$$

where P_{p+n-1} belongs to Π_{p+n-1} as a function of t. (Note that $(1-\overline{\alpha_n}t)$ is a factor both in $\pi_p(t)$ and in $\pi_n(t)$, and also in the numerator.)
It follows that we may write

$$f(t) = \frac{P_{p+n-1}(z,t)}{\omega_p(t)\pi_{n-1}(t)}, \tag{4.7}$$

hence $f(t) \in \mathcal{L}_{n-1} + \mathcal{L}_{p*} \subset \mathcal{L}_{n-1} + \mathcal{L}_{(n-1)*}$, by partial fraction decomposition. (Note that $\omega_p(t)$ and $\pi_{n-1}(t)$ have no common factors.) Since $f(\xi_m^{(n)}(u,v)) = -D(\xi_m^{(n)}(u,v),z)\varphi_n(z,u,v)$, as $\varphi_n(\xi_m^{(n)}(u,v),u,v)$ equals zero, application of Theorem 4.1 and Formula (2.11) yields

$$\psi_n(z,u,v) = -\varphi_n(z,u,v)\sum_{m=1}^{n}\lambda_m^{(n)}(u,v)D(\xi_m^{(n)}(u,v),z), \tag{4.8}$$

which is equivalent to (4.2). □

Since (4.1) is valid for $L = 1$, the following equality holds:

$$\sum_{m=1}^{n}\lambda_m^{(n)}(u,v) = \mu_0. \tag{4.9}$$

Theorem 4.3 *Let* $|u_n| = |v_n|$ *for* $n = 1,2,,,$. *Then the sequence* $\{R_n(z,u_n,v_n)\}$ *converges locally uniformly on* $\boldsymbol{D}$ *to* $-\Omega_\mu(z)$.

Proof:

It easily follows by (4.2) and (4.9) that the functions $R_n(z,u,v)$, $|u| = |v|$, are uniformly bounded on every compact subset of $\boldsymbol{D}$, and thus form a normal family. So there exist subsequences of $\{R_n(z,u_n,v_n)\}$ converging locally uniformly on $\boldsymbol{D}$. Let $\nu_n(t,u_n,v_n)$ be the measure on $\boldsymbol{T}$ having masses $\lambda_m^{(n)}(u_n,v_n)$ at the points $\xi_m^{(n)}(u_n,v_n)$. By Theorem 4.2 we may then write

$$R_n(z,u_n,v_n) = -\int_{\boldsymbol{T}} D(t,z)\, d\nu_n(t,u_n,v_n). \tag{4.10}$$

A standard argument shows that a subsequence of $\{R_n(z,u_n,v_n)\}$ converges locally uniformly on $\boldsymbol{D}$ to a function $F(z)$ if and only if the corresponding subsequence of $\{\nu_n(t,u_n,v_n)\}$ converges to a measure ν such that $F(z) = -\Omega_\nu(z)$.

Furthermore $\int_{\boldsymbol{T}} \frac{d\nu_n(u_n,v_n,t)}{\omega_m(t)}$ converges to $\int_{\boldsymbol{T}} \frac{d\nu(t)}{\omega_m(t)}$ for $m = 0,1,2,\ldots$. On the other hand Theorem 3.2 shows that $R_n(z,u_n,v_n) + \Omega_\mu(z) = g_n(z) z \omega_{n-1}(z)$, where $g_n(z)$ is analytic in $\boldsymbol{D}$. It follows from this and (4.10) that $\int_{\boldsymbol{T}} \frac{d\nu_n(t,u_n,v_n)}{\omega_m(t)} = \int_{\boldsymbol{T}} \frac{d\mu(t)}{\omega_m(t)}$ for $m = 0,1,\ldots,n-1$.

Consequently $\int_{\boldsymbol{T}} \frac{d\nu(t)}{\omega_m(t)} = \int_{\boldsymbol{T}} \frac{d\mu(t)}{\omega_m(t)}$ for $m = 0,1,2,\ldots$. (Cf. [7,8] where related problems are treated.) It is known that the measure giving rise to the moments $\mu_m = \int_{\boldsymbol{T}} \frac{d\mu(t)}{\omega_m(t)}$ is unique when $\sum_{m=1}^{\infty}(1-|\alpha_n|) = \infty$ (this follows e.g. from the convergence result in [3], Section 21). This is the case in the cyclic situation. Thus $\nu = \mu$ and the whole sequence $\{R_n(z,u_n,v_n)\}$ converges to $-\Omega_\mu(z)$. □

For convergence properties of the rational approximants $R_n(z,0,1)$ and $R_n(z,1,0)$ see [3]. For a more detailed study of convergence of multipoint Padé approximants, see especially [15].

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