Boundary asymptotics for orthogonal rational functions on the unit circle

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Let $w(\theta)$ be a positive weight function on the unit circle of the complex plane. For a sequence of points $\{\alpha_k\}_{k=1}^{\infty}$ included in a compact subset of the unit disk, we consider the orthogonal rational functions ϕ_n that are obtained by orthogonalization of the sequence $\{1, z/\pi_1, z^2/\pi_2, \ldots\}$ where $\pi_k(z) = \prod_{j=1}^{k}(1 - \overline{\alpha}_j z)$, with respect to the inner product $\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}w(\theta)\mathrm{d}\theta$. We discuss in this paper the behaviour of $\phi_n(t)$ for $|t| = 1$ and $n \to \infty$ under certain conditions. The main condition on the weight is that it satisfies a Lipschitz-Dini condition and that it is bounded away from zero. This generalizes a theorem given by Szegő in the polynomial case, that is when all $\alpha_k = 0$.

Keywords: orthogonal rational functions, asymptotics

AMS Subject classification: 42C05, 42A10

1. Introduction

The asymptotics of orthogonal polynomials on the unit circle have been discussed in many papers and monographs. Consider the inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\mathrm{d}\mu(\theta),$$

where μ represents a positive measure on the interval $[-\pi, \pi]$. By Gram-Schmidt orthogonalization, we can orthogonalize the sequence $\{1, z, z^2, \ldots\}$ and obtain orthogonal polynomials $\{\phi_k\}_{k=0}^{\infty}$.

An important problem in the theory of orthogonal polynomials is to describe the limiting behaviour of $\phi_n(z)$ as $n \to \infty$. These results can be obtained under various conditions on the measure μ. Typically, there is a distinction between the cases where $z \in \mathbb{D}$, $z \in \mathbb{T}$ or $z \in \mathbb{E}$ where

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}, \quad \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}, \quad \mathbb{E} = \{z \in \mathbb{C} : |z| > 1\}.$$

In $\mathbb{D}$, the polynomials converge locally uniformly to zero. This holds under rather weak conditions like $\mu' > 0$ a.e. in $[-\pi, \pi]$.

The behaviour in $\mathbb{E}$ is more interesting. Therefore we need the definition of the the Szegő function or spectral factor of the measure.

* This work is partially supported by the Fund for Scientific Research (FWO), project "Orthogonal systems and their applications", grant #G.0278.97 and the Belgian Programme on Interuniversity Poles of Attraction, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture. The scientific responsibility rests with the authors.

Assuming that $\log \mu' \in L^1$ (Szegő's condition), then the spectral factor is defined by

$$S(z) = \exp\left\{ \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \mu'(\theta) \mathrm{d}\theta \right\}.$$

This is an outer function in $H^2(\mathbb{D})$, which implies that S and $1/S$ are analytic functions in $\mathbb{D}$ and that the radial limit of $S(re^{i\theta})$ for $r \to 1$ exists a.e. It is defined up to a unimodular constant factor. By taking it equal to 1, we obtain the normalization $S(0) > 0$. If we denote the boundary function again as $S(e^{i\theta})$, or $s(\theta)$, thus $s(\theta) = \lim_{r\to 1^-} S(re^{i\theta}) = S(e^{i\theta})$, then it holds that $\mu'(\theta) = |s(\theta)|^2$ almost everywhere on $[-\pi, \pi]$.

A typical behaviour is that $\phi_n(z)/z^n$ converges locally uniformly in $\mathbb{E}$ to the function $1/S_*(z)$, where for any function f, we define its para-hermitian conjugate f_* as $f_*(z) = \overline{f(1/\overline{z})}$. This can also be expressed as $z^n \phi_{n*}(z) \to 1/S(z)$ uniformly in compact subsets of the unit disk $\mathbb{D}$.

These results are standard and so are the conditions on the measure under which these results hold.

For the asymptotics of $\phi_n(t)$ when $t \in \mathbb{T}$, the literature is more subtle on the conditions that should be imposed on the measure. Under various conditions, various asymptotics were derived. See for example [5–8].

Many results for orthogonal polynomials were generalized to orthogonal rational functions. These generalizations have shown usefull for numerical quadrature and in several signal processing and system theoretic applications. A survey of currently available results and some applications are given in the recent monograph [3].

The polynomial situation is generalized as follows. Given a sequence of complex points $\{\alpha_k\}_{k=1}^{\infty} \subset \mathbb{D}$, one constructs the Blaschke factors $\zeta_k(z)$ and the finite Blaschke products $B_n(z)$ as follows:

$$\zeta_k(z) = z_k \frac{z - \alpha_k}{1 - \overline{\alpha}_k z}, \quad k = 1, 2, \ldots \quad \text{with} \quad z_k = \begin{cases} \overline{\alpha}_k/|\alpha_k| \text{ if } \alpha_k \neq 0 \\ 1 \qquad \text{otherwise} \end{cases}$$

$$B_0 = 1, \quad B_k(z) = \zeta_1(z) \cdots \zeta_k(z), \quad k = 1, 2, \ldots$$

The space Π_n of the polynomials of degree at most n is replaced by the space $\mathcal{L}_n$ of rational functions:

$$\mathcal{L}_n = \operatorname{span}\{B_0, \ldots, B_n\}.$$

When we introduce the notation

$$\pi_n(z) = \prod_{k=1}^{n} (1 - \overline{\alpha}_k z) \quad \text{and} \quad \pi_n^*(z) = \prod_{k=1}^{n} (z - \alpha_k), \quad k = 1, 2, \ldots$$

then we can also express $\mathcal{L}_n$ as $\mathcal{L}_n = \{p_n/\pi_n : p_n \in \Pi_n\}$. By orthogonalization of the sequence $\{B_0, B_1, \ldots\}$, one obtains the orthogonal rational functions $\{\phi_0, \phi_1, \ldots\}$. Note that if all the $\alpha_k = 0$, then the rational situation reduces to the polynomial case. In analogy with the polynomial situation, we shall call the coefficient a_n in

$$f(z) = a_n B_n(z) + a_{n-1} B_{n-1}(z) + \cdots + a_0 \in \mathcal{L}_n$$

the leading coefficient of f (with respect to the basis B_k). We also denote for a function $f_n \in \mathcal{L}_n \backslash \mathcal{L}_{n-1}$ the reciprocal function

$$f_n^*(z) = B_n(z) f_{n*}(z) = \overline{a}_n + \overline{a}_{n-1} B_n(z)/B_{n-1}(z) + \cdots + \overline{a}_0 B_n.$$

The leading coefficient of ϕ_n will be denoted as κ_n: $\phi_n = \kappa_n B_n + \cdots$. Note that $\overline{\kappa}_n = \phi_n^*(\alpha_n)$. We will use the normalization that κ_n is real and positive. The notation introduced in this section is consistently used throughout this paper.

The previous theorems about the asymptotics for the orthogonal polynomials were generalized to the rational case (see e.g. [3]). Inside $\mathbb{D}$ the orthogonal rational functions converge locally uniformly to zero, while the behaviour of ϕ_n in $\mathbb{E}$ is expressed by the behaviour of ϕ_n^* in $\mathbb{D}$. We find, under rather rather mild conditions like $\log \mu' \in L_1$, and divergence of the Blaschke product with zeros α_k, that (1.1) holds for appropriate $\rho_n \in \mathbb{T}$.

$$\rho_n \frac{(1-\overline{\alpha}_n z)\phi_n^*(z)}{\sqrt{1-|\alpha_n|^2}} \to \frac{1}{S(z)} \quad \text{locally uniformly in } \mathbb{D}. \tag{1.1}$$

However, in the monograph [3], there is no theorem giving the asymptotics of ϕ_n on $\mathbb{T}$. In this paper we shall give the rational form of a theorem that can be found in Szegő's book [8, p. 297] for the polynomial case. If we suppose that $\mathrm{d}\mu(\theta) = w(\theta)\mathrm{d}\theta$ is absolutely continuous and the weight w satisfies $0 < m \le w(\theta) \le M < \infty$ uniformly in $[-\pi, \pi]$, and if it satisfies the Lipschitz-Dini condition then with some additional constraints on the asymptotics of the prescribed points $\{\alpha_i\}$ we shall prove that (1.1) also holds uniformly on $\mathbb{T}$ and we shall give the rate of convergence.

For the asymptotics of the points $\alpha = \{\alpha_k : k \in \mathbb{N}\}$, we shall assume some limiting distribution that is contained in a compact subset of $\mathbb{D}$. Thus $1 - |\alpha_k| \ge d > 0$ for some $d > 0$ that does not depend on k. Moreover, we assume that the counting measure, that is the discrete measure $\nu_n^\alpha := \frac{1}{n}\sum_{i=1}^n \delta_{\alpha_i}$ that assigns a mass $1/n$ to the points α_k for $k = 1, \ldots, n$, has a weak star limit ν^α, that is $\lim_{n\to\infty} \int f(z)\mathrm{d}\nu_n^\alpha(z) = \int f(z)\mathrm{d}\nu^\alpha(z)$ for all functions f, continuous in $\mathbb{D}$. We shall denote this as $\nu_n^\alpha \xrightarrow[n]{*} \nu^\alpha$.

Unless stated otherwise, we shall assume the following conditions

(AC) The measure μ is *absolutely continuous*: $\mathrm{d}\mu(\theta) = w(\theta)\mathrm{d}\theta$, with

$$0 < m \le w(\theta) \le M < \infty \text{ for } \theta \in [-\pi, \pi].$$

(LD) The 2π-periodic function $w(\theta)$ satisfies a *Lipschitz-Dini* condition:

$$\exists L > 0\ \exists\lambda > 0\ \forall\theta > 0\ \forall\delta > 0: \quad |w(\theta+\delta) - w(\theta)| < L|\log\delta|^{-1-\lambda},$$

where L and λ do not depend on θ or δ.

(CI) The point set $\alpha = \{\alpha_k : k \in \mathbb{N}\}$ is *compactly included* in $\mathbb{D}$, i.e. $\alpha \subset C$ with C a compact subset of $\mathbb{D}$, and the associated counting measures ν_n^α converge to some ν^α in weak star sense: $\nu_n^\alpha \xrightarrow[n]{*} \nu^\alpha$.

Note that (AC) implies that the Szegő condition $\log w(\theta) \in L_{2\pi}^1$ is satisfied. From (CI) it also follows that the Blaschke condition $\sum(1 - |\alpha_k|) = \infty$ is satisfied, which means that the infinite Blaschke product $B(z) = \prod_{k=1}^\infty \zeta_k(z)$ diverges. The requirement $\nu_n^\alpha \xrightarrow[n]{*} \nu^\alpha$ allows us to write the root asymptotics for the polynomials π_n^* in terms of logarithmic potentials of the ν^α. Recall that for a measure ν, its logarithmic potential is defined by $V_\nu(z) = -\int \log|z - \xi|\mathrm{d}\nu(\xi)$. It can be shown [2] that $\nu_n^\alpha \xrightarrow[n]{*} \nu^\alpha$ implies the root asymptotics

$$\lim_{n\to\infty} |\pi_n^*(z)|^{1/n} = \exp\{-V_{\nu^\alpha}(z)\}, \quad z \in \mathbb{C} - \operatorname{supp}(\nu^\alpha),$$

and

$$\limsup_{n\to\infty} |\pi_n^*(z)|^{1/n} \le \exp\{-V_{\nu^\alpha}(z)\}, \quad z \in \mathbb{C}$$

uniformly in each compact subset of the indicated regions.

The following is then a generalization of Theorem 12.1.3 of [8, p. 297].

Theorem 1.1. *Suppose that the conditions (AC), (LD), and (CI) are satisfied.*

Then there exist a sequence of unimodular constants ρ_n such that the orthogonal rational functions ϕ_n satisfy

$$\rho_n \frac{(1-\overline{\alpha}_n z)\phi_n^*(z)}{\sqrt{1-|\alpha_n|^2}} \to \frac{1}{S(z)} \quad \textit{uniformly for } z \in \mathbb{D} \cup \mathbb{T}.$$

The rate of convergence is $O(\log n)^{-\lambda}$. Thus also, with the same rate of convergence,

$$\frac{P(t,\alpha_n)}{|\phi_n(t)|^2} \to |S(t)|^2 = w(\theta), \quad \textit{with } t = e^{i\theta} \textit{ and } \quad P(t,w) = \frac{1-|w|^2}{|t-w|^2},$$

the Poisson kernel.

The proof of this theorem will be given in the next sections and it essentially repeats Szegő's proof of the polynomial case (see [8]); only the conditions on the α_k and related technicalities are different. The idea is as follows.

First we generalize the notion of trigonometric or Laurent polynomial. Note that a Laurent polynomial of degree n can be defined as an element from the space $\Lambda_n = \Pi_n \cdot \Pi_{n*}$ with $\Pi_{n*} = \{p_* : p \in \Pi_n\}$. For its rational generalization, we consider elements from the space $\mathcal{R}_n = \mathcal{L}_n \cdot \mathcal{L}_{n*}$ where $\mathcal{L}_{n*} = \{f : f_* \in \mathcal{L}_n\}$. Thus if $f \in \mathcal{R}_n$, then $f(z) = q(z)/[\pi_n(z)\pi_n^*(z)]$ where $q \in \Pi_{2n}$ is a polynomial of degree at most $2n$. If all α_k are zero, then $\mathcal{R}_n = \Lambda_n$ is the space of Laurent polynomials of degree at most n.

Then the inverse of the weight function: $1/w(\theta)$ is approximated arbitrary close by a positive trigonometric rational function, say $g_n(\theta)$. Thus $w(\theta) \approx w_n(\theta) = 1/g_n(\theta) = 1/|h_n(\theta)|^2$. It is meant that the approximant g_n is of the form $g_n(\theta) = G_n(e^{i\theta})$ with $G_n \in \mathcal{R}_{n-1}$ and $h_n(\theta) = H_n(e^{i\theta})$ where $H_n \in \mathcal{L}_{n-1}$ is the outer spectral factor of g_n. The nth orthogonal rational function ψ_n for the weight w_n can be explicitly written in terms of H_n and it can be shown that H_n converges to the inverse $1/S$ of the spectral factor of w, not only in $\mathbb{D}$ but also on $\mathbb{T}$. This gives the asymptotics for ψ_n. It then remains to show that ψ_n and ϕ_n have the same asymptotics.

2. Approximation of the weight

We first state a simple lemma, whose proof is obvious, but we formulate it separately for easy reference later. It says that if (AC) and (LD) hold then also the inverse $1/w$ satisfies a Lipschitz-Dini condition of the same order.

Lemma 2.1. *Assume that (AC) and (LD) hold, then there is some constant K such that for all $\delta > 0$*

$$\left|\frac{1}{w(\theta+\delta)} - \frac{1}{w(\theta)}\right| < K|\log\delta|^{-1-\lambda}$$

holds.

Proof. This is obvious since

$$\left|\frac{1}{w(\theta+\delta)} - \frac{1}{w(\theta)}\right| = \left|\frac{w(\theta+\delta)-w(\theta)}{w(\theta)w(\theta+\delta)}\right| < Lm^{-2}|\log\delta|^{-1-\lambda}.$$

Thus the lemma holds with $K = Lm^{-2}$. □

We now want to find an approximation for the orthogonal rational functions (**ORF**) with respect to the given weight function w. We know that the space $\bigcup_{n=0}^{\infty} \mathcal{R}_n$ is dense in $C(\mathbb{T})$ with respect to the supremum norm if and only if the Blaschke product diverges [3, Theorem 7.1.2]. Thus, it should be possible to find some approximant in $\mathcal{R}_n$ for $1/w$ that is as close to $1/w$ as we want.

The next theorem is a Jackson III type of theorem [4, p. 144]. It says how good such an approximation is as a function of n. It depends on the smoothness of $1/w$. In the polynomial case it states the following. Let $f \in C_{2\pi}$ be a continuous 2π-periodic function with modulus of continuity $\omega(f;\delta)$, that is $\omega(f;\delta) = \sup\{|f(x_1) - f(x_2)| : |x_1 - x_2| < \delta\}$. Then the best approximation in the set of trigonometric polynomials of degree $\leq n$ gives an error that is at most $\omega(f; \frac{2\pi}{n})$ in $[-\pi,\pi]$. A similar theorem was obtained in [2, Lemma 4.6] for the rational case. It requires some extra assumption on the distribution of the α_k so that asymptotics $|\pi_n^*(z)|^{1/n}$ can be estimated. This is where the condition $\nu_n^\alpha \xrightarrow[n]{*} \nu^\alpha$ comes in.

We include it here without proof.

Theorem 2.2. *Suppose the point set α satisfies the condition (CI). Then every real 2π-periodic continuous function $f \in C_{2\pi}$ can be approximated by a trigonometric rational function $r_n(\theta) = R_n(e^{i\theta})$ with $R_n \in \mathcal{R}_{n-1}$ such that for n large enough there is some constant K_1 such that*

$$\sup_{[-\pi,\pi]} |f(\theta) - r_n(\theta)| \leq K_1 \omega(f; \frac{\pi}{n})$$

where $\omega(f;\delta)$ denotes the modulus of continuity for f.

Obviously the function $r_n(\theta)$ in this theorem is of the form $r_n = T_{n-1}(\theta)/|\pi_{n-1}(e^{i\theta})|^2$ with T_{n-1} a trigonometric polynomial of degree at most $n-1$.

According to this theorem we can find a function $G_n \in \mathcal{R}_{n-1}$ such that $g_n(\theta) := G_n(e^{i\theta})$ satisfies

$$\left|\frac{1}{w(\theta)} - g_n(\theta)\right| \leq K_1 \omega\Big(\frac{1}{w}; \frac{\pi}{n}\Big). \tag{2.1}$$

This function $g_n(\theta)$ is uniformly bounded and positive if w is uniformly bounded and positive. This follows directly from (2.1). So we can find positive constants g, G and n_0 such that for $n \geq n_0$

$$0 < g \leq g_n(\theta) \leq G < \infty. \tag{2.2}$$

Next we show how good the approximation of $w(\theta)$ by $1/g_n(\theta)$ is.

Lemma 2.3. *Suppose the conditions (AC), (LD) and (IC) hold. Then there is a constant K' depending only on m, M, λ and L so that for n large*

$$|w(\theta) - g_n^{-1}(\theta)| < K'(\log n)^{-1-\lambda}.$$

Proof. First note that

$$\left|\frac{1}{w(\theta)} - g_n(\theta)\right| = \left|\frac{w(\theta) - g_n^{-1}(\theta)}{w(\theta) g_n^{-1}(\theta)}\right| \geq \frac{|w(\theta) - g_n^{-1}(\theta)|}{\max_\theta |w(\theta)| \max_\theta |g_n^{-1}(\theta)|}.$$

Because the denominator on the right-hand side is bounded, this inequality and (2.1) imply that there exists a positive constant K_2 such that

$$\left|w(\theta) - \frac{1}{g_n(\theta)}\right| \leq K_2\, \omega\Big(\frac{1}{w}; \frac{\pi}{n}\Big).$$

From Lemma 2.1 we get $\omega(\frac{1}{w}, \frac{\pi}{n}) < K \log(\frac{\pi}{n})^{-1-\lambda}$. Hence if n is large enough, the result follows. □

Let us now introduce the spectral factors

$$S(z) = \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log w(\theta) \mathrm{d}\theta\right\}$$

and

$$H_n(z) = \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log g_n(\theta) \mathrm{d}\theta\right\}.$$

Their boundary functions are denoted as $s(\theta) = S(e^{i\theta})$ and $h_n(\theta) = H_n(e^{i\theta})$. So $w(\theta) = |s(\theta)|^2$ and $g_n(\theta) = |h_n(\theta)|^2$.

We now prove that $1/H_n(z)$ converges to $S(z)$ uniformly in $|z| \leq 1$. More precisely, we prove

Lemma 2.4. *Assume that (AC), (LD), and (CI) hold. With the notation just introduced we have*

$$|S(z) - 1/H_n(z)| < Q(\log n)^{-\lambda}$$

uniformly in $|z| \leq 1$. The constant Q depends on L, λ, as well as on m and M, but not on n or z.

Proof. Because of analyticity, it is sufficient to give the proof for $z \in \mathbb{T}$. So we switch to the notation $h_n(\theta)$ and $s(\theta)$. Thanks to the previous lemma's, the rest of the proof is almost a copy of the proof in the polynomial case [8, p. 279-280]. The convergence of the moduli and the arguments is treated separately.

First it is noted that

$$\Big||s(\theta)| - |h_n^{-1}(\theta)|\Big| = \frac{|w(\theta) - g_n^{-1}(\theta)|}{\Big||s(\theta)| + |h_n^{-1}(\theta)|\Big|} = O(\log n)^{-1-\lambda}, \quad n \to \infty \tag{2.3}$$

is trivial, because of Lemma 2.3, condition (LD) and (2.2).

Szegő then argues that the convergence of the arguments is exactly like the convergence of the Cauchy principal value of the integral

$$\int_{-\pi}^{\pi} \log[w(\omega)g_n(\omega)] \cot\frac{\theta-\omega}{2} \mathrm{d}\omega.$$

which he splits into two terms

$$\int_{E_n} \log\frac{w(\omega)g_n(\omega)}{w(\theta)g_n(\theta)} \cot\frac{\theta-\omega}{2}\mathrm{d}\omega + \int_{E_n'} \log[w(\omega)g_n(\omega)] \cot\frac{\theta-\omega}{2}\mathrm{d}\omega.$$

The set $E_n \subset [-\pi, \pi]$ is the neighborhood of θ where one has $|\theta - \omega| < n^{-1}$ modulo 2π and $E_n' = [-\pi, \pi] \setminus E_n$. Because of Lemma 2.3 and (2.2), we can bound the absolute value of the second integral as in [8] by

$$O(\log n)^{-1-\lambda} \int_{E_n'} \left|\cot\frac{\theta-\omega}{2}\right| \mathrm{d}\omega = O(\log n)^{-\lambda}.$$

For bounding the Cauchy principal value of the first integral, Szegő uses Bernstein's theorem for trigonometric polynomials. This can be generalized, but we do not really need it here. Indeed, w satisfies the (LD) condition $|w(\omega) - w(\theta)| \leq L|\log|\omega - \theta||^{-1-\lambda}$, and therefore, for $|\omega - \theta|$ small, also $|\log(w(\omega)/w(\theta))|$ has a bound of the order $O(|\log|\omega - \theta||^{-1-\lambda})$. Because of Lemma 2.1, we have the same bound for $|\log(g_n(\omega)/g_n(\theta))|$. Thus also the first integral is bounded by

$$O(1) \int_{E_n} |\log|\theta - \omega||^{-1-\lambda} |\theta - \omega|^{-1} \mathrm{d}\omega = O(\log n)^{-\lambda}.$$

The remaining details of the proof are exactly as in [8, p. 279-280] □

3. Two systems of orthogonal rational functions

We now have an approximation $w_n = 1/g_n$ for w. We consider the orthogonal rational functions (**ORF**) ϕ_k for w and the **ORF** ψ_{nk} for w_n. We then derive the asymptotics for ϕ_n from the asymptotics for $\psi_n = \psi_{nn}$.

So, we suppose that we have an approximant $w_n = 1/g_n$ as described in the previous section. Assume that the spectral factors of w and g_n are denoted as before by $S(z)$ and $H_n(z)$, and that we set $s(\theta) := S(e^{i\theta})$ and $h_n(\theta) := H_n(e^{i\theta})$. Thus

$$w_n(\theta) = \frac{1}{H_n(e^{i\theta})H_{n*}(e^{i\theta})} = \frac{1}{|H_n(e^{i\theta})|^2} = \frac{1}{|h_n(\theta)|^2}. \tag{3.1}$$

For ease of the notation below, we shall assume that $g_n \in \mathcal{R}_{n-1}$ (note the index!), so that the outer spectral factor H_n has the form $H_n(z) = q_{n-1}(z)/\pi_{n-1}(z) \in \mathcal{L}_{n-1}$. Assume that we denote by $\psi_{nk} \in \mathcal{L}_k$ the kth orthogonal rational function for the weight $w_n(\theta)$.

We first give an explicit expression for the functions $\psi_{nk}(z)$. (Recall the notation B_k and z_k from the introduction.)

Theorem 3.1. *Suppose $H_n \in \mathcal{L}_{n-1}$ is given and $w_n(\theta) = 1/|h_n(\theta)|^2$ with $h_n(\theta) = H_n(e^{i\theta})$ as defined above. Then the kth orthonormal rational function ψ_{nk}, orthogonal with respect to the weight function w_n is given by*

$$\psi_{nk}(z) = \rho_{nk} z_k \frac{\sqrt{1-|\alpha_k|^2}}{1-\bar{\alpha}_k z} z B_{k-1}(z) H_{n*}(z), \quad k \geq n.$$

The constant $\rho_{nk} \in \mathbb{T}$ is for normalization, which we choose to be $H_n(\alpha_k)/|H_n(\alpha_k)|$.

Proof. First, it is clear that $\psi_{nk} \in \mathcal{L}_k$. We have to show that $\psi_{nk} \perp_{w_n} \mathcal{L}_{k-1}$ and that $\|\psi_{nk}\|_{w_n} = 1$.

Suppose $l < k$. Any function $f_l \in \mathcal{L}_l$ can be written as $f_l(z) = p_l(z)/\pi_l(z)$, with $p_l \in \Pi_l$. Furthermore, we set $p_n^*(z) = z^n p_{n*}(z)$ if $p_n \in \Pi_n \setminus \Pi_{n-1}$ and we assume that $H_n = q_{n-1}/\pi_{n-1}$. Also recall the definition of $z_k = \bar{\alpha}_k/|\alpha_k|$ if $\alpha_k \neq 0$ and $z_k = 1$ otherwise. We set $\eta_k = \prod_{i=1}^k z_i$.

- Orthogonality: According to Cauchy's theorem, we get ($t = e^{i\theta}$)

$$\begin{aligned}
&\langle \psi_{nk}, f_l \rangle_{w_n} \\
&= \rho_{nk} z_k \frac{\sqrt{1-|\alpha_k|^2}}{2\pi} \int_{-\pi}^{\pi} H_{n*}(t) \frac{t B_{k-1}(t)}{1-\bar{\alpha}_k t} \frac{p_l^*(t)}{\pi_l^*(t)} \frac{\mathrm{d}\theta}{H_n(t)H_{n*}(t)} \\
&= \rho_{nk} \eta_k \frac{\sqrt{1-|\alpha_k|^2}}{2\pi} \int_{-\pi}^{\pi} t \frac{\prod_{i=l+1}^{k-1}(t-\alpha_i)}{\prod_{i=n}^{k}(1-\bar{\alpha}_i t)} \frac{p_l^*(t)}{q_{n-1}(t)} \mathrm{d}\theta
\end{aligned}$$

$$= \rho_{nk}\eta_k\sqrt{1-|\alpha_k|^2}\frac{1}{2\pi i}\oint_{|t|=1} t\frac{\prod_{i=l+1}^{k-1}(t-\alpha_i)}{\prod_{i=n}^{k}(1-\bar{\alpha}_i t)}\frac{p_l^*(t)}{q_{n-1}(t)}\frac{\mathrm{d}t}{t} = 0.$$

- Normalization:

$$\begin{aligned}\langle\psi_{nk},\psi_{nk}\rangle_{w_n} \\ &= \frac{1-|\alpha_k|^2}{2\pi}\int_{-\pi}^{\pi} H_{n*}(t)\frac{tB_{k-1}(t)}{1-\bar{\alpha}_k t}\frac{H_n(t)}{(t-\alpha_k)B_{k-1}(t)}\frac{\mathrm{d}\theta}{H_n(t)H_{n*}(t)} \\ &= (1-|\alpha_k|^2)\frac{1}{2\pi i}\oint_{|t|=1}\frac{t}{(1-\bar{\alpha}_k t)(t-\alpha_k)}\frac{\mathrm{d}t}{t} = \frac{1-|\alpha_k|^2}{1-|\alpha_k|^2} = 1.\end{aligned}$$

□

Now we have the **ORF** $\{\phi_k\}$ with respect to the weight w and we have the **ORF** $\{\psi_{nk}\}$ with respect to the weight w_n. We define the functions $\psi_n := \psi_{nn}$. Our next step is to compare ψ_n and ϕ_n.

We introduce the notation $\kappa_n = \overline{\phi_n^*(\alpha_n)}$ and $\kappa_n' = \overline{\psi_n^*(\alpha_n)}$ for the leading coefficients in their expansions with respect to the basis $\{B_k\}$. We can find the explicit form of κ_n'

$$\overline{\kappa_n'} = \psi_n^*(\alpha_n) = \overline{\rho}_n\frac{H_n(\alpha_n)}{\sqrt{1-|\alpha_n|^2}} = \frac{|H_n(\alpha_n)|}{\sqrt{1-|\alpha_n|^2}} > 0. \tag{3.2}$$

Where $\rho_n = \rho_{nn}$. By the choice of ρ_{nk}, we find that $\kappa_n' > 0$. Recall that we also used the normalization $\kappa_n > 0$. We now express ϕ_n in the basis $\{\psi_{nk}\}$.

Lemma 3.2. *The* ***ORF*** ϕ_k *with respect to* w *and the* ***ORF*** ψ_{nk} *with respect to the weight* w_n *are both supposed to be normalized by making their leading coefficients* κ_k*, respectively* κ_k'*, positive. Then* ϕ_n *can be expressed in terms of the* ψ_{nk} *by*

$$\phi_n(z) = a_n\psi_n(z) + \sum_{k=0}^{n-1} a_{nk}\psi_{nk}(z) \tag{3.3}$$

where

$$a_n = \kappa_n\sqrt{1-|\alpha_n|^2}/|H_n(\alpha_n)| \tag{3.4}$$

and

$$a_{nk} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\phi_n(t)\overline{\psi_{nk}(t)}(w_n(\theta)-w(\theta))\mathrm{d}\theta, \quad k=0,\ldots,n-1.$$

Proof. This is easy to work out since by orthogonality we have for $k = 0,\ldots,n-1$

$$\begin{aligned}a_{nk} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\phi_n(t)\overline{\psi_{nk}(t)}w_n(\theta)\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\phi_n(t)\overline{\psi_{nk}(t)}\left(w_n(\theta)-w(\theta)\right)\mathrm{d}\theta \\ &\quad+\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi_n(t)\overline{\psi_{nk}(t)}w(\theta)\mathrm{d}\theta.\end{aligned}$$

The last term vanishes because $\phi_n \perp_w \mathcal{L}_{n-1}$. So we get the expressions for a_{nk}, $k=0,\ldots,n-1$. The form of a_n follows from $a_n = \kappa_n/\kappa_n'$ and (3.2). □

4. The asymptotics

Our strategy is now to develop the asymptotics for ψ_n, and then show that the asymptotics for ϕ_n and ψ_n are the same, thus we shall have to show that $\sum_{k=0}^{n-1} a_{nk}\psi_{nk} \to 0$ and $a_n \to 1$. Note that the expression for ψ_n contains H_{n*} and the asymptotics for the latter are already known (see Lemma 2.4).

We need to introduce first the reproducing kernels (or equivalently the Chistoffel functions) for $\mathcal{L}_n$. For the weight w, the reproducing kernels are $k_n(z,u) = \sum_{k=0}^{n} \phi_k(z)\overline{\phi_k(u)}$. The kernels $k_n(z,u)$ feature in the following optimization result that holds in any reproducing kernel Hilbert space.

Lemma 4.1. *For a positive measure μ on $\mathbb{T}$, consider the rational function spaces $\mathcal{L}_n$ as subspaces of $L^2(\mathbb{T},\mu)$, let $k_n(z,u)$ be the reproducing kernel for $\mathcal{L}_n$, then for fixed $u \in \mathbb{D}\cup\mathbb{T}$*

$$\min_{f_n\in\mathcal{L}_n,\ f_n(u)=1} ||f_n||_\mu^2 = \frac{1}{k_n(u,u)}. \tag{4.1}$$

The minimum is reached for $f_n(z) = k_n(z,u)/k_n(u,u)$.

The next lemma is from [3, Theorem 9.6.4]. Under very mild conditions on α (which are satisfied when (CI) holds) and when $\log w \in L^1$, it gives the asymptotics for the reproducing kernels.

Lemma 4.2. *Let $k_n(z,u)$ denote the reproducing kernel for $\mathcal{L}_n$ for the measure μ with $\log\mu' \in L^1$ and S the spectral factor for μ. Then if (CI) holds, we have for $z,u \in \mathbb{D}$*

$$\lim_{n\to\infty} k_n(z,u) = \frac{1}{(1-\bar{u}z)S(z)\overline{S(u)}}.$$

This convergence is uniform for z and u in compact subsets of $\mathbb{D}$.

Moreover it holds that [3, Theorem 2.2.3] $k_n(z,\alpha_n) = \kappa_n\phi_n^*(z)$ and $k_n(\alpha_n,\alpha_n) = \kappa_n^2$. (Recall that $\kappa_n > 0$.) Thus, the previous lemma gives

$$\lim_{n\to\infty} \kappa_n^2(1-|\alpha_n|^2)|S(\alpha_n)|^2 = 1. \tag{4.2}$$

We are now ready to bound κ_n^{-2}.

Lemma 4.3. *Under the conditions (AC), (LD), and (CI), assume that S is the outer spectral factor of w. Let $k_n(z,u)$ be the reproducing kernel for $\mathcal{L}_n$ and let $\phi_k(z) = \kappa_k B_k(z) + \cdots \in \mathcal{L}_k$ be the* ***ORF****. Furthermore, let H_n be the outer spectral factor of g_n, the approximant of w^{-1} as defined by the Jackson III theorem. Then*

$$(1-|\alpha_n|^2)|S(\alpha_n)|^2 \le \frac{1}{\kappa_n^2} \le \frac{1-|\alpha_n|^2}{|H_n(\alpha_n)|^2}\left(1+O(\log n)^{-1-\lambda}\right). \tag{4.3}$$

Proof. Because $\mathcal{L}_n \subset \mathcal{L}_{n+1}$, the minimum of Lemma 4.1 does not increase if n increases

$$\frac{1}{k_n(u,u)} \ge \frac{1}{k_{n+1}(u,u)} \ge \ldots \ge (1-|u|^2)|S(u)|^2.$$

The last bound is by Lemma 4.2. If we put $u = \alpha_n$, we get the lower bound.

For the upper bound, we consider the **ORF** ψ_n for the weight function w_n. Let us define

$$f_n(z) := \frac{\psi_n^*(z)}{\psi_n^*(\alpha_n)} = \frac{\psi_n^*(z)}{\kappa_n'} = \frac{(1-|\alpha_n|^2)H_n(z)}{(1-\bar{\alpha}_n z)H_n(\alpha_n)} \in \mathcal{L}_n. \tag{4.4}$$

This follows because of Theorem 3.1 and the fact that $H_n \in \mathcal{L}_{n-1}$. Because $f_n(z)$ is in $\mathcal{L}_n$ and satisfies $f_n(\alpha_n) = 1$, we find from (2.3), (4.1) and (4.4)

$$\begin{aligned}
\frac{1}{\kappa_n^2} &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} w(\theta)|f_n(t)|^2 \mathrm{d}\theta, \quad t = e^{i\theta} \\
&= \frac{(1-|\alpha_n|^2)^2}{|H_n(\alpha_n)|^2}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{S(t)H_n(t)}{t-\alpha_n}\right|^2 \mathrm{d}\theta \\
&= \frac{(1-|\alpha_n|^2)^2}{|H_n(\alpha_n)|^2}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta}{|t-\alpha_n|^2}\left(1+O(\log n)^{-1-\lambda}\right) \\
&= \frac{(1-|\alpha_n|^2)^2}{|H_n(\alpha_n)|^2}\frac{1}{2\pi i}\oint_{|t|=1}\frac{t\mathrm{d}t}{(t-\alpha_n)(1-\bar{\alpha}_n t)t}\left(1+O(\log n)^{-1-\lambda}\right) \\
&= \frac{1-|\alpha_n|^2}{|H_n(\alpha_n)|^2}\left(1+O(\log n)^{-1-\lambda}\right).
\end{aligned}$$

So we find the result. □

It now follows that we have as in (12.4.3) of [8]

Lemma 4.4. *Let (CA), (LD), and (CI) hold, then the coefficient a_n from Lemma 3.2 satisfies*

$$a_n = 1 + O(\log n)^{-\lambda}.$$

(Recall the normalization of the ϕ_n and the ψ_n.)

Proof. Because S and H_n are outer in Lemma 2.4 and because of (4.3),

$$\kappa_n = \frac{1}{|S(\alpha_n)|\sqrt{1-|\alpha_n|^2}} + O(\log n)^{-\lambda}.$$

From (3.4) we find

$$|a_n| = 1 + O(\log n)^{-\lambda}.$$

Because of the normalization of ϕ_n and ψ_n, this holds without the modulus bars. □

We now try to find a bound for the second term in (3.3). The idea is to find a uniform bound for $|\phi_n(t)|$. The remaining integral to be bounded is then of the form $\frac{1}{2\pi}\int_{-\pi}^{\pi}|k_{n-1}(w_n, z, t)|\mathrm{d}\theta$ where $t = e^{i\theta}$ and $k_{n-1}(w_n, z, t) = \sum_{i=0}^{n-1}\psi_{ni}(z)\overline{\psi_{ni}(t)}$ is the reproducing kernel for $\mathcal{L}_{n-1}$ with respect to the weight w_n.

We first search for an upper bound for $\psi_n(t)$ if $t = e^{i\theta} \in \mathbb{T}$.

$$\begin{aligned}
|\psi_n(t)| = |\psi_n^*(t)| &= \sqrt{1-|\alpha_n|^2}\left|\frac{H_n(t)}{1-\bar{\alpha}_n t}\right| \\
&= \sqrt{1-|\alpha_n|^2}\left|\frac{h_n(\theta)}{t-\alpha_n}\right| \leq \frac{\sqrt{G}}{d} = O(1)
\end{aligned} \tag{4.5}$$

The last inequality follows from the fact that we choose $\{\alpha_i\}$ to lie in a compact subset of $\mathbb{D}$: $1-|\alpha_k| \geq d$, and that $g_n = |h_n|^2 \leq G$ (see (2.2)).

We will use the Bernstein inequality for rational functions [1, Theorem 7.1.7] to bound the kernel $k_{n-1}(w_n, z, t)$.

Lemma 4.5. *For every rational function $f \in \mathcal{L}_n$, we have*

$$|f'(z)| \leq \sum_{k=1}^{n} \frac{1-|\alpha_k|^2}{|1-\bar{\alpha}_k z|^2} ||f||_{\mathbb{T}}, \quad z \in \mathbb{T},$$

where $||.||_{\mathbb{T}}$, denotes the sup-norm on the unit circle.

We also make use of the Christoffel-Darboux-relation (see [3, Theorem 3.1.3]).

$$k_{n-1}(w_n, z, t) = \frac{\overline{\psi_{n-1}^*(t)}\psi_{n-1}^*(z) - \overline{\psi_{n-1}(t)}\psi_{n-1}(z)}{1-\overline{\zeta_{n-1}(t)}\zeta_{n-1}(z)}. \tag{4.6}$$

These allow us to prove the following lemma.

Lemma 4.6. *Assume (AC), (LD) and (CI) hold. Then we can find the following upper bounds for the kernel $k_{n-1}(w_n, z, t)$.*

$$|k_{n-1}(w_n, z, t)| \leq \frac{8(n-1)G}{d^5} = O(n), \quad z, t \in \mathbb{D} \cup \mathbb{T}$$

and

$$|k_{n-1}(w_n, z, t)| \leq \frac{4\pi G}{d^3|\theta-\gamma|} = \frac{O(1)}{|\theta-\gamma|}, \quad t = e^{i\theta}, z = e^{i\gamma} \in \mathbb{T}.$$

The constants d and G are such that $1-|\alpha_k| \geq d > 0$ and $g_n(\theta) \leq G$.

Proof. First we note from (4.6) that for $t \in \mathbb{T}$

$$k_{n-1}(w_n, z, t) = -\frac{(t-\alpha_{n-1})(1-\bar{\alpha}_{n-1}z)}{1-|\alpha_{n-1}|^2} \frac{f(z)}{z-t}$$

where, as a function of z,

$$f(z) = \overline{\psi_{n-1}^*(t)}\psi_{n-1}^*(z) - \overline{\psi_{n-1}(t)}\psi_{n-1}(z) \in \mathcal{L}_{n-1}.$$

Moreover $f(z)$ vanishes for $z = t$, $t \in \mathbb{T}$.

Since the first factor in the previous expression for $k_{n-1}(w_n, z, t)$ is uniformly bounded (by $4/d$) for $z \in \mathbb{D} \cup \mathbb{T}$, $t \in \mathbb{T}$, it is only a matter of bounding

$$\frac{f(z)}{z-t} = \frac{f(z)-f(t)}{z-t} = \int_0^1 f'(\xi z + (1-\xi)t)\mathrm{d}\xi. \tag{4.7}$$

By Lemma 4.5, we get for $z \in \mathbb{T}$,

$$|f'(z)| \leq \sum_{k=1}^{n-1} \frac{1-|\alpha_k|^2}{|1-\bar{\alpha}_k z|^2} ||f||_{\mathbb{T}} \leq \frac{n-1}{d^2} ||f||_{\mathbb{T}}.$$

But, because f' is analytic in $\mathbb{D}$, the previous inequality extends to the interior of the disc. By (4.5), $||f||_{\mathbb{T}} = O(1)$ which shows together with (4.7) that the kernel $k_{n-1}(w_n, z, t)$, $t \in \mathbb{T}$ is of order $O(n)$ for $z \in \mathbb{D} \cup \mathbb{T}$.

Taking the complex conjugate of $k_{n-1}(w_n, z, t)$ interchanges the role of t and z, so that the result also holds for $t \in \mathbb{D}$.

We can get the other bound for t, $z \in \mathbb{T}$ by using the Christoffel-Darboux-relation (4.6) as follows

$$|k_{n-1}(w_n, z, t)| = \left| \frac{\overline{\psi_{n-1}^*(t)}\psi_{n-1}^*(z) - \overline{\psi_{n-1}(t)}\psi_{n-1}(z)}{1-\overline{\zeta_{n-1}(t)}\zeta_{n-1}(z)} \right|$$

$$\leq \frac{2\|\psi_{n-1}\|_{\mathbb{T}}^2}{|t-z|} \frac{|t-\alpha_{n-1}||z-\alpha_{n-1}|}{1-|\alpha_{n-1}|^2}.$$

So we find from (4.5)

$$|k_{n-1}(w_n, z, t)| \leq \frac{8G}{d^3|t-z|} = \frac{4G}{d^3|\sin(\frac{\theta-\gamma}{2})|} \leq \frac{4\pi G}{d^3|\theta-\gamma|}.$$

□

We have now prepared all the material to proceed along the same lines as in the proof given by Szegő for the polynomial case. In fact we could just refer to [8, p. 303-304], but for the sake of completeness, and to show where all the previous results fall into place, we do include some of the details.

So our next step is to derive an upper bound for the second part of (3.3). We shall prove below that $\phi_n(t)$ is uniformly bounded. The remaining integral can be bounded as follows.

Lemma 4.7. *Assume that (AC), (LD) and (CI) hold. Then we have the following bound uniformly in z:*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |k_{n-1}(w_n, z, t)| \mathrm{d}\theta \leq O(\log n), \quad z \in \mathbb{D} \cup \mathbb{T}, \quad t = e^{i\theta}.$$

Proof. The proof is along the same lines as the proof of (12.4.7) in [8]. Since $k_{n-1}(w_n, z, t)$ is analytic in $z \in \mathbb{D}$, it reaches its maximum modulus on the boundary. Thus if we have $\sup_{s\in\mathbb{T}} |k_{n-1}(w_n, s, t)| = |k_{n-1}(w_n, e^{i\gamma}, t)|$, we can bound for $z \in \mathbb{D} \cup \mathbb{T}$

$$\int_{-\pi}^{\pi} |k_{n-1}(w_n, z, t)| \mathrm{d}\theta \leq \int_{-\pi}^{\pi} |k_{n-1}(w_n, e^{i\gamma}, t)| \mathrm{d}\theta.$$

Therefore, it is sufficient to prove the theorem for $z \in \mathbb{T}$. For $z \in \mathbb{T}$, we split the integral in two parts

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |k_n(w_n, z, t)| \mathrm{d}\theta = \frac{1}{2\pi} \int_{E_n} |k_n(w_n, z, t)| \mathrm{d}\theta + \frac{1}{2\pi} \int_{E_n'} |k_n(w_n, z, t)| \mathrm{d}\theta.$$

Where E_n refers to the part where $|\theta - \gamma| \leq n^{-1}$ holds modulo 2π and $E_n' = [-\pi, \pi] \setminus E_n$. For the first integral we use the first upper bound from Lemma 4.6 which gives an upper bound of the form $O(1)$. For the second we use the second upper bound from Lemma 4.6. This gives

$$O(1) \int_{E_n'} \frac{\mathrm{d}\theta}{|\theta-\gamma|} = O(\log n).$$

□

We now try to find a bound for $|\phi_n(z)|$ if $z \in \mathbb{D} \cup \mathbb{T}$.

Lemma 4.8. *Suppose (AC), (LD) and (CI) hold. Then there is an absolute constant U such that* $|\phi_n(z)| \leq U$ *uniformly in n and* $z \in \mathbb{D} \cup \mathbb{T}$.

Proof. This proof is again completely similar to the proof given by Szegő [8, p. 304]. Since ϕ_n has all its poles in $\mathbb{E}$, it is sufficient to prove this for $z = t \in \mathbb{T}$. Suppose

$$\max_{|t|=1} |\phi_n(t)| = U = U(n).$$

From Lemma 2.3 and Lemma 3.2, we find ($\tau = e^{i\theta}$)

$$U \leq |a_n| \|\psi_n(t)\|_{\mathbb{T}} + UK'(\log n)^{-1-\lambda} \max_{|t|=1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{i=0}^{n-1} \overline{\psi_{ni}(\tau)} \psi_{ni}(t) \right| \mathrm{d}\theta. \tag{4.8}$$

Using (4.5) and Lemma 4.7, we can bound (4.8) as

$$U \leq O(1) + U\, O(\log n)^{-1-\lambda}\, O(\log n). \tag{4.9}$$

This implies $U = O(1)$. □

5. Proof of the main theorem

We now have all the ingredients to prove our main result. As we have said before, the previous lemma's generalize all the polynomial results used by Szegő to the rational case, and the proof is now essentially reduced to a transcription of Szegő's proof.

Proof. *(of Theorem 1.1)* We write the superstar conjugate of (3.3) as

$$\phi_n^*(z) = a_n \psi_n^*(z) + B_n(z) E_{n*}(z), \quad E_n(z) = \sum_{k=0}^{n-1} a_{nk} \psi_{nk}(z).$$

Recall that $a_n > 0$ by the normalization of the **ORF**. Substitute the expression for $\psi_n = \psi_{nn}$ from Theorem 3.1 and multiply with $\rho_n(1 - \overline{\alpha}_n z)/\sqrt{1 - |\alpha_n|^2}$ to get

$$\rho_n \frac{1 - \overline{\alpha}_n z}{\sqrt{1 - |\alpha_n|^2}} \phi_n^*(z) = a_n H_n(z) + \rho_n \frac{1 - \overline{\alpha}_n z}{\sqrt{1 - |\alpha_n|^2}} B_n(z) E_{n*}(z).$$

Let us call the second term in this expression $r_n(z)$. Because of the (CI) condition and because of Lemmas 4.7 and 4.8 we find that $r_n(z) = O(\log n)^{-\lambda}$.

Furthermore, we know that $a_n = 1 + O(\log n)^{-\lambda}$ by Lemma 4.4. By construction S and H_n are outer spectral factors with the same normalization, namely $S(0), H_n(0) > 0$. By this and Lemma 2.4 we have that $H_n(z) = 1/S(z) + O(\log n)^{-\lambda}$ uniformly in $\mathbb{D}$. This can be extended to $\mathbb{T}$ because S and H_n have continuous boundary values.

Thus

$$\begin{aligned} \rho_n \frac{1 - \overline{\alpha}_n z}{\sqrt{1 - |\alpha_n|^2}} \phi_n^*(z) &= (1 + O(\log n)^{-\lambda})(1/S(z) + O(\log n)^{-\lambda}) + r_n(z) \\ &= 1/S(z) + r_n(z), \quad r_n(z) = O(\log n)^{-\lambda}. \end{aligned}$$

The unimodular constants ρ_n were defined as $H_n(\alpha_n)/|H_n(\alpha_n)|$, which can not be defined in terms of ϕ_n. However, since we know that a sequence of unimodular constants exists, we can as well take any other sequence of unimodular constants, as long as the left-hand side converges to a positive constant for $z = 0$ because that is how $S(z)$ is normalized: $S(0) > 0$. Thus we can choose $\rho_n = |\phi_n^*(0)|/\phi_n^*(0)$. This proves the theorem, since the second formula for the modulus square is trivial. □

6. Acknowledgement

We thank the anonymous referees for their constructive remarks. Thanks to their suggestions, a shorter proof could be given for Lemma 4.6.

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