Interpolation by rational functions with nodes on the unit circle

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Abstract. From the Erdős-Turán theorem [10], it is known that if f is a continuous function on $\mathbb{T} = \{z : |z| = 1\}$ and $L_n(f, z)$ denotes the unique Laurent polynomial interpolating f at the $(2n+1)$th roots of unity, then

$$\lim_{n\to\infty} \int_{\mathbb{T}} |f(z) - L_n(f,z)|^2 |\mathrm{d}z| = 0.$$

Several years later, Walsh and Sharma [14] gave a similar result but now considering a function analytic in $\mathbb{D} = \{z : |z| < 1\}$ and continuous on $\mathbb{D} \cup \mathbb{T}$ and making use of algebraic interpolating polynomials in the roots of unity.

In this paper, the above results will be generalized in two directions. On the one hand, more general rational functions than polynomials or Laurent polynomials will be used as interpolants, and on the other hand, the interpolation points will be zeros of certain para-orthogonal functions with respect to a given measure on $\mathbb{T}$.

Keywords: orthogonal rational functions, interpolation, R-Szegő quadrature

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1. Introduction

The analysis of the mean convergence of a polynomial interpolation process on a finite interval of the real line has been studied throughout the century. On the other hand, consider an integral of the form $I_\mu(f) = \int_a^b f(x)\mathrm{d}\mu(x)$, $(-\infty < a < b < +\infty)$ where in general μ is a complex measure on $[a, b]$, and suppose it is approximated by a numerical quadrature formula of the form $I_n(f) = \sum_{j=1}^n A_j f(x_j)$ such that $I_\mu(f) = I_n(f)$ for all $f \in \Pi_{n-1}$ (the space of polynomials of degree at most $n-1$), where the mutually distinct nodes $x_i \in [a, b]$ are given in advance. Then the convergence of the interpolation process and the convergence of the quadrature process are closely related, since $I_n(f) = I_\mu(P_n(f, \cdot))$, where $P_n(f, \cdot) \in \Pi_{n-1}$ is the polynomial interpolating f in the nodes $\{x_j\}_{j=1}^n$ (see e.g. [11]). Thus

$$|I_\mu(f) - I_n(f)| \le \int_a^b |f(x) - P_n(f,x)||\mathrm{d}\mu(x)|.$$

It follows that the L^p-convergence $(p > 1)$ with respect to $|\mathrm{d}\mu|$, of sequences of interpolants, implies convergence of the quadrature formulas to $I_\mu(f)$. Thus an adequate selection of the nodes $\{x_j\} = \{x_{j,n}\}_{j=1}^n$ turns out to be fundamental. So far, investigations were concerned with systems of nodes that were roots of orthogonal systems (for further details see [12] and the references found therein).

Here, instead of interpolating a function on an interval $[a, b]$, we shall be mainly concerned with functions on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The following notation shall be used.

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The open unit disk is denoted by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and the external of the closed unit disk is $\mathbb{E} = \{z \in \mathbb{C} : |z| > 1\}$. For any function f, the para-hermitian conjugate is defined as $f_*(z) = \overline{f(1/\overline{z})}$. By $\Lambda_{p,q} = \{\sum_{j=p}^{q} a_j z^j : a_j \in \mathbb{C}\}$, we denote the space of Laurent polynomials (L-polynomials), if p and q are integers such that $p \leq q$. Λ is the set of all L-polynomials and Π the set of all polynomials. Finally, if $P \in \Pi_n \setminus \Pi_{n-1}$, then $P^*(z) = z^n P_*(z)$.

In contrast to the algebraic polynomial interpolation, where a large number of different node systems has been used, for the interpolation of functions on $\mathbb{T}$, most of the results make use of the roots of unity as interpolation nodes. (For other sets of nodes, see the recent paper [8].)

Assume that f is continuous on $\mathbb{T}$, i.e., $f \in C(\mathbb{T})$. Take into account that Λ is dense in $C(\mathbb{T})$. Then it seems natural to find sequences of interpolating Laurent polynomials with nodes on $\mathbb{T}$ and converging to f in a certain norm. Thus, from the classical Erdős-Turán Theorem [10], it can be deduced that

THEOREM 1.1. *Let $f \in C(\mathbb{T})$ and $\{x_{j,2n+1}\}$ be the $(2n+1)$th roots of unity. Let $L_n(f, \cdot)$ denote the L-polynomial in $\Lambda_{-n,n}$ interpolating f at the nodes $\{x_{j,2n+1}\}_{j=1}^{2n+1}$. Then*

$$\lim_{n\to\infty} \int_{\mathbb{T}} |f(z) - L_n(f, z)|^2 |\mathrm{d}z| = 0.$$

On the other hand, several years later, Walsh and Sharma [14] gave a similar result but now considering a function f analytic in $\mathbb{D}$ and continuous on $\mathbb{D} \cup \mathbb{T}$.

THEOREM 1.2. *Let f be analytic in $\mathbb{D}$ and continuous on $\mathbb{D}\cup\mathbb{T}$, and let $p_n \in \Pi_n$ be the polynomial of degree n coinciding with f in the $(n+1)$th roots of unity. Then the sequence $\{p_n\}$ converges to f on $\mathbb{T}$ in the mean convergence of second order, i.e.,*

$$\lim_{n\to\infty} \int_{\mathbb{T}} |f(z) - p_n(z)|^2 |\mathrm{d}z| = 0.$$

Consequently, it holds that $\lim_{n\to\infty} p_n(z) = f(z)$ uniformly on compact subsets of $\mathbb{D}$.

In this paper, Theorem 1.1 and 1.2 will be extended in two directions. Firstly, more general rational functions with prescribed poles will be used instead of polynomials or L-polynomials, and secondly, the roots of unity as interpolation nodes will be replaced by the zeros of certain para-orthogonal functions (see Section 2).

The paper is organized as follows. In Section 2, some auxiliary results concerning orthogonal rational functions and quadrature formulas will be given. They are needed in the subsequent sections. Section 3 is devoted to the study of the convergence of the interpolation process in the L_2-norm with respect to a given weight function. In Section 4, the interpolation of functions analytic in a domain containing $\mathbb{T}$ will be considered and the uniform convergence shall be studied.

2. Preliminary results

Let us first fix our spaces of interpolating functions. For a given sequence $\mathbb{A} = \{\alpha_j\}_{j=1}^{\infty} \subset \mathbb{D}$, we consider for $n = 0, 1, 2, \ldots$ the nested spaces $\mathcal{L}_n$ of rational functions of degree n (numerator and denominator degree at most n) which are spanned by the basis of partial Blaschke products $\{B_k\}_{k=0}^{n}$ where $B_0 = 1$, $B_n = B_{n-1}\zeta_n$, for $n = 1, 2, \ldots$ and the Blaschke factors are defined as

$$\zeta_n(z) = \frac{\overline{\alpha}_n}{|\alpha_n|} \frac{\alpha_n - z}{1 - \overline{\alpha}_n z}.$$

By convention, we set $\overline{\alpha}_n/|\alpha_n| = -1$ for $\alpha_n = 0$. If we set

$$\eta_n = (-1)^n \prod_{j=1}^{n} \frac{\overline{\alpha}_j}{|\alpha_j|}, \quad \text{then} \quad B_n(z) = \eta_n \frac{\omega_n(z)}{\pi_n(z)}$$

where

$$\omega_0 = 1, \quad \omega_n(z) = \prod_{j=1}^{n}(z-\alpha_j), \quad \text{and} \quad \pi_0 = 1, \quad \pi_n(z) = \prod_{j=1}^{n}(1-\overline{\alpha}_j z).$$

Clearly, $\mathcal{L}_n$ is the space of rational functions with poles among the prescribed points $1/\overline{\alpha}_i$, $i = 1, 2, \ldots, n$, which are all in $\mathbb{E}$. Thus

$$\mathcal{L}_n = \text{span}\{B_k : k = 0, 1, \ldots, n\} = \left\{\frac{p}{\pi_n} : p \in \Pi_n\right\}.$$

The space $\mathcal{L}$ is the closed linear span of all $\{B_k\}_{k=0}^{\infty}$. With the para-hermitian conjugate $f_*(z)$, we define for $f_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ the superstar conjugate

$$f_n^*(z) = B_n(z) f_{n*}(z) \in \mathcal{L}_n.$$

In general, our interpolating function spaces will be of the form $\mathcal{R}_{p,q} = \mathcal{L}_{p*} + \mathcal{L}_q$, p and q are nonnegative integers and $\mathcal{L}_{p*} = \{f : f_* \in \mathcal{L}_p\} = \text{span}\{1, B_{1*}, \ldots, B_{p*}\} = \text{span}\{1, 1/B_1, \ldots, 1/B_p\}$. Therefore,

$$\mathcal{R}_{p,q} = \text{span}\left\{\frac{1}{B_p}, \ldots, \frac{1}{B_1}, 1, B_1, \ldots, B_q\right\} = \left\{\frac{P(z)}{\omega_p(z)\pi_q(z)} : P \in \Pi_{p+q}\right\}.$$

The closed linear span of $\{\ldots, 1/B_q, \ldots, 1, \ldots, B_p, \ldots\}$ is denoted as $\mathcal{R}$, i.e., $\mathcal{R} = \mathcal{L} + \mathcal{L}_*$. Observe that $\mathcal{R}_{0,n} = \mathcal{L}_n$. Furthermore, if all the α_i are equal to zero, then $B_k(z) = z^k$ and one has $\mathcal{R}_{p,q} = \Lambda_{-p,q}$. Writing $\mathbb{A} = \{\alpha_i\}_{i=1}^{\infty}$ and $\mathbb{A}^* = \{1/\overline{\alpha}_i : \alpha_i \in \mathbb{A}\}$, and $B_{-k} = 1/B_k = B_{k*}$, it is easily seen that $\{B_k\}_{k=-p}^{q}$ represents a Chebyshev system [9] on any set $X \subset \mathbb{C} \setminus (\mathbb{A} \cup \mathbb{A}^*)$. Thus, given $n+1$ distinct nodes $\{x_j\}_{j=1}^{n+1} \subset \mathbb{C} \setminus (\mathbb{A} \cup \mathbb{A}^*)$, and $n+1$ arbitrary complex numbers $\{y_j\}_{j=1}^{n+1}$ there exists a unique $R_n \in \mathcal{R}_{p,q}$, $(p+q=n)$ such that $R_n(x_j) = y_j$, $j = 1, \ldots, n+1$. When $y_j = f(x_j)$, we will sometimes write $R_n(f, x)$.

Now that the scenery has been prepared, the first step will be to consider sets of interpolation nodes which we will assume to lie on $\mathbb{T}$. In analogy with the polynomial interpolation on intervals of the real line, where the interpolation nodes are all in the interval (they are taken to be zeros of orthogonal polynomials w.r.t. some positive measure), we shall take interpolation points on the circle $\mathbb{T}$. However, if μ is a positive measure on $[0, 2\pi)$ and if $\{\phi_n\}_{n=0}^{\infty}$ is a sequence of orthogonal rational functions obtained by orthogonalization of the sequence $\{B_n\}_{n=0}^{\infty}$, with respect to the inner product

$$\langle f, g\rangle_\mu = \int_0^{2\pi} f(x)\overline{g(x)}\mathrm{d}\mu(\theta), \quad x = e^{i\theta},$$

then it is known [6] that the zeros of $\phi_n(z)$ lie inside $\mathbb{D}$ for each n. Thus these zeros cannot be taken as interpolation nodes. However, we can overcome this drawback by introducing the functions (called para-orthogonal functions)

$$Q_n(z, \tau) = \phi_n(z) + \tau\phi_n^*(z) \in \mathcal{L}_n, \quad \tau \in \mathbb{T}. \tag{2.1}$$

The following theorem holds [2].

THEOREM 2.1. *The para-orthogonal functions Q_n satisfy*

1. *$Q_n(z, \tau)$ has n simple zeros which lie on $\mathbb{T}$.*
2. *Let $x_1, \ldots, x_n$ be the zeros of $Q_n(z, \tau)$. Then there exist positive numbers $A_1, \ldots, A_n$ such that the quadrature formula*

$$I_n(f) = \sum_{j=1}^{n} A_j f(x_j)$$

is exact in $\mathcal{R}_{n-1,n-1}$, that is, $I_n(f) = I_\mu(f) = \int_0^{2\pi} f(e^{i\theta})\mathrm{d}\mu(\theta)$ for all $f \in \mathcal{R}_{n-1,n-1}$.

Moreover, it was also proved in [2] that the only quadrature formulas $I_n(f)$ with distinct nodes $\{x_j\}$ on $\mathbb{T}$, exact in $\mathcal{R}_{n-1,n-1}$ are just the ones given in Theorem 2.1, so that

$$I_n(f) = I_\mu(R_{n-1}(f,\cdot))$$

where $R_{n-1}(f,\cdot)$ is the interpolant from $\mathcal{R}_{p,q}$ (p and q arbitrary nonnegative integers such that $p+q=n-1$) of the function f at the nodes $\{x_j\}$. The formula $I_n(f)=\sum_{j=1}^n A_j f(x_j)$ as defined above is called an n-point Rational Szegő quadrature formula (R-Szegő formula for short). In [3] the following was proved:

THEOREM 2.2. *Let $\{x_j\}_{j=1}^n$ be the zeros of $Q_n(z,\tau)=\phi_n(z)+\tau\phi_n^*(z)$, $\tau\in\mathbb{T}$. Let $H_{2n-1}^k(f,z)$ be the quasi-Hermite interpolant in $\mathcal{R}_{n-1,n-1}$ satisfying*

$$\begin{aligned} &H_{2n-1}^k(f,x_j) = f(x_j), \quad j=1,\ldots,n \\ &[H_{2n-1}^k]'(f,x_j) = f'(x_j), \; j=1,\ldots,n;\; j\neq k,\; 1\le k\le n. \end{aligned}$$

Then the corresponding n-point R-Szegő formula is given by

$$I_n(f) = \sum_{j=1}^n A_j f(x_j) = I_\mu(H_{2n-1}^k(f,\cdot)).$$

The following result concerning density will also be required. See [1, 6].

THEOREM 2.3. *With the notation introduced, we have*

1. *The space $\mathcal{R}$ is dense in $L^p(\mathbb{T})$, $1\le p<\infty$ and in $C(\mathbb{T})$ iff $\sum_{k=1}^\infty(1-|\alpha_k|)=\infty$.*
2. *The space $\mathcal{L}$ is dense in the class of functions continuous on $\mathbb{D}\cup\mathbb{T}$ and analytic in $\mathbb{D}$ iff $\sum_{k=1}^\infty(1-|\alpha_k|)=\infty$.*

3. Mean square convergence

We first assume that the function f to be interpolated is analytic in $\mathbb{D}$ and continuous on $\mathbb{D}\cup\mathbb{T}$. By Theorem 2.3, we know that f can be uniformly approximated by functions in $\mathcal{L}$, provided that the set $\mathbb{A}=\{\alpha_k\}$ satisfies

$$\sum_{k=1}^\infty (1-|\alpha_k|) = \infty. \tag{3.2}$$

Throughout this section and the following one, we shall always assume that this condition is satisfied. Our objective is to construct a sequence of interpolants in $\mathcal{L}$, converging to f on compact subsets of $\mathbb{D}$ and whose interpolation nodes are on $\mathbb{T}$. In the polynomial case, i.e., if all $\alpha_k=0$ so that $\mathcal{L}=\Pi$, then the solution is given in Theorem 1.2. In this theorem the interpolation nodes are the roots of unity. The latter can also be considered as zeros of para-orthogonal functions. Indeed, the orthogonal polynomials for the Lebesgue measure $\mathrm{d}\mu(\theta)=\mathrm{d}\theta/(2\pi)$, are $\phi_k(z)=z^k$. Therefore, the corresponding para-orthogonal polynomials are $Q_n(z,\tau)=\phi_n(z)+\tau\phi_n^*(z)=z^n+\tau$, $\tau\in\mathbb{T}$. Thus, when taking $\tau=-1$, we see that the roots of unity are zeros of para-orthogonal polynomials for the Lebesgue measure.

Let μ be a finite positive Borel measure on $[0,2\pi)$ that is absolutely continuous so that $\mathrm{d}\mu(\theta)=\omega(\theta)\mathrm{d}\theta$, $\omega(\theta)>0$ a.e. on $[0,2\pi)$ i.e., $\omega(\theta)$ is a weight function. So, in the rest of the paper, we shall have orthogonality with respect to $\omega(\theta)$ rather than orthogonality with respect to μ.

THEOREM 3.1. *Let f be a function analytic in $\mathbb{D}$ and continuous on $\mathbb{D}\cup\mathbb{T}$. Let $\omega(\theta)$ be a weight function on $[0,2\pi)$, generating the para-orthogonal functions $\{Q_n(z,\tau_n)\}_{n=1}^\infty$, with $\tau_n\in\mathbb{T}$ as in*

(2.1). Let $\{x_{j,n+1}\}_{j=1}^{n+1}$ be the zeros of $Q_{n+1}(z,\tau_{n+1})$ and $L_n(z)$ the interpolant for $f(z)$ from $\mathcal{L}_n$ in the nodes $\{x_{j,n+1}\}_{j=1}^{n+1}$. Then

$$\lim_{n\to\infty}\int_0^{2\pi}|f(x)-L_n(x)|^2\omega(\theta)\mathrm{d}\theta=0.$$

Proof. Let $T_n\in\mathcal{L}_n$ denote the best uniform approximant from $\mathcal{L}_n$ to f on $\mathbb{T}$, i.e.,

$$\max_{x\in\mathbb{T}}|f(x)-T_n(x)|:=\|f-T_n\|_{\mathbb{T}}\le\|f-R\|_{\mathbb{T}},\quad\forall R\in\mathcal{L}_n.$$

Note that this best approximant always exists because $\mathcal{L}_n$ is finite dimensional. We introduce the notation

$$\gamma_n(x)=f(x)-T_n(x)\quad\text{and}\quad\gamma_n(f)=\max_{x\in\mathbb{T}}|\gamma_n(x)|.$$

Denote by $P_n(x)$ the element in $\mathcal{L}_n$, interpolating $\gamma_n(x)$ at the nodes $\{x_{j,n+1}\}_{j=1}^{n+1}$. We have for $j=1,\ldots,n+1$,

$$\begin{aligned}P_n(x_{j,n+1}) &= \gamma_n(x_{j,n+1})=f(x_{j,n+1})-T_n(x_{j,n+1})\\ &= L_n(x_{j,n+1})-T_n(x_{j,n+1}).\end{aligned}$$

By a unicity argument, we get $P_n(x)=L_n(x)-T_n(x)$, and therefore

$$\begin{aligned}\int_0^{2\pi}|f(x)-L_n(x)|^2\omega(\theta)\mathrm{d}\theta &= \int_0^{2\pi}|\gamma_n(x)-P_n(x)|^2\omega(\theta)\mathrm{d}\theta\\ &\le 2\int_0^{2\pi}|\gamma_n(x)|^2\omega(\theta)\mathrm{d}\theta+2\int_0^{2\pi}|P_n(x)|^2\omega(\theta)\mathrm{d}\theta.\end{aligned}$$

Clearly, for the first integral, it holds that $\int_0^{2\pi}|\gamma_n(x)|^2\omega(\theta)\mathrm{d}\theta\le[\gamma_n(f)]^2$. (Here we assume without loss of generality that $\int_0^{2\pi}\omega(\theta)\mathrm{d}\theta=1$.) As for the second integral, we observe that for $x\in\mathbb{T}$, $|P_n(x)|^2\in\mathcal{R}_{n,n}=\mathcal{L}_{n*}+\mathcal{L}_n$. Therefore, by Theorem 2.1 (2), one has

$$\begin{aligned}\int_0^{2\pi}|P_n(x)|^2\omega(\theta)\mathrm{d}\theta &= \sum_{j=1}^{n+1}A_{j,n+1}|P_n(x_{j,n+1})|^2\\ &= \sum_{j=1}^{n+1}A_{j,n+1}|\gamma_n(x_{j,n+1})|^2\\ &\le [\gamma_n(f)]^2\sum_{j=1}^{n+1}A_{j,n+1}=[\gamma_n(f)]^2.\end{aligned}$$

Thus it follows that

$$\int_0^{2\pi}|f(x)-L_n(x)|^2\omega(\theta)\mathrm{d}\theta\le 4[\gamma_n(f)]^2.$$

Since by Theorem 2.3, $\lim_{n\to\infty}\gamma_n(f)=0$, we can conclude the proof. □

COROLLARY 3.2. *Let $\omega(\theta)$ be a weight function on $[0,2\pi)$ such that $\int_0^{2\pi}[\omega(\theta)]^{-1}\mathrm{d}\theta<+\infty$. Then, under the same conditions as in Theorem 3.1, one has $\lim_{n\to\infty}L_n(z)=f(z)$, uniformly on compact subsets of $\mathbb{D}$.*

Proof. By the Cauchy integral formula, it holds that

$$f(z)-L_n(z)=\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{f(t)-L_n(t)}{t-z}\mathrm{d}t,\quad z\in\mathbb{D}.$$

Let K be a compact in $\mathbb{D}$, so that $\mathrm{dist}(K, \mathbb{T}) = \delta > 0$. Thus, for all $t \in K$ and for all $z \in \mathbb{T}$, $|t - z| \geq \delta$. Hence,

$$\begin{aligned} |f(z) - L_n(z)| &\leq \frac{1}{2\pi}\int_{\mathbb{T}} \frac{|f(t) - L_n(t)|}{|t - z|}|\mathrm{d}t| \\ &\leq \frac{1}{2\pi\delta}\int_{\mathbb{T}} |f(t) - L_n(t)||\mathrm{d}t| \\ &= \frac{1}{2\pi\delta}\int_{\mathbb{T}} |f(t) - L_n(t)|\frac{\sqrt{\omega(\theta)}}{\sqrt{\omega(\theta)}}|\mathrm{d}t|. \end{aligned}$$

Making use of the Cauchy-Schwarz inequality for integrals, it follows that

$$|f(z) - L_n(z)| \leq \frac{1}{2\pi\delta}\left[\int_{\mathbb{T}} |f(t) - L_n(t)|^2\omega(\theta)|\mathrm{d}t|\right]^{1/2}\left[\int_0^{2\pi}\frac{\mathrm{d}\theta}{\omega(\theta)}\right]^{1/2},$$

and the proof is complete. □

Assume now that f is continuous on $\mathbb{T}$, i.e. $f \in C(\mathbb{T})$. By (1) of Theorem 2.3, it seems natural to look for interpolants in $\mathcal{R}$ with nodes on $\mathbb{T}$ converging to f in a certain norm. To this end, observe that if $R \in \mathcal{R}_{n,n}$, then $R(x) = P(x)/[\omega_n(x)\pi_n(x)]$ with $P \in \Pi_{2n}$. Therefore ($x = e^{i\theta}$): $|R(x)|^2 = Q(x)/[\omega_n^2(x)\pi_n^2(x)]$, with $Q \in \Pi_{4n}$. Thus, a quadrature formula integrating exactly $|R(x)|^2$ is required. From the sequence $\mathbb{A} = \{\alpha_k\}_{k=1}^{\infty} \subset \mathbb{D}$, define $\tilde{\mathbb{A}} = \{\alpha_1, \alpha_1, \alpha_2, \alpha_2, \ldots\} = \{\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots\}$. For a given weight function $\omega(\theta)$, let $\{\tilde{\phi}_n(z)\}_{n=1}^{\infty}$ denote the corresponding sequence of orthogonal functions, i.e. $\tilde{\phi}_n(z) = p_n(z)/\tilde{\pi}_n(z) \in \tilde{\mathcal{L}}_n$ with $\tilde{\pi}_n(z) = \prod_{j=1}^{n}(1 - \overline{\tilde{\alpha}}_j z)$. For $\tau \in \mathbb{T}$, set $\tilde{Q}_n(z, \tau) = \tilde{\phi}_n(z) + \tau\tilde{\phi}_n^*(z)$, and let $\{x_{j,2n+1}\}_{j=1}^{2n+1}$ be the zeros of $\tilde{Q}_{2n+1}(z, \tau_{2n+1})$, $\tau_{2n+1} \in \mathbb{T}$. By (2) in Theorem 2.1, there exist positive weights $\{\tilde{A}_{j,2n+1}\}$ so that ($x = e^{i\theta}$)

$$\int_0^{2\pi} f(x)\omega(\theta)\mathrm{d}\theta = \sum_{j=1}^{2n+1}\tilde{A}_{j,2n+1}f(x_{j,2n+1}), \quad \forall f \in \tilde{\mathcal{R}}_{2n,2n}. \tag{3.3}$$

(Here $\tilde{\mathcal{R}}_{p,q} = \tilde{\mathcal{L}}_{p*} + \tilde{\mathcal{L}}_q$ and $\tilde{\mathcal{L}}_{p*} = \{f : f_* \in \tilde{\mathcal{L}}_p\}$.) For each $n = 1, 2, \ldots$, let R_n be the element in $\mathcal{R}_{n,n}$ interpolating $f \in C(\mathbb{T})$ at the nodes $\{x_{j,2n+1}\}$ as given above. Take $T_n \in \mathcal{R}_{n,n}$ such that $\max_{x\in\mathbb{T}}|f(x) - T_n(x)| = \|f - T_n\|_{\mathbb{T}} \leq \|f - R\|_{\mathbb{T}}$ for any $R \in \mathcal{R}_{n,n}$. Define $\gamma_n(x) = f(x) - T_n(x)$ and $\gamma_n(f) = \max_{x\in\mathbb{T}}|\gamma_n(x)|$. By (1) in Theorem 2.3, $\lim_{n\to\infty}\gamma_n(f) = 0$. Thus, making use of (3.3) and proceeding as in the proof of Theorem 3.1, one can easily deduce that

$$\int_0^{2\pi} |f(x) - R_n(x)|^2\omega(\theta)\mathrm{d}\theta \leq 4[\gamma_n(f)]^2. \tag{3.4}$$

Hence the following theorem can be proved.

THEOREM 3.3. *Let f be a continuous function on $\mathbb{T}$ and let $R_n \in \mathcal{R}_{n,n}$ be the interpolant for f at the nodes $\{x_{j,2n+1}\}_{j=1}^{2n+1}$ as given above, i.e., the zeros of $\tilde{Q}_{2n+1}(z, \tau_{2n+1}) = \tilde{\phi}_{2n+1}(z) + \tau_{2n+1}\tilde{\phi}_{2n+1}^*(z)$, $\tau_{2n+1} \in \mathbb{T}$, where $\{\tilde{\phi}_k\}_{k=0}^{\infty}$ is the sequence of orthogonal rational functions associated with the sequence $\tilde{A} = \{\tilde{\alpha}_j\}_{j=1}^{\infty}$ and with respect to a given weight function $\omega(\theta)$. Then*

$$\lim_{n\to\infty}\int_0^{2\pi} |f(x) - R_n(x)|^2\omega(\theta)\mathrm{d}\theta = 0, \quad x = e^{i\theta}.$$

REMARK 3.1. Theorem 3.3 was earlier proved by the authors in [5] (see also [7]). However, the proof presented here is more constructive in the sense that an error bound like (3.4) for the L^2_ω-norm has been given.

EXAMPLE 3.1. Assume that we take the normalized Lebesgue measure $\mathrm{d}\mu(\theta) = \omega(\theta)\mathrm{d}\theta = \frac{\mathrm{d}\theta}{2\pi}$. Then it is known that the orthonormal functions are given by

$$\phi_n(z) = \kappa_n \frac{zB_n(z)}{z - \alpha_n}, \quad \kappa_n = \sqrt{1 - |\alpha_n|^2} > 0.$$

This basis is known as the Malmquist basis [13]. Observe that $\omega(\theta)$ trivially satisfies the conditions of Corollary 3.2. Therefore the zeros of the corresponding para-orthogonal functions can be used as interpolation nodes both in Theorem 3.1 and Corollary 3.2. In fact, we have

$$Q_n(z, \tau_n) = \phi_n(z) + \tau_n \phi_n^*(z), \quad |\tau_n| = 1$$

and since $\phi_n^*(z) = \kappa_n/(1 - \overline{\alpha}_n z)$, it follows that

$$Q_n(z, \tau_n) = \kappa_n \left[\frac{zB_n(z)}{z - \alpha_n} + \frac{\tau_n}{1 - \overline{\alpha}_n z} \right].$$

Noting that $B_n(z) = \eta_n \omega_n(z)/\pi_n(z)$, with $\eta_n \in \mathbb{T}$, and because $\tau_n \in \mathbb{T}$ is arbitrary, we can choose it as $\tau_n = \eta_n$, so that $Q_n(z, \tau_n)$ becomes

$$Q_n(z, \tau_n) = \kappa_n \tau_n \left[\frac{z\omega_{n-1}(z) + \pi_{n-1}(z)}{\pi_n(z)} \right].$$

As for Theorem 3.3, interpolation nodes can be chosen as the zeros of

$$\tilde{Q}_{2n+1}(z, \tau_n) = \lambda_n \left[\frac{z\tilde{\omega}_{2n}(z) + \tilde{\pi}_{2n}(z)}{\tilde{\pi}_{2n+1}(z)} \right] = \frac{\lambda_n}{1 - \overline{\alpha}_{n+1} z} \left[\frac{z\omega_n^2(z) + \pi_n^2(z)}{\pi_n^2(z)} \right],$$

with $\lambda_n \neq 0$, i.e., as the nodes $\{x_{j,2n+1}\}_{j=1}^{2n+1}$ that are the roots of the equation

$$z\omega_n^2(z) + \pi_n^2(z) = 0.$$

4. Uniform convergence

Throughout this section, we shall assume that $f(z)$ is analytic in a neighborhood of $\mathbb{T}$. We also assume that the set $\mathbb{A}$ lies in a compact subset of $\mathbb{D}$, i.e., $|\alpha_k| \leq \rho < 1$ for all k. Note that condition (3.2) trivially holds in this case. Then, there exist numbers $r, R > 0$ such that $0 < \rho < r < 1 < R < 1/\rho$ defining an annulus $\mathbb{B} = \{z \in \mathbb{C} : r < |z| < R\}$ where f is analytic and which excludes all the points α_k and $1/\overline{\alpha}_k$. We denote its boundary as $\partial\mathbb{B} = C$. Let $\{x_j\}_{j=1}^{2n+1}$ be $2n+1$ arbitrary distinct points on $\mathbb{T}$ and let $R_n \in \mathcal{R}_{n,n}$ be the interpolant in these nodes. Define the node polynomial $P_{2n+1}(z) = \prod_{j=1}^{2n+1}(z - x_j) \in \Pi_{2n+1}$ and $Q_{2n+1}(z) = P_{2n+1}(z)/[\pi_n^2(z)(1 - \overline{\alpha}_{n+1} z)]$. We then have

LEMMA 4.1. *For any $z \in \mathbb{B}$, $z \neq \alpha_j$ and $z \neq 1/\overline{\alpha}_j$, $j = 1, \ldots, n$, it holds that*

$$f(z) - R_n(z) = \frac{1}{2\pi i} \int_C \frac{1 - \overline{\alpha}_{n+1} z}{1 - \overline{\alpha}_{n+1} t} \frac{Q_{2n+1}(z)}{Q_{2n+1}(t)} \frac{B_n(t)}{B_n(z)} \frac{f(t)}{t - z} \mathrm{d}t.$$

Proof. Making use of the formula for the error of the rational interpolant with prescribed poles (see p. 182 of [13]), one can write for $z \in \mathbb{B}$, $z \neq \alpha_j$ and $z \neq 1/\overline{\alpha}_j$

$$f(z) - R_n(z) = \frac{1}{2\pi i} \int_C \frac{P_{2n+1}(z)\omega_n(t)\pi_n(t)}{P_{2n+1}(t)\omega_n(z)\pi_n(z)} \frac{f(t)}{t - z} \mathrm{d}t.$$

From here, the lemma follows. □

For the given sequence $\mathbb{A} = \{\alpha_j\}_{j=1}^{\infty}$, let $\nu_n^{\mathbb{A}} = \frac{1}{n}\sum_{j=1}^{n}\delta_{\alpha_j}$ be the normalized counting measure which assigns a mass at α_j proportional to its multiplicity. Assume that $\{\nu_n^{\mathbb{A}}\}_{n=1}^{\infty}$ converges to some $\nu^{\mathbb{A}}$ in the weak star topology (for the space of functions continuous in $\{z \in \mathbb{C} : |z| \leq \rho\}$). Define

$$\lambda(z) = \int \log|\xi_z(x)|\mathrm{d}\nu^{\mathbb{A}}(x), \quad \xi_z(x) = \frac{x-z}{1-\overline{x}z}, \quad z \in \mathbb{B}. \tag{4.5}$$

Let $\omega(\theta)$ be a weight function on $[0, 2\pi)$ and $\{\tau_n\}$ a given sequence of complex numbers in $\mathbb{T}$. According to (2.1), let $\tilde{Q}_{2n+1}(z, \tau_{2n+1}) = \tilde{Q}_{2n+1}(z) = P_{2n+1}(z)/\tilde{\pi}_{2n+1}(z)$, where as before $\tilde{\pi}_{2n+1}(z) = \prod_{j=1}^{2n+1}(1-\overline{\alpha}_j z) = \pi_n^2(z)(1-\overline{\alpha}_{n+1}z)$. Let $\{x_{j,2n+1}\}_{j=1}^{2n+1}$ be the zeros of $\tilde{Q}_{2n+1}(z)$ and $R_n \in \mathcal{R}_{n,n}$ be the interpolant for $f(z)$ in these nodes. We then have

THEOREM 4.2. *Let $f(z)$ be an analytic function in the annulus $\mathbb{B} = \{z \in \mathbb{C} : r < |z| < R\}$ with $0 < r < 1 < R$ and suppose $|\alpha_k| \leq \rho < r$ for all k. Then*

$$\limsup_{n\to\infty} |f(z) - R_n(z)|^{1/2n+1} \leq \delta < 1.$$

uniformly on $\mathbb{T}$, where $\delta = \max\{\delta_r, \delta_R\}$ with

$$\delta_r = \max_{t\in C_r}\{\exp(\lambda(t)/2)\} < 1 \quad \text{and} \quad \delta_R = \max_{t\in C_{1/R}}\{\exp(\lambda(t)/2)\} < 1,$$

where $\lambda(t)$ is given by (4.5) and $C_\rho = \{z \in \mathbb{C} : |z| = \rho\}$.

Proof. By Lemma 4.1, one has for any $z \in \mathbb{B}$,

$$f(z) - R_n(z) = \frac{1}{2\pi i}\int_{C_r} h_n(t)\mathrm{d}t + \frac{1}{2\pi i}\int_{C_R} h_n(t)\mathrm{d}t = I_1 + I_2,$$

with

$$h_n(t) = \frac{1-\overline{\alpha}_{n+1}z}{1-\overline{\alpha}_{n+1}t}\frac{\tilde{Q}_{2n+1}(z)}{\tilde{Q}_{2n+1}(t)}\frac{B_n(t)}{B_n(z)}\frac{f(t)}{t-z}.$$

Obviously, $|f(z) - R_n(z)| \leq |I_1| + |I_2|$. Define

$$M_r(f) = \max_{t\in C_r}|f(t)|, \quad M_R(f) = \max_{t\in C_R}|f(t)|, \quad M_n = \max_{z\in\mathbb{T}}|\tilde{Q}_{2n+1}(z)|.$$

Furthermore, $|B_n(z)| = 1$ for $z \in \mathbb{T}$, so that for any $z \in \mathbb{T}$ we have

$$|I_1| \leq \frac{r(1+\rho)M_r(f)}{(1-r)(1-\rho r)}M_n\max_{t\in C_r}|g_n(t)| = K(f)M_n\max_{t\in C_r}|g_n(t)|, \tag{4.6}$$

where $K(f)$ is a positive constant depending on f and $g_n(t) = B_n(t)/\tilde{Q}_{2n+1}(t)$, which is a continuous function on C_r. (Recall that we assume that $|\alpha_k| \leq \rho < 1$ and that $0 < \rho < r < 1 < R < 1/\rho$.) Thus, from (4.6) we get

$$|I_1|^{1/2n+1} \leq [K(f)]^{1/2n+1}M_n^{1/2n+1}\left[\max_{t\in C_r}|g_n(t)|\right]^{1/2n+1}.$$

Now by Theorem 4.9 in [4] we have $\lim_{n\to\infty} M_n^{1/2n+1} = 1$. On the other hand if $\max_{t\in C_r}|g_n(t)| = |g_n(t_n)|$, for some $t_n \in C_r$, then

$$[\max_{t\in C_r}|g_n(t)|]^{1/2n+1} = |g_n(t_n)|^{1/2n+1} \leq \max_{t\in C_r}|g_n(t)|^{1/2n+1}.$$

Therefore

$$\limsup_{n\to\infty}|I_1|^{1/2n+1} \leq \limsup_{n\to\infty}\left[\max_{t\in C_r}|g_n(t)|^{1/2n+1}\right]$$

$$\begin{aligned}
&\le \max_{t\in C_r}\left[\limsup_{n\to\infty}|g_n(t)|^{1/2n+1}\right]\\
&= \max_{t\in C_r}\limsup_{n\to\infty}\frac{|B_n(t)|^{1/2n+1}}{|\tilde{Q}_{2n+1}(t)|^{1/2n+1}}\\
&\le \max_{t\in C_r}\frac{\limsup_{n\to\infty}|B_n(t)|^{1/2n+1}}{\liminf_{n\to\infty}|\tilde{Q}_{2n+1}(t)|^{1/2n+1}}. \qquad (4.7)
\end{aligned}$$

Now by Theorem 4.8 in [4] we have $\lim_{n\to\infty}|\tilde{Q}_{2n+1}(t)|^{1/2n+1}=1$ for all $z\in\mathbb{D}$ and $\lim_{n\to\infty}|B_n(t)|^{1/n}=\exp(\lambda(t))$, for all $t\in\mathbb{D}\setminus(\{0\}\cup\operatorname{supp}(\nu^{\mathbb{A}}))$. Hence, from (4.7), it follows that

$$\begin{aligned}
\limsup_{n\to\infty}|I_1|^{1/2n+1} &\le \max_{t\in C_r}\lim_{n\to\infty}|B_n(t)|^{1/2n+1}\\
&= \max_{t\in C_r}\lim_{n\to\infty}[|B_n(t)|^{1/n}]^{\frac{n}{2n+1}}\\
&= \max_{t\in C_r}[\exp(\lambda(t)/2)]<1. \qquad (4.8)
\end{aligned}$$

As for the integral I_2, we can similarly obtain that

$$\limsup_{n\to\infty}|I_2|^{1/2n+1}\le\max_{t\in C_R}\frac{\limsup_{n\to\infty}|B_n(t)|^{1/2n+1}}{\liminf_{n\to\infty}|\tilde{Q}_{2n+1}(t)|^{1/2n+1}}.$$

Now $t\in C_R\subset\mathbb{E}$ and because of Theorem 4.8 in [4],

$$\lim_{n\to\infty}|\tilde{Q}_{2n+1}(t)|^{1/2n+1}=\exp(\lambda(t)).$$

This yields

$$\limsup_{n\to\infty}|I_2|^{1/2n+1}\le\max_{t\in C_R}\frac{\exp(\frac{1}{2}\lambda(t))}{\exp(\lambda(t))}=\max_{t\in C_R}[\exp(-\tfrac{1}{2}\lambda(t))].$$

Set $t=1/\overline{x}=\hat{x}$. If $t\in C_R$, then $x\in C_{1/R}=\{z\in\mathbb{C}:|z|=1/R\}\subset\mathbb{D}$. Thus, $\limsup_{n\to\infty}|I_2|^{1/2n+1}\le\max_{x\in C_{1/R}}\exp[-\frac{1}{2}\lambda(\hat{x})]$. Since $\lambda(\hat{x})=-\lambda(x)$, we conclude that

$$\limsup_{n\to\infty}|I_2|^{1/2n+1}\le\max_{t\in C_{1/R}}\exp[\frac{1}{2}\lambda(t)]<1.$$

Finally, for any $z\in\mathbb{T}$, we have

$$\begin{aligned}
\limsup_{n\to\infty}|f(z)-R_n(z)|^{1/2n+1}&\le\limsup_{n\to\infty}[|I_1|+|I_2|]^{1/2n+1}\\
&\le\max\left\{\limsup_{n\to\infty}|I_1|^{1/2n+1},\limsup_{n\to\infty}|I_2|^{1/2n+1}\right\}.
\end{aligned}$$

Putting our results together proves the theorem. □

COROLLARY 4.3. *Under the above conditions, the sequence $\{R_n(z)\}$ of interpolants converges to $f(z)$ uniformly and geometrically on $\mathbb{T}$.*

EXAMPLE 4.1. Let us consider the simple illustrative situation where $\lim_{k\to\infty}\alpha_k=a\in\mathbb{D}$. Then $\nu^{\mathbb{A}}=\delta_a$. Since $\lambda(z)=\int\log|\xi_z(x)|d\nu^{\mathbb{A}}(x)$ with $\xi_z(x)=\frac{x-z}{1-\overline{z}x}$, we obtain $\lambda(z)=\log\left|\frac{z-a}{1-\overline{z}a}\right|$. Hence, $\exp(\frac{1}{2}\lambda(z))=|\frac{z-a}{1-a\overline{z}}|^{1/2}$ and we get

$$\delta_r=\max_{t\in C_r}\left|\frac{t-a}{1-\overline{t}a}\right|^{1/2}<1,\quad \delta_R=\max_{t\in C_{1/R}}\left|\frac{t-a}{1-\overline{t}a}\right|^{1/2}<1.$$

If we assume that $rR<1$, i.e., $r<1/R$, then $\delta_r<\delta_R$ and $\delta=\delta_R$. On the other hand, if $a=0$, then $\delta_r=\sqrt{r}$ and $\delta_R=1/\sqrt{R}$. These situations of the rate of convergence were obtained in [8] in the polynomial situation, i.e., when all $\alpha_k=0$.

So far, we started from the sequence $\mathbb{A} = \{\alpha_k : k \in \mathbb{N}\}$ and we have constructed for each $n \in \mathbb{N}$ an interpolant $R_n \in \mathcal{R}_{n,n}$ for $f(z)$ at the nodes $\{x_{j,2n+1}\}_{j=1}^{2n+1}$ which were the zeros of the para-orthogonal function $\tilde{Q}_{2n+1}(z) \in \tilde{\mathcal{L}}_{2n+1} = \{P/\tilde{\pi}_{2n+1} : P \in \Pi_{2n+1}\}$ and $\tilde{\pi}_{2n+1}(z) = \prod_{j=1}^{2n+1}(1 - \overline{\tilde{\alpha}}_j z)$, associated with the sequence

$$\tilde{\mathbb{A}} = \{\tilde{\alpha}_k\}_{k=1}^{\infty} = \{\alpha_1, \alpha_1, \alpha_2, \alpha_2, \ldots\}.$$

We can generalize Theorem 4.2 to the asymmetric case and consider more general spaces of rational interpolants $\mathcal{R}_n = \mathcal{R}_{p(n),q(n)}$ where $\{p(n)\}$ and $\{q(n)\}$ are nondecreasing sequences of nonnegative integers such that $p(n) + q(n) = n$ and $\lim_{n\to\infty} \frac{p(n)}{n} = s \in (0,1)$. For this purpose, set

$$\begin{aligned} r(n) &= \max\{p(n), q(n)\}, \quad s(n) = \min\{p(n), q(n)\}, \quad \alpha_0 = 0,\\ \tilde{\mathbb{A}}_n &= \{\alpha_0, \alpha_1, \alpha_1, \alpha_2, \alpha_2, \ldots, \alpha_{s(n)}, \alpha_{s(n)}, \alpha_{s(n)+1}, \ldots, \alpha_{r(n)}\}\\ &= \{\tilde{\alpha}_0, \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_4, \ldots \qquad\qquad \ldots, \tilde{\alpha}_n \ \}. \end{aligned}$$

Since increasing n to $n+1$ increases either $p(n)$ or $q(n)$ by one, this increases either $r(n)$ or $s(n)$ by one. The numbering of the $\tilde{\alpha}_k$ is such that $\tilde{\alpha}_n$ is a repeated point $\alpha_{s(n)+1}$ or a new point $\alpha_{r(n)+1}$. This defines the sequence $\tilde{\mathbb{A}} = \{\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots\}$ uniquely. The quantities related to $\tilde{\mathbb{A}}$ are as before denoted with a tilde. (For further details see [7].) Thus, let $\{x_{j,n+1}\}_{j=1}^{n+1}$ be the zeros of the para-orthogonal function $\tilde{Q}_{n+1}(z, \tau_{n+1}) = \tilde{\phi}_{n+1}(z) + \tau_{n+1}\tilde{\phi}^*_{n+1}(z)$, $\tau_{n+1} \in \mathbb{T}$, and let $R_n(z)$ be the interpolant in $\mathcal{R}_n = \mathcal{R}_{p(n),q(n)}$ for $f(z)$ at such nodes. Proceeding as in the proof of Theorem 4.2, we can obtain the following

THEOREM 4.4. *Let $f(z)$ be an analytic function in the annulus $\mathbb{B} = \{z \in \mathbb{C} : r < |z| < R\}$ with $0 < r < 1 < R$. Let $R_n(z)$ be the interpolant in $\mathcal{R}_n = \mathcal{R}_{p(n),q(n)}$ for $f(z)$ at the nodes $\{x_{j,n+1}\}_{j=1}^{n+1}$ as given above, and where $p(n)$ and $q(n)$ are nondecreasing sequences of nonnegative integers such that $p(n) + q(n) = n$ and $\lim_{n\to\infty} \frac{p(n)}{n} = s \in (0,1)$. Then*

$$\limsup_{n\to\infty} |f(z) - R_n(z)|^{1/n} \le \delta(s) < 1$$

uniformly on $\mathbb{T}$ where $\delta(s) = \max\{\delta_r(s), \delta_R(s)\}$ with

$$\delta_r(s) = \max_{t\in C_r}\{\exp(s\lambda(t))\}, \quad \textit{and} \quad \delta_R(s) = \max_{t\in C_{1/R}}\{\exp((1-s)\lambda(t))\}$$

where $\lambda(t)$ is given by (4.5).

As can be observed both from Theorem 4.2 and 4.4, our interpolants belong to the subspaces of rational functions $\mathcal{R}_n = \mathcal{R}_{p(n),q(n)}$ associated with the sequence $\mathbb{A} = \{\alpha_k\}$. However, the interpolation nodes come from the para-orthogonal functions associated to a new sequence $\tilde{\mathbb{A}} = \{\tilde{\alpha}_k\}$. One may wonder whether it is possible to use the zeros of the para-orthogonal functions associated with $\mathbb{A}$ as interpolation nodes. Therefore we consider the following quasi-Hermite interpolation problem. For a given sequence of complex numbers $\tau_n \in \mathbb{T}$, $n = 1, 2, \ldots$, let us consider the corresponding sequence of para-orthogonal functions $Q_n(z, \tau_n) = \phi_n(z) + \tau_n \phi^*_n(z)$. Let $\{x_{j,n+1}\}_{j=1}^{n+1}$ be the zeros of $Q_{n+1}(z, \tau_n) = Q_{n+1}(z)$. Take x a fixed complex number on $\mathbb{T}$. Then, by varying $\tau_n \in \mathbb{T}$, we can place for example $x_{1,n+1}$ at that position for any n. Under these conditions, let $f(z)$ be an analytic function in a region $\mathbb{B}$ containing $\mathbb{T}$ and let $H_{2n+1}(z)$ be the unique interpolant in $\mathcal{R}_{n,n}$ satisfying

$$\begin{aligned} H_{2n+1}(x_{j,n+1}) &= f(x_{j,n+1}), \quad j = 1, \ldots, n+1\\ H'_{2n-1}(x_{j,n+1}) &= f'(x_{j,n+1}), \quad j = 1, \ldots, n+1, \ x_{j,n+1} \ne x. \end{aligned} \tag{4.9}$$

Set $Q_{n+1}(z) = P_{n+1}(z)/\pi_{n+1}(z)$ with $P_{n+1}(x_{j,n+1}) = 0$, $j = 1, \ldots, n+1$ and write $\hat{P}_n(z) = P_{n+1}(z)/(z - x)$. Define $\hat{\mathbb{A}} = \{\hat{\alpha}_k\}_{k=1}^{\infty}$ such that $\hat{\alpha}_k = \alpha_{k+1}$, $k = 1, 2, \ldots$ and consider the corresponding quantities associated with $\hat{\mathbb{A}}$ with a hat, i.e., $\hat{\mathcal{L}}_n$, $\hat{\mathcal{R}}_{n,n}$, $\hat{\pi}_n$, $\hat{\omega}_n$, $\hat{B}_n$,.... We first need the following.

LEMMA 4.5. *With the above notation,* $\hat{Q}_n(z) = \hat{P}_n(z)/\hat{\pi}_n(z) \in \hat{\mathcal{L}}_n$ *is a para-orthogonal function with respect to the weight function* $\tilde{\omega}(\theta) = |\frac{z-x}{1-\overline{\alpha}_1 z}|^2\omega(\theta)$, $z = e^{i\theta}$.

Proof. We have to prove that $\hat{Q}_n$ is para-orthogonal, which means that it is not orthogonal to 1 or $\hat{B}_n$ while it should be orthogonal to the space $\hat{\mathcal{L}}_n(\alpha_n) \cap \hat{\mathcal{L}}_{n-1}$ where $\hat{\mathcal{L}}_n(\alpha_n) = \{f \in \hat{\mathcal{L}}_n : f(\hat{\alpha}_n) = 0\}$. (For this and other properties of para-orthogonal rational functions see [6].) Let us start by proving the latter orthogonality property. Take $f \in \hat{\mathcal{L}}_n(\alpha_n) \cap \hat{\mathcal{L}}_{n-1}$, then

$$f(z) = \frac{(z-\hat{\alpha}_n)P(z)}{\hat{\pi}_{n-1}(z)} = \frac{(z-\alpha_{n+1})P(z)}{\hat{\pi}_{n-1}(z)}, \quad \text{with} \quad P \in \Pi_{n-2}.$$

Hence

$$\begin{aligned}
\int_0^{2\pi} \hat{Q}_n(z)\overline{f(z)}\tilde{\omega}(\theta)\mathrm{d}\theta &= \int_0^{2\pi} \frac{\hat{P}_n(z)}{\hat{\pi}_n(z)}\left|\frac{z-x}{1-\overline{\alpha}_1 z}\right|^2 \overline{f(z)}\omega(\theta)\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{\hat{P}_n(z)}{\hat{\pi}_n(z)}\left[\frac{z-x}{1-\overline{\alpha}_1 z}\right]\overline{\left[\frac{z-x}{1-\overline{\alpha}_1 z}\right]}\overline{f(z)}\omega(\theta)\mathrm{d}\theta \\
&= \int_0^{2\pi} Q_{n+1}(z)\overline{g(z)}\omega(\theta)\mathrm{d}\theta
\end{aligned}$$

where

$$g(z) = \frac{(z-x)f(z)}{(1-\overline{\alpha}_1 z)} = \frac{(z-x)(z-\alpha_{n+1})P(z)}{(1-\overline{\alpha}_1 z)\hat{\pi}_{n-1}(z)} = \frac{(z-\alpha_{n+1})q(z)}{\pi_n(z)},$$

with $q \in \Pi_{n-1}$. This means that $g \in \mathcal{L}_{n+1}(\alpha_{n+1}) \cap \mathcal{L}_n$, and by para-orthogonality of Q_{n+1}, this means that $\int_0^{2\pi} Q_{n+1}(z)\overline{g(z)}\omega(\theta)\mathrm{d}\theta = 0$. This proves orthogonality of $\hat{Q}_n$ to $\hat{\mathcal{L}}_n(\alpha_n) \cap \hat{\mathcal{L}}_{n-1}$. But $\hat{Q}_n$ should not be orthogonal to 1 or $\hat{B}_n$. Therefore we consider

$$\begin{aligned}
\int_0^{2\pi} \hat{Q}_n(z)\tilde{\omega}(\theta)\mathrm{d}\theta &= \int_0^{2\pi} \frac{\hat{P}_n(z)}{\hat{\pi}_n(z)}\left|\frac{z-x}{1-\overline{\alpha}_1 z}\right|^2 \omega(\theta)\mathrm{d}\theta \\
&= \int_0^{2\pi} Q_{n+1}(z)\overline{\left(\frac{z-x}{1-\overline{\alpha}_1 z}\right)}\omega(\theta)\mathrm{d}\theta \\
&= \tau_{n+1}\int_0^{2\pi} \phi_{n+1}^*(z)\overline{\left(\frac{z-x}{1-\overline{\alpha}_1 z}\right)}\omega(\theta)\mathrm{d}\theta, \quad \tau_{n+1} \in \mathbb{T}.
\end{aligned}$$

Thus

$$\begin{aligned}
\overline{\int_0^{2\pi} \hat{Q}_n(z)\tilde{\omega}(\theta)\mathrm{d}\theta} &= \overline{\tau}_{n+1}\int_0^{2\pi} \phi_{n+1}(z)\overline{B_{n+1}(z)}\left(\frac{z-x}{1-\overline{\alpha}_1 z}\right)\omega(\theta)\mathrm{d}\theta \\
&= \overline{\eta}_{n+1}\overline{\tau}_{n+1}\int_0^{2\pi} \phi_{n+1}(z)\frac{\pi_{n+1}(z)}{\omega_{n+1}(z)}\left(\frac{z-x}{1-\overline{\alpha}_1 z}\right)\omega(\theta)\mathrm{d}\theta \\
&= \overline{\eta}_{n+1}\overline{\tau}_{n+1}\int_0^{2\pi} \phi_{n+1}(z)f_n(z)\omega(\theta)\mathrm{d}\theta,
\end{aligned}$$

where $f_n(z) \in \mathcal{L}_{n+1} \setminus \mathcal{L}_n$. Therefore, we see that $\int_0^{2\pi} \hat{Q}_n(z)\tilde{\omega}(\theta)\mathrm{d}\theta \neq 0$, $(z = e^{i\theta})$. Similarly, it can be proved that $\int_0^{2\pi} \hat{Q}_n(z)\hat{B}_n(z)\tilde{\omega}(\theta)\mathrm{d}\theta \neq 0$ and the proof can be concluded. □

Proceeding now as in Lemma 4.1, we can obtain an error expression for the quasi-Hermite rational interpolant satisfying (4.9) Indeed, if $f(z)$ is an analytic function in a region $\mathbb{B}$ containing $\mathbb{T}$, it holds that for any $z \in \mathbb{B}$, $z \neq \alpha_j$, $z \neq 1/\overline{\alpha}_j$, $j = 1, \ldots, n$

$$f(z) - H_{2n+1}(z) = \qquad (4.10)$$

$$\frac{1}{2\pi i}\int_C \left[\frac{1-\overline{\alpha}_{n+1}z}{1-\overline{\alpha}_{n+1}t}\right]^2 \frac{Q_{n+1}(z)}{Q_{n+1}(t)}\frac{\hat{Q}_n(z)}{\hat{Q}_n(t)}\frac{B_n(t)}{B_n(z)}\left[\frac{1-\overline{\alpha}_1 t}{1-\overline{\alpha}_1 z}\right]\frac{f(t)}{t-z}\mathrm{d}t,$$

with Q_{n+1} and $\hat{Q}_n$ para-orthogonal functions as defined above. From (4.10) and following rather closely the proof of Theorem 4.2, we can easily prove the following.

THEOREM 4.6. *Let $f(z)$ be an analytic function in the annulus $\mathbb{B} = \{z \in \mathbb{C} : r < |z| < R\}$ with $0 < r < 1 < R$. Let $H_{2n+1}(z)$ be the interpolant from (4.9) at the nodes $\{x_{j,n+1}\}$ which are the zeros of $Q_{n+1}(z) = \phi_{n+1}(z) + \tau_{n+1}\phi^*_{n+1}(z)$, $|\tau_{n+1}| = 1$, a para-orthogonal function in $\mathcal{L}_{n+1}$ with respect to a weight function $\omega(\theta)$. Then*

$$\limsup_{n\to\infty} |f(z) - H_{2n+1}(z)|^{1/2n+1} \leq \delta < 1,$$

uniformly on $\mathbb{T}$, where δ is given in Theorem 4.2.

Now, from Theorem 4.6 and Theorem 2.2, we can immediately give an estimate of the rate of convergence of certain sequences of R-Szegő formulas. Indeed, set $I_\omega(f) = \int_0^{2\pi} f(e^{i\theta})\omega(\theta)\mathrm{d}\theta$ with $\omega(\theta)$ a given weight function on $[0, 2\pi)$ and let $I_n(f) = \sum_{j=1}^n A_{j,n} f(x_{j,n})$ denote the corresponding n-point R-Szegő formula. By Theorem 2.2, one can write $I_n(f) = I_\omega(H_{2n-1})$ where H_{2n-1} interpolates f according to (4.9) with n replaced by $n-1$, where $f(z)$ is a function analytic in a region $\mathbb{B}$ containing $\mathbb{T}$. As usual, we shall assume that $\mathbb{B} = \{z \in \mathbb{C} : r < |z| < R\}$ with $0 < \rho < r < 1 < R < 1/\rho$, where ρ is a positive number so that $|\alpha_j| \leq \rho$, $j = 1, 2, \ldots$. Set

$$E_n(f) = I_\omega(f) - I_n(f) = \int_0^{2\pi} (f(x) - H_{2n-1}(x))\omega(\theta)\mathrm{d}\theta, \quad x = e^{i\theta}.$$

Recalling that we imposed that $\int_0^{2\pi} \omega(\theta)\mathrm{d}\theta = 1$, we have $|E_n(f)| \leq \max_{x\in\mathbb{T}} |f(x) - H_{2n-1}(x)|$. Thus

$$\begin{aligned} |E_n(f)|^{1/n} &\leq [\max_{x\in\mathbb{T}} |f(x) - H_{2n-1}(x)|]^{1/n} \\ &\leq \max_{x\in\mathbb{T}} |f(x) - H_{2n-1}(x)|^{1/n}. \end{aligned}$$

Hence

$$\begin{aligned} \limsup_{n\to\infty} |E_n(f)|^{1/n} &\leq \limsup_{n\to\infty} \max_{x\in\mathbb{T}} |f(x) - H_{2n-1}(x)|^{1/n} \\ &\leq \max_{x\in\mathbb{T}} \limsup_{n\to\infty} |f(x) - H_{2n-1}(x)|^{1/n}. \end{aligned}$$

By Theorem 4.6, it results that

$$\limsup_{n\to\infty} |E_n(f)|^{1/n} \leq \max_{x\in\mathbb{T}} \limsup_{n\to\infty} [|f(x) - H_{2n-1}(x)|^{1/2n-1}]^{\frac{2n-1}{n}} \leq \delta^2$$

with δ as in Theorem 4.2. Thus we have proved the following.

THEOREM 4.7. *Let $E_n(f)$ denote the error for the n-point R-Szegő formula, i.e., $E_n(f) = \int_0^{2\pi} f(e^{i\theta})\omega(\theta)\mathrm{d}\theta - \sum_{j=1}^n A_{j,n} f(x_{j,n}) = I_\omega(f) - I_n(f)$, where f is a function analytic in an annulus $\mathbb{B} = \{z \in \mathbb{C} : r < |z| < R\}$ such that $0 < r < 1 < R$. Then*

$$\limsup_{n\to\infty} |E_n(f)|^{1/n} \leq \max\{\tilde{\delta}_r, \tilde{\delta}_R\} < 1$$

where

$$\tilde{\delta}_r = \max_{t\in C_r}\{\exp(\lambda(t))\}, \quad \textit{and} \quad \tilde{\delta}_R = \max_{t\in C_{1/R}}\{\exp(\lambda(t))\},$$

where $\lambda(t)$ is given by (4.5).

REMARK 4.1. The estimate of the rate of convergence given in Theorem 4.7 was obtained earlier by the authors in Theorem 4.4 of [4] in connection with multipoint Padé approximants to the Riesz-Herglotz transform of the measure $\mathrm{d}\mu(\theta) = \omega(\theta)\mathrm{d}\theta$.

REMARK 4.2. During the last years, the authors have studied extensively orthogonal rational functions and quadrature formulas on the unit circle along with other related topics such as moment problems, multipoint versions of Padé-like rational approximation etc. (see e.g., [6]). The rational interpolation problems that were considered in our previous papers were mainly investigated in the connection with the construction of quadrature formulas. Therefore we used rational interpolants (usually not in $\mathcal{L}_n$) for the Riesz-Herglotz transform of a positive measure on the unit circle. The interpolation points were the α_k's or their reciprocals. Use is made of orthogonal rational functions (in $\mathcal{L}_n$) to construct the rational interpolants. The nodes of the quadrature formulas were the zeros of the para-orthogonal rational functions.

The problem considered here is quite different. Here we construct a rational interpolant from $\mathcal{L}_n$ for an arbitrary function. We consider the choice of a set of interpolation points such that we get good convergence properties. In this paper they were chosen to be the zeros of the para-orthogonal rational functions.

Of course, interpolation by using rational functions with prescribed poles is a classical problem (see [13]) which is presently receiving more and more attention. In this respect, making use of the zeros of para-orthogonal rational functions as interpolation nodes can be viewed as a first step in the analysis of the convergence process when dealing with functions defined on $\mathbb{T}$. We think that more general results than those given in Section 3 for L^2 convergence and also for L^p $(p > 2)$ convergence will be obtained in forthcoming papers.

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