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SOLUTION OF A MULTIPLE NEVANLINNA-PICK PROBLEM FOR SCHUR FUNCTIONS VIA ORTHOGONAL RATIONAL FUNCTIONS

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Abstract An interpolation problem of Nevanlinna-Pick type for complex-valued Schur functions in the open unit disk is considered. We prescribe the values of the function and its derivatives up to a certain order at finitely many points. Primarily, we study the case that there exist many Schur functions fulfilling the required conditions. For this situation, an application of the theory of orthogonal rational functions is used to characterize the set of all solutions of the problem in question. Moreover, we treat briefly the case of exactly one solution and present an explicit description of the unique solution in that case.

Keywords: Nevanlinna-Pick interpolation problem, Schur functions, orthogonal rational functions, Schur-Nevanlinna algorithm, Szegő parameters, Schur parameters

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1. Introduction

Starting with Carathéodory's paper [8] interpolation problems and particularly Taylor coefficient problems for classes of holomorphic functions have been studied widely. Probably, one of the most important classical works on this topic is Schur's paper [24]. It contains already an algorithm to check if the given data in a Taylor coefficient problem correspond to a holomorphic functions in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}$ of the complex plane $\mathbb{C}$ which is bounded by one in modulus. (In the sequel, we call a holomorphic function in $\mathbb{D}$ which is bounded by one in modulus *Schur function* and the notation $\mathcal{S}$ stands for the set of all Schur functions.) Some years later, Nevanlinna presented a generalization of Schur's algorithm and used this to solve an interpolation problem for Schur functions where some values of the function are prescribed (see, e.g., [20]). Concerning the history it has to be mentioned that Pick was the first considering such a kind of interpolation problems (see, e.g., [21]). Today, there is an extensive liter-

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ature on several types of such problems for holomorphic functions (see, e.g., the books [1], [13], [15], [3], [12], [23], [16]).

The present paper is another contribution to this topic and deals with an interpolation problem for Schur functions including both the Taylor coefficient problem treated by Schur and the classical interpolation problem studied by Nevanlinna and Pick as special cases. More precisely, the problem we are going to study is a multiple point interpolation problem, i.e. a problem where along with the values of the function the values of its derivatives up to a certain order are prescribed at some points as follows:

(MNP) Given are mutually distinct points $z_1, z_2, \ldots, z_n \in \mathbb{D}$, numbers $l_1, l_2, \ldots, l_n \in \mathbb{N}$, and some $w_{js} \in \mathbb{C}$, $s = 0, 1, \ldots, l_j - 1$, $j = 1, 2, \ldots, n$, where $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$. (These data are represented by Δ, see (2.1) below.) Find necessary and sufficient conditions for the existence of a $g \in \mathcal{S}$ such that

$$\frac{1}{s!} g^{(s)}(z_j) = w_{js}, \quad s = 0, 1, \ldots, l_j - 1, \; j = 1, 2, \ldots, n. \tag{1.1}$$

Moreover, describe the solution set $\mathcal{S}_\Delta$ of all $g \in \mathcal{S}$ fulfilling (1.1).

Note that the terminology of such interpolation problems is not unified in the literature (cf., e.g., [22, Section 2.6], [15, Section X.5], [27], and [28]).

We continue here the investigations stated in [7, Section 6]. As a main result there, [7, Theorem 6.3] gives a description of $\mathcal{S}_\Delta$, if there is more than one Schur function g fulfilling (1.1), by means of the linear fractional transformation

$$g(z) = \frac{\delta_m^{[m]}(z) + b_{\alpha_m}(z)\gamma_m(z)h(z)}{\gamma_m^{[m]}(z) + b_{\alpha_m}(z)\delta_m(z)h(z)}, \quad z \in \mathbb{D}. \tag{1.2}$$

In doing so, δ_m and γ_m are special rational functions recursively defined via some Schur parameters corresponding to an algorithm of Schur-Nevanlinna type, $\delta_m^{[m]}$ and $\gamma_m^{[m]}$ denote their adjoint rational functions (according to (3.9)), b_{α_m} is a Blaschke factor, and if the parameter function h runs through the Schur class $\mathcal{S}$ then the function g runs through the solution set $\mathcal{S}_\Delta$ of Problem (MNP). In particular, the functions γ_m and δ_m occurring in (1.2) can be constructed from the interpolation data Δ, but only indirectly. One needs to determine the corresponding Schur parameters first, and these are not easy to compute from Δ by sole utilization of the algorithm of Schur-Nevanlinna type in general.

The main objective of this paper is to show a way out. In fact, we will see that the theory of orthogonal rational functions on the unit circle $\mathbb{T} := \{u \in \mathbb{C} : |u| = 1\}$ pointed out in [6] can be used to detect the required Schur parameters or the rational functions γ_m and δ_m directly. The details about this can be found in Section 5. Based on this fact, we obtain in Section 6 a further description of $\mathcal{S}_\Delta$ similar to the linear fractional transformation of (1.2), but with expressions that are given more explicitly in terms of the underlying data Δ in the problem. In the sections preceeding those results, we explain how the theory of orthogonal rational functions on $\mathbb{T}$ comes into play starting from the given data Δ. In other words, using the generalized Schwarz-Pick matrix $\mathbf{P}_\Delta$ which can be computed from the data Δ given in Problem (MNP), we introduce some spaces of rational functions in Section 3 and on this basis we bring in orthogonal rational functions on $\mathbb{T}$ in Section 4. Particularly, in Section 4 we will get that in the non-degenerate case, i.e. if $\mathbf{P}_\Delta$ is a positive Hermitian matrix, this generalized Schwarz-Pick matrix (with $w_{10} := 0$ in (1.1)) can be parametrized by some special parameters $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{D}$ which appear in the recurrence relations for the orthogonal rational functions (cf. [11] for the case of positive Hermitian block Toeplitz matrices). Since the matrix $\mathbf{P}_\Delta$ is a crucial tool, we start in Section 2 with some basics on this generalized Schwarz-Pick matrix.

We focus mostly on the situation where Problem (MNP) has infinitely many solutions, i.e. when the associated matrix $\mathbf{P}_\Delta$ is positive Hermitian. Starting from those results, we shall obtain in Section 7 an explicit description of the unique solution for the degenerate case as well, i.e. if $\mathbf{P}_\Delta$ is a non-negative Hermitian matrix and $\det \mathbf{P}_\Delta = 0$.

Note that there exist other approaches to solve Problem (MNP) (see, e.g., [14], [15], [27], and [4]). The essential new feature of this paper is that we make substantial use of the orthogonal rational functions on $\mathbb{T}$ which were introduced by Djrbashian [10] (see also [5, 6] and other papers cited there). The role of the orthogonal rational functions here is analogous to the role of the orthogonal polynomials in the study of the trigonometric moment problem or the Taylor coefficient problem for Carathéodory functions. Schur's classical algorithm delivers some coefficients (so-called Schur parameters) that turned out to be exactly the complex conjugates of the coefficients (so-called Szegő parameters) that appeared in the recurrence relations for the orthogonal polynomial as formulated by Szegő [25]. This classical result goes back to Geronimus (see [17] and for a matrix extension [11]). In view of the results presented below, we prove and apply a similar connection between an algorithm of Schur-Nevanlinna type and orthogonal rational functions on $\mathbb{T}$ (cf. [6, Section 6.4] and [18, Section 6]).

The interplay between interpolation problems of Nevanlinna-Pick type and orthogonal rational functions on $\mathbb{T}$ plays already a central role in [6]. In this monograph some particular solutions of such interpolation problems are obtained and many other related questions are discussed, but the precise Problem (MNP) is not. We are guided by the investigations in [18], where a similar problem for Carathéodory functions is solved via orthogonal rational functions on $\mathbb{T}$. We deal here with Schur functions and moreover without normalizing the first interpolation point z_1 to be 0, so we need to rephrase some known results in detail since some concrete formulas will differ from the corresponding formulas in [18]. Finally, we should also mention in this context that recently in [2] a parametrization of the solution set for a rational moment problem based on orthogonal rational functions with respect to the real line case is exposed.

2. The generalized Schwarz-Pick matrix $\mathbf{P}_\Delta$

In view of Problem (MNP), we assume from now on that the following data are given: an $n \in \mathbb{N}$, mutually different points $z_1, z_2, \ldots, z_n \in \mathbb{D}$, numbers $l_1, l_2, \ldots, l_n \in \mathbb{N}$ and some $w_{js} \in \mathbb{C}$, $s = 0, 1, \ldots, l_j - 1$, $j = 1, 2, \ldots, n$. We denote this data set by Δ, i.e.

$$\Delta := \left\{ \left(z_j, l_j, (w_{js})_{s=0}^{l_j-1} \right)_{j=1}^{n} \right\}, \tag{2.1}$$

and put additionally

$$m := \sum_{j=1}^{n} l_j - 1.$$

Furthermore, the complex $(l_j \times l_k)$-matrices

$$\mathbf{P}_{jk} := \left(p_{jk}^{s,t}\right)_{\substack{s=0,1,\dots,l_j-1\\ t=0,1,\dots,l_k-1}}, \quad j,k = 1,2,\dots,n,$$

are constructed from Δ, for each $s \in \{0,1,\dots,l_j-1\}$ and each $t \in \{0,1,\dots,l_k-1\}$, by the entries (compare with (2.6) below)

$$\begin{aligned} p_{jk}^{s,t} \;:=\; & \sum_{r=0}^{\min\{s,t\}} \frac{(s+t-r)!}{(s-r)!r!(t-r)!} \frac{z_j^{t-r}\overline{z_k}^{s-r}}{(1-z_j\overline{z_k})^{s+t-r+1}} \\ & - \sum_{\ell=0}^{s}\sum_{h=0}^{t}\sum_{r=0}^{\min\{\ell,h\}} \frac{(h+\ell-r)!}{(\ell-r)!r!(h-r)!} \frac{z_j^{h-r}\overline{z_k}^{h-r}}{(1-z_j\overline{z_k})^{h+\ell-r+1}} w_{j,s-\ell}\overline{w_{k,t-h}}. \end{aligned} \tag{2.2}$$

Therewith, the *generalized Schwarz-Pick matrix* of size $(m+1)\times(m+1)$ is defined by

$$\mathbf{P}_\Delta := (\mathbf{P}_{jk})_{j,k=1}^{n}.$$

Note that in the case $l_1 = l_2 = \dots = l_n = 1$ we obtain the classical Schwarz-Pick matrix

$$\mathbf{P}_\Delta = \left(\frac{1 - w_{j0}\overline{w_{k0}}}{1 - z_j\overline{z_k}}\right)_{j,k=1}^{n},$$

which plays a crucial role in the original Nevanlinna-Pick interpolation problem for Schur functions (see, e.g., [21] and [20]). In general, if we define the $(l_j \times l_k)$-matrices

$$\mathbf{Z}_{jk} := \left(\frac{1}{s!t!}\frac{\partial^{s+t}}{\partial v^s \partial w^t}\frac{1}{1-vw}\,\Big|_{\substack{v=z_j\\ w=\overline{z_k}}}\right)_{\substack{s=0,1,\dots,l_j-1\\ t=0,1,\dots,l_k-1}}, \quad j,k = 1,2,\dots,n,$$

the $(l_j \times l_j)$-matrices

$$\mathbf{W}_{jj} := \begin{pmatrix} w_{j0} & 0 & \cdots & 0 \\ w_{j1} & w_{j0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ w_{j,l_j-1} & \cdots & w_{j1} & w_{j0} \end{pmatrix}, \quad j = 1,2,\dots,n, \tag{2.3}$$

and the matrices

$$\mathbf{Z} := (\mathbf{Z}_{jk})_{j,k=1}^{n} \quad \text{and} \quad \mathbf{W} := \operatorname{diag}\big(\mathbf{W}_{11}, \mathbf{W}_{22}, \dots, \mathbf{W}_{nn}\big), \tag{2.4}$$

then a straightforward calculation leads to the relations

$$\mathbf{P}_{jk} = \mathbf{Z}_{jk} - \mathbf{W}_{jj}\mathbf{Z}_{jk}\mathbf{W}_{kk}^{*}, \quad j,k = 1,2,\dots,n, \tag{2.5}$$

or, equivalently,

$$\mathbf{P}_\Delta = \mathbf{Z} - \mathbf{W}\mathbf{Z}\mathbf{W}^*.$$

Observe that the generalized Schwarz-Pick matrix $\mathbf{P}_\Delta$ is uniquely determined by the given data Δ except for a constant of modulus one as explained in the following.

Remark 2.1. Let the data set Δ be given as in (2.1), let $\widetilde{w}_{js}$ be a complex number for each $s = 0, 1, \ldots, l_j - 1$ and each $j = 1, 2, \ldots, n$, and let

$$\widetilde{\Delta} := \left\{ \left(z_j, l_j, (\widetilde{w}_{js})_{s=0}^{l_j-1} \right)_{j=1}^{n} \right\}.$$

The equality $\mathbf{P}_\Delta = \mathbf{P}_{\widetilde{\Delta}}$ is fulfilled if and only if there is a $u \in \mathbb{T}$ satisfying $w_{js} = u\widetilde{w}_{js}$ for all $s = 0, 1, \ldots, l_j - 1$ and all $j = 1, 2, \ldots, n$.

If g is a Schur function fulfilling (1.1) then, in view of (2.2), we have

$$p_{jk}^{s,t} = \frac{1}{s!t!} \frac{\partial^{s+t}}{\partial v^s \partial w^t} \frac{1 - g(v)\overline{g(\overline{w})}}{1 - vw} \Bigg|_{\substack{v=z_j \\ w=\overline{z_k}}}. \tag{2.6}$$

It is a well-known fact (see, e.g., [6, Theorem 1.2.4]), that the Cayley transform gives a connection between the Schur class $\mathcal{S}$ and the *Carathéodory class* $\mathcal{C}$, where $\mathcal{C}$ is the set of all functions which map the open unit disk $\mathbb{D}$ holomorphically into the closed right half plane. Consequently, it is not hard to accept that the generalized Schwarz-Pick matrix of a Schur function can be computed by the appropriate values and derivatives of its Cayley transform and vice versa. In fact, the relations given in Remark 2.2 below do hold (cf. (2.5), [12, Lemma 1.1.21 and Lemma 1.3.12]). To give the relations, we define for a function h holomorphic in $\mathbb{D}$, the following $(l_j \times l_j)$-matrices similar to (2.3):

$$\mathbf{W}_{jj}^{(h)} := \begin{pmatrix} h(z_j) & 0 & \cdots & 0 \\ \frac{1}{1!}h^{(1)}(z_j) & h(z_j) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \frac{1}{(l_j-1)!}h^{(l_j-1)}(z_j) & \cdots & \frac{1}{1!}h^{(1)}(z_j) & h(z_j) \end{pmatrix}, \quad j = 1, 2, \ldots, n.$$

Remark 2.2. If $\Omega \in \mathcal{C}$ then $g := \dfrac{1 - \Omega}{1 + \Omega} \in \mathcal{S}$, where $\Omega = \dfrac{1 - g}{1 + g}$ and

$$\left(\frac{1}{s!t!} \frac{\partial^{s+t}}{\partial v^s \partial w^t} \frac{1 - g(v)\overline{g(\overline{w})}}{1 - vw} \Bigg|_{\substack{v=z_j \\ w=\overline{z_k}}} \right)_{\substack{s=0,1,\ldots,l_j-1 \\ t=0,1,\ldots,l_k-1}}$$

$$= 2\big(\mathbf{I} + \mathbf{W}_{jj}^{(\Omega)}\big)^{-1} \left(\frac{1}{s!t!} \frac{\partial^{s+t}}{\partial v^s \partial w^t} \frac{\Omega(v) + \overline{\Omega(\overline{w})}}{1 - vw} \Bigg|_{\substack{v=z_j \\ w=\overline{z_k}}} \right)_{\substack{s=0,1,\ldots,l_j-1 \\ t=0,1,\ldots,l_k-1}} \Big(\big(\mathbf{I} + \mathbf{W}_{kk}^{(\Omega)}\big)^{-1} \Big)^*.$$

Inversely, if $g \in \mathcal{S}$ with $g(z_0) \neq -1$ for some $z_0 \in \mathbb{D}$ then $\Omega := \dfrac{1-g}{1+g} \in \mathcal{C}$, $g = \dfrac{1-\Omega}{1+\Omega}$, and

$$\left(\frac{1}{s!t!} \frac{\partial^{s+t}}{\partial v^s \partial w^t} \frac{\Omega(v) + \overline{\Omega(\overline{w})}}{1 - vw} \Bigg|_{\substack{v=z_j \\ w=\overline{z_k}}} \right)_{\substack{s=0,1,\dots,l_j-1 \\ t=0,1,\dots,l_k-1}}$$
$$= 2\big(\mathbf{I} + \mathbf{W}_{jj}^{(g)}\big)^{-1} \left(\frac{1}{s!t!} \frac{\partial^{s+t}}{\partial v^s \partial w^t} \frac{1 - g(v)\overline{g(\overline{w})}}{1 - vw} \Bigg|_{\substack{v=z_j \\ w=\overline{z_k}}} \right)_{\substack{s=0,1,\dots,l_j-1 \\ t=0,1,\dots,l_k-1}} \Big(\big(\mathbf{I} + \mathbf{W}_{kk}^{(g)}\big)^{-1}\Big)^*.$$

Here and in the sequel, the identity matrix is denoted as $\mathbf{I}$. Furthermore, we always write shortly 0 for the zero matrix of appropriate size and if $\mathbf{A}$, $\mathbf{B}$ are Hermitian matrices of the same size then $\mathbf{A} \geq \mathbf{B}$ (resp., $\mathbf{A} > \mathbf{B}$) means that $\mathbf{A} - \mathbf{B}$ is a non-negative (resp., positive) Hermitian matrix.

Theorem 2.3. *The set $\mathcal{S}_\Delta$ is non-empty if and only if $\mathbf{P}_\Delta \geq 0$.*

A proof of Theorem 2.3 can be found for instance in [26]. Based on Remark 2.2 one can also derive Theorem 2.3 from a corresponding result for Carathéodory functions (which is proved, e.g., in [9, Section 2]).

The maximum modulus principle for holomorphic functions proves that if $|w_{k0}| = 1$ for some $k = 1, 2, \dots, n$ then there exists a solution of Problem (MNP) if and only if $w_{j0} = w_{k0}$ for each $j = 1, 2, \dots, n$ as well as $w_{js} = 0$ for each $s = 1, 2, \dots, l_j - 1$ and each $j = 1, 2, \dots, n$, where the constant function with value w_{k0} is the unique solution. We mostly exclude this trivial case in the further course of this paper and suppose that

$$|w_{j0}| \neq 1, \quad j = 1, 2, \dots, n. \tag{2.7}$$

Moreover, we shall use the normalization $w_{10} := 0$ in the following sections. The final remark of this section shows that this restriction is actually without loss of generality.

Remark 2.4. A $g \colon \mathbb{D} \to \mathbb{C}$ fulfilling $g(z_1) \in \mathbb{D}$ belongs to $\mathcal{S}$ if and only if the function

$$f(z) := \frac{g(z) - g(z_1)}{1 - \overline{g(z_1)}g(z)}, \quad z \in \mathbb{D},$$

belongs to $\mathcal{S}$ (cf. [6, Theorem 1.2.3]). In particular, if $g \in \mathcal{S}$ with $g(z_1) \in \mathbb{D}$ then $f(z_1) = 0$ and, in view of (2.6), (2.5), and [12, Lemma 1.1.21], a straightforward calculation yields

$$\frac{1}{1 - |g(z_1)|^2} \left(\frac{1}{s!t!} \frac{\partial^{s+t}}{\partial v^s \partial w^t} \frac{1 - f(v)\overline{f(\overline{w})}}{1 - vw} \Bigg|_{\substack{v=z_j \\ w=\overline{z_k}}} \right)_{\substack{s=0,1,\dots,l_j-1 \\ t=0,1,\dots,l_k-1}}$$
$$= \big(\mathbf{I} - \overline{g(z_1)}\mathbf{W}_{jj}^{(g)}\big)^{-1} \left(\frac{1}{s!t!} \frac{\partial^{s+t}}{\partial v^s \partial w^t} \frac{1 - g(v)\overline{g(\overline{w})}}{1 - vw} \Bigg|_{\substack{v=z_j \\ w=\overline{z_k}}} \right)_{\substack{s=0,1,\dots,l_j-1 \\ t=0,1,\dots,l_k-1}} \Big(\big(\mathbf{I} - \overline{g(z_1)}\mathbf{W}_{kk}^{(g)}\big)^{-1}\Big)^*.$$

3. Spaces of rational functions associated with the data set $\boldsymbol{\Delta}$

For a given data set Δ as in (2.1), we construct in this section a space of rational functions which are holomorphic in the closed unit disk $\mathbb{D} \cup \mathbb{T}$. In fact, we put

$$e_{js}(v) := \frac{v^s}{(1-\overline{z_j}v)^{s+1}}, \quad s = 0,1,\dots,l_j-1,\ j = 1,2,\dots,n, \tag{3.1}$$

and denote the linear span of these rational functions by $\mathfrak{H}_\Delta$. Moreover, we set

$$f_{js}(v) := e_{js}(v) + \sum_{r=0}^{s} \overline{w_{j,s-r}} e_{jr}(v), \quad s = 0,1,\dots,l_j-1,\ j = 1,2,\dots,n. \tag{3.2}$$

In (3.1) and (3.2) the independent variable v is mostly taken from the open unit disk $\mathbb{D}$. However, such relations for the rational functions belonging to $\mathfrak{H}_\Delta$ can also be considered outside of $\mathbb{D}$ in all points where the functions involved are holomorphic; that is for each $v \notin \{\frac{1}{\overline{z_1}}, \frac{1}{\overline{z_2}}, \dots, \frac{1}{\overline{z_n}}\}$, where we use the convention $\frac{1}{0} := \infty$.

Remark 3.1. A rational function x belongs to $\mathfrak{H}_\Delta$ if and only if the representation

$$x(v) = \frac{p(v)}{(1-\overline{z_1}v)^{l_1}(1-\overline{z_2}v)^{l_2}\cdots(1-\overline{z_n}v)^{l_n}}$$

is satisfied for some (unique) polynomial p of degree not greater than m. Moreover, for each $j \in \{1,2,\dots,n\}$, the equality (3.2) can also be written in matricial form as

$$\big(f_{j0}(v)\ \ f_{j1}(v)\ \cdots\ f_{j,l_j-1}(v)\big) = \big(e_{j0}(v)\ \ e_{j1}(v)\ \cdots\ e_{j,l_j-1}(v)\big)\big(\mathbf{I} + \mathbf{W}_{jj}^*\big). \tag{3.3}$$

Thus, in the case of (2.7), the linear span of the functions f_{js} coincides with the linear span of the functions e_{js}, $s = 0,1,\dots,l_j-1$ and $j = 1,2,\dots,n$, i.e. the system of functions $f_{10}, f_{11}, \dots, f_{1,l_1-1}, \dots\dots, f_{n0}, f_{n1}, \dots, f_{n,l_n-1}$ is also a basis of $\mathfrak{H}_\Delta$.

Henceforth, we assume (2.7) and we normalize the first value by

$$w_{10} := 0.$$

This normalization is always possible by a conformal mapping of the open unit disk $\mathbb{D}$ (see Remark 2.4). Using the entries of the generalized Schwarz-Pick matrix $\mathbf{P}_\Delta$ introduced in Section 2, we define an inner product on $\mathfrak{H}_\Delta$ by the relations

$$\Big\langle f_{kt}, f_{js} \Big\rangle := p_{jk}^{s,t}, \quad s=0,1,\dots,l_j-1,\ t=0,1,\dots,l_k-1,\ j,k=1,2,\dots,n. \tag{3.4}$$

Suppose for a moment that the data set Δ belongs to a Schur function g. In view of (2.7) and the Cayley transform (see Remark 2.2) we find an associated Carathéodory function Ω. Therefore, according to a theorem of F. Riesz and G. Herglotz there exists a unique Borel measure μ on $\mathbb{T}$ such that

$$\Omega(z) = \mathrm{i}\,\Im\mathfrak{m}\,\Omega(0) + \int_{\mathbb{T}} \frac{u+z}{u-z}\,\mu(\mathrm{d}u), \quad z \in \mathbb{D}. \tag{3.5}$$

We call μ the *associated measure of* g.

Lemma 3.2. *Let $g \in \mathcal{S}_\Delta$ and let μ be the associated measure of g. Then*

$$\left\langle x, y \right\rangle = \int_{\mathbb{T}} \overline{y(u)} x(u) \, \mu(\mathrm{d}u), \quad x, y \in \mathfrak{H}_\Delta.$$

Proof. An easy calculation shows that

$$\frac{1}{1 - vw} \left(\frac{u+v}{u-v} + \frac{\overline{u}+w}{\overline{u}-w} \right) = \frac{2}{(1 - v\overline{u})(1 - wu)}, \quad v, w \in \mathbb{D}, \; u \in \mathbb{T}.$$

Consequently, for each $u \in \mathbb{T}$, we obtain in view of (3.1) the identity

$$\frac{1}{s!t!} \frac{\partial^{s+t}}{\partial v^s \partial w^t} \frac{1}{1 - vw} \left(\frac{u+v}{u-v} + \frac{\overline{u}+w}{\overline{u}-w} \right) \Bigg|_{\substack{v=z_j \\ w=\overline{z_k}}} = \frac{2\overline{u}^s u^t}{(1 - z_j\overline{u})^{s+1}(1 - \overline{z_k}u)^{t+1}} = 2\overline{e_{js}(u)} e_{kt}(u)$$

and hence, if Ω stands for the Cayley transform of g as in Remark 2.2, (3.5) implies

$$\frac{1}{s!t!} \frac{\partial^{s+t}}{\partial v^s \partial w^t} \frac{\Omega(v) + \overline{\Omega(\overline{w})}}{1 - vw} \Bigg|_{\substack{v=z_j \\ w=\overline{z_k}}} = 2 \int_{\mathbb{T}} \overline{e_{js}(u)} e_{kt}(u) \, \mu(\mathrm{d}u)$$

for each $s = 0, 1, \ldots, l_j - 1$, $t = 0, 1, \ldots, l_k - 1$, and $j, k = 1, 2, \ldots, n$. Therefore, from (3.4), (2.6), Remark 2.2, and (3.3) it follows that

$$\left\langle f_{kt}, f_{js} \right\rangle = p_{jk}^{s,t} = \frac{1}{s!t!} \frac{\partial^{s+t}}{\partial v^s \partial w^t} \frac{1 - g(v)\overline{g(\overline{w})}}{1 - vw} \Bigg|_{\substack{v=z_j \\ w=\overline{z_k}}} = \int_{\mathbb{T}} \overline{f_{js}(u)} f_{kt}(u) \, \mu(\mathrm{d}u)$$

for each $s = 0, 1, \ldots, l_j - 1$, $t = 0, 1, \ldots, l_k - 1$, and $j, k = 1, 2, \ldots, n$. By virtue of Remark 3.1, we can finally conclude the assertion. □

Remark 3.3. Using the arguments in the proof of Lemma 3.2 in a slightly modified order, one can also verify that conversely if $g \in \mathcal{S}$ such that the condition $g(z_1) = 0$ is satisfied and that the associated measure μ of g fulfills the equality

$$p_{jk}^{s,t} = \int_{\mathbb{T}} \overline{f_{js}(u)} f_{kt}(u) \, \mu(\mathrm{d}u)$$

for each $j, k = 1, 2, \ldots, n$, $s = 0, 1, \ldots, l_j - 1$, and $t = 0, 1, \ldots, l_k - 1$ then $g \in \mathcal{S}_\Delta$.

Observe that Lemma 3.2 implies the necessity of the condition $\mathbf{P}_\Delta \geq 0$ for the existence of a solution $g \in \mathcal{S}_\Delta$. Moreover, Lemma 3.2 paves the way to the treatment on orthogonal rational functions in [6]. The following notes gives a detailed explanation.

Suppose again that a data set Δ as in (2.1) is given. With the points $z_1, z_2, \ldots, z_n$ in Δ we form now a sequence $(\alpha_k)_{k=0}^m$ in which z_j appears according to its multiplicity l_j. For instance, we can choose $\alpha_k := \beta_k$ with

$$\beta_k := z_j \quad \text{if} \quad \sum_{r=1}^{j-1} l_r \leq k < \sum_{r=1}^{j} l_r, \quad j = 1, 2, \ldots, n. \tag{3.6}$$

The treatment in [18] was restricted to this special choice to simplify matters. However putting equal points together was not essential in [18], nor is it in this paper. So, in the following, for an arbitrary permutation π of $\{0, 1, \ldots, m\}$ satisfying $\pi(0) = 0$ we can put

$$\alpha_k := \beta_{\pi(k)}, \quad k = 0, 1, \ldots, m. \tag{3.7}$$

Actually, it is tacitly understood that we always take such a sequence $(\alpha_k)_{k=0}^m$ whenever we fix a data set Δ. To emphasize that the data of the set Δ are introduced in a certain order as characterized by the ordering corresponding to the sequence $(\alpha_k)_{k=0}^m$ given by the permutation π, we shall denote the data by Δ_π where appropriate. Thus, we associate with Δ_π a fixed sequence $(\alpha_k)_{k=0}^m$, and all the related objects are ordered accordingly. For example, we renumber the rational functions defined by (3.1) to the sequence $(e_k)_{k=0}^m$ satisfying the following two conditions:

- For each $k \in \{0, 1, \ldots, m\}$, the function e_k has a pole $\big($at most$\big)$ at the point $\frac{1}{\alpha_k}$.
- If $0 \le k_1 < k_2 \le m$ and if $e_{k_1} = e_{js_1}$, $e_{k_2} = e_{js_2}$ for certain $s_1, s_2 \in \{0, 1, \ldots, l_j - 1\}$ and some $j \in \{1, 2, \ldots, n\}$ then $s_1 < s_2$.

In particular, by the special choice $\alpha_k := \beta_k$, $k = 0, 1, \ldots, m$, and (3.6) this means that

$$e_k(v) = e_{js}(v), \quad k = s + \sum_{r=1}^{j-1} l_r, \quad s = 0, 1, \ldots, l_j - 1,\ j = 1, 2, \ldots, n.$$

In general, we have the identity $e_0(v) = e_{10}(v)$ and there is a (uniquely determined) permutation matrix $\mathbf{U}$ of size $(m+1) \times (m+1)$ fulfilling

$$\big(e_0(v)\ e_1(v)\ \cdots\ e_m(v)\big) = \big(e_{10}(v)\, e_{11}(v) \cdots e_{1,l_1-1}(v)\ \cdots\ e_{n0}(v)\, e_{n1}(v) \cdots e_{n,l_n-1}(v)\big)\mathbf{U}.$$

Similarly, we renumber the rational functions defined by (3.2) to $(f_k)_{k=0}^m$ such that

$$\big(f_0(v)\ f_1(v)\ \cdots\ f_m(v)\big) = \big(f_{10}(v)\, f_{11}(v) \cdots f_{1,l_1-1}(v)\ \cdots\ f_{n0}(v)\, f_{n1}(v) \cdots f_{n,l_n-1}(v)\big)\mathbf{U}$$

and in view of (2.4) we set moreover

$$\mathbf{W}_m := \mathbf{U}^* \mathbf{W} \mathbf{U}. \tag{3.8}$$

Also because of this new enumeration we write $\mathbf{P}_r := (p_{jk})_{j,k=0}^r$, $r = 0, 1, \ldots, m$, where

$$\Big\langle f_k, f_j \Big\rangle = p_{jk}, \quad j, k = 0, 1, \ldots, r.$$

Furthermore, the notation $\mathfrak{H}_k$, $k = 0, 1, \ldots, m$, stands for the linear span of all rational functions $f_0, f_1, \ldots, f_k$ with inner product given by the corresponding Gram matrix $\mathbf{P}_k$. In particular, we have $\mathfrak{H}_m = \mathfrak{H}_\Delta$ and $\mathbf{P}_m = \mathbf{U}^* \mathbf{P}_\Delta \mathbf{U}$.

The notation b_{ς_0} stands for the *elementary Blaschke factor* corresponding to $\varsigma_0 \in \mathbb{D}$, i.e.

$$b_{\varsigma_0}(v) := \begin{cases} v & \text{if } \varsigma_0 = 0, \\ \dfrac{\overline{\varsigma_0}}{|\varsigma_0|} \dfrac{\varsigma_0 - v}{1 - \overline{\varsigma_0} v} & \text{if } \varsigma_0 \neq 0. \end{cases}$$

For $k \in \{0, 1, \dots, m\}$, we introduce the *Blaschke product* (with zeros $\alpha_0, \alpha_1, \dots, \alpha_k$)

$$B_k(v) := \prod_{j=0}^{k} b_{\alpha_j}(v)$$

and we also use for technical reasons the setting

$$\eta_k := \begin{cases} -1 & \text{if } \alpha_k = 0, \\ \dfrac{\overline{\alpha_k}}{|\alpha_k|} & \text{if } \alpha_k \neq 0. \end{cases}$$

We define the *adjoint rational function* $x^{[k]}$ of some $x \in \mathfrak{H}_k$, $k = 0, 1, \dots, m$, as suggested in [6], i.e. $x^{[k]}$ stands for the rational function which is uniquely determined via

$$x^{[k]}(v) = \frac{1}{v} B_k(v) \overline{x\Big(\frac{1}{\overline{v}}\Big)}. \tag{3.9}$$

Some information on further interrelations between the adjoint rational function $x^{[k]}$ and the underlying rational function x can be found in [6, Section 2.2] for the special case $\alpha_0 = 0$. Nevertheless, an analog argumentation implies that for all $x, y \in \mathfrak{H}_k$, $k = 0, 1, \dots, m$, the properties below are satisfied:

(I) $x^{[k]} \in \mathfrak{H}_k, \ (x^{[k]})^{[k]} = x$.

(II) $x^{[k]}(\alpha_k) = 0 \Longleftrightarrow x \in \mathfrak{H}_{k-1}, \quad k \neq 0$.

(III) $\Big\langle x, y \Big\rangle = \Big\langle y^{[k]}, x^{[k]} \Big\rangle$.

Henceforth, we suppose that the generalized Schwarz-Pick matrix $\mathbf{P}_\Delta$ is positive Hermitian, i.e. $\mathbf{P}_\Delta > 0$. This implies immediately $\mathbf{P}_m > 0$. Subject to this limitation, the reproducing kernel K for the space $(\mathfrak{H}_\Delta, \langle \cdot, \cdot \rangle)$ is given by

$$K(v, w) = \big(f_0(v) \ \ f_1(v) \ \cdots \ f_m(v)\big) \, \mathbf{P}_m^{-1} \big(f_0(w) \ \ f_1(w) \ \cdots \ f_m(w)\big)^*$$

in analogy to [6, Theorem 2.2.2]. (The inverse $\mathbf{P}_m^{-1}$ exists due to $\mathbf{P}_m > 0$.) We set

$$K_w(v) := K(v, w).$$

If $m = 0$ then by definition (note (3.2), (2.2), $\alpha_0 = z_1$, and $w_{10} = 0$), we have

$$K_w(v) = \frac{1 - |\alpha_0|^2}{(1 - \overline{\alpha_0} v)(1 - \alpha_0 \overline{w})}.$$

The case $m \geq 1$ is considered in the next statement, where the same argumentation as in [6, Theorem 2.2.3 and Theorem 2.2.4] (see also [18, Proposition 3.1]) based on the properties (I), (II), and (III) of adjoint rational functions apply to yield a proof. We omit the details. Recall that $\mathfrak{H}_m = \mathfrak{H}_\Delta$.

Proposition 3.4. *In the case of $m \geq 1$, the following relations are satisfied:*

(a) $K_w \in \mathfrak{H}_m$ and $\left\langle x, K_w \right\rangle = x(w), \quad x \in \mathfrak{H}_m.$

(b) $$K(v,w) = \frac{-1}{\det \mathbf{P}_m} \begin{vmatrix} p_{00} & p_{01} & \cdots & p_{0m} & \overline{f_0(w)} \\ p_{10} & p_{11} & \cdots & p_{1m} & \overline{f_1(w)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{m0} & p_{m1} & \cdots & p_{mm} & \overline{f_m(w)} \\ f_0(v) & f_1(v) & \cdots & f_m(v) & 0 \end{vmatrix}.$$

(c) $K_w^{[m]}(v) = K_v^{[m]}(w).$

(d) $$K_{\alpha_m}^{[m]}(v) = \frac{f_m^{[m]}(\alpha_m)}{\det \mathbf{P}_m} \begin{vmatrix} p_{00} & \cdots & p_{0,m-1} & p_{0m} \\ \vdots & \ddots & \vdots & \vdots \\ p_{m-1,0} & \cdots & p_{m-1,m-1} & p_{m-1,m} \\ f_0(v) & \cdots & f_{m-1}(v) & f_m(v) \end{vmatrix}.$$

(e) $K(\alpha_m, \alpha_m) = \left| f_m^{[m]}(\alpha_m) \right|^2 \dfrac{\det \mathbf{P}_{m-1}}{\det \mathbf{P}_m}.$

(f) $\left\langle K_{\alpha_m}^{[m]}, x \right\rangle = 0 \Longleftrightarrow x \in \mathfrak{H}_{m-1}.$

4. Orthogonal rational functions with respect to $\mathfrak{H}_\Delta$ and $\mathbf{P}_\Delta$

The considerations in the present section are directly related with the spaces of rational functions introduced in Section 3. In fact, we discuss some special systems of rational functions which are orthogonal with respect to these spaces. Here, in view of Lemma 3.2, the following definition of orthonormal systems is in line with [6]. A sequence $(\varphi_k)_{k=0}^m$ of elements of $\mathfrak{H}_\Delta$ is called an *orthonormal system corresponding to* Δ_π (with $w_{10} = 0$) if the two conditions below are satisfied:

(O1) $\varphi_k \in \mathfrak{H}_k,\ k = 0, 1, \ldots, m.$

(O2) $\left\langle \varphi_j, \varphi_k \right\rangle = \delta_{jk},\ j, k = 0, 1, \ldots, m.$

In (O2) and in the sequel, the notation δ_{jk} stands for the Kronecker symbol, i.e. $\delta_{jk} = 0$ if $j \neq k$ and $\delta_{jk} = 1$ if $j = k$.

Note that the conditions (O1) and (O2) depend not only on Δ, but also on the ordering of the α_k's according to (3.6) and (3.7) (which is uniquely determined by the permutation π). Moreover (cf. [18, Remark 4.1] and [6, Theorem 2.2.4]), the assumption $\mathbf{P}_\Delta > 0$ is necessary and sufficient for the existence of an orthonormal system corresponding to Δ_π (for any ordering of the α_k's).

As already mentioned above, the statements in [6] (and in [18] as well) are elaborated for the special case $\alpha_0 = 0$. In fact, either one has to adapt the proofs presented there to the slightly modified situation here or one can also use the following connection between both cases (cf. [6, Corollary 6.2.4]).

Remark 4.1. For each $k \in \{0,1,\dots,m\}$, let $\varphi_k \in \mathfrak{H}_k$ and

$$\widetilde{\varphi_k}(v) := \frac{1-\overline{\alpha_0}v}{\sqrt{1-|\alpha_0|^2}}\varphi_k(v).$$

Furthermore, let $g \in \mathcal{S}_\Delta$, let μ be the associated measure of g, and let

$$\mu_1(A) := \int_A \frac{1-|\alpha_0|^2}{|u-\alpha_0|^2}\,\mu(\mathrm{d}u), \quad A \in \mathfrak{B},$$

where $\mathfrak{B}$ stands for the σ-algebra of all Borel subsets of $\mathbb{T}$. Then μ_1 is also a Borel measure on $\mathbb{T}$ and $(\varphi_k)_{k=0}^m$ is an orthonormal system corresponding to Δ if and only if

$$\int_{\mathbb{T}} \overline{\widetilde{\varphi_k}(u)}\widetilde{\varphi_j}(u)\,\mu_1(\mathrm{d}u) = \delta_{jk}, \quad j,k = 0,1,\dots,m.$$

Observe that the assumption $\mathbf{P}_\Delta > 0$ implies actually that $\mathbf{P}_k > 0$, $k = 0,1,\dots,m$. Hence, for each $k \in \{0,1,\dots,m\}$, we may define a kernel K_k similar as K with respect to the principal submatrix $\mathbf{P}_k$ and the corresponding rational $(k+1)$-row vector function $\begin{pmatrix} f_0 & f_1 & \cdots & f_k\end{pmatrix}$ instead of $\mathbf{P}_m$ and $\begin{pmatrix} f_0 & f_1 & \cdots & f_m\end{pmatrix}$, i.e.

$$K_k(v,w) := \begin{pmatrix} f_0(v) & f_1(v) & \cdots & f_k(v)\end{pmatrix}\mathbf{P}_k^{-1}\begin{pmatrix} f_0(w) & f_1(w) & \cdots & f_k(w)\end{pmatrix}^*,$$

$$K_{k,w}(v) := K_k(v,w).$$

If the underlying sequence $(\alpha_k)_{k=0}^m$ is fixed then the relations (O1) and (O2) above determine the orthonormal system $(\varphi_k)_{k=0}^m$ only up to constant factors of modulus one. In [6] mostly the following orthonormal system $(\phi_k)_{k=0}^m$ is considered:

$$\phi_k(v) := \frac{1}{\sqrt{K_k(\alpha_k,\alpha_k)}}K_{k,\alpha_k}^{[k]}(v), \quad k = 0,1,\dots,m.$$

Then $\phi_0(v) = -\eta_0\frac{\sqrt{1-|\alpha_0|^2}}{1-\overline{\alpha_0}v}$ and (cf. Proposition 3.4 (d), (e)) we have

$$\phi_k(v) = \frac{f_k^{[k]}(\alpha_k)}{\left|f_k^{[k]}(\alpha_k)\right|}\frac{1}{\sqrt{\det[\mathbf{P}_{k-1}\mathbf{P}_k]}}\begin{vmatrix} p_{00} & \cdots & p_{0,k-1} & p_{0k}\\ \vdots & \ddots & \vdots & \vdots\\ p_{k-1,0} & \cdots & p_{k-1,k-1} & p_{k-1,k}\\ f_0(v) & \cdots & f_{k-1}(v) & f_k(v)\end{vmatrix}, \quad k = 1,2,\dots,m.$$

For an arbitrary orthonormal system $(\varphi_k)_{k=0}^m$ corresponding to Δ_π there exist numbers $u_0,u_1,\dots,u_m \in \mathbb{T}$ such that

$$\varphi_k = u_k\phi_k, \quad k = 0,1,\dots,m. \tag{4.1}$$

In [6, Section 4.1] recurrence relations for the system $(\phi_k)_{k=0}^m$ were proved. We fix here the system $(\varphi_k)_{k=0}^m$ similar to [18], that is we fix the constants u_k, $k = 0,1,\dots,m$, in (4.1) by a special choice, and use corresponding recursions. In fact, we set

$$\varphi_0(v) := \frac{\sqrt{1-|\alpha_0|^2}}{1-\overline{\alpha_0}v} \tag{4.2}$$

and choose φ_k recursively such that

$$\arg\left[\varphi_k^{[k]}(\alpha_{k-1})\right] = \arg\left[\frac{\eta_k\overline{\eta_{k-1}}}{1-\overline{\alpha_k}\alpha_{k-1}}\varphi_{k-1}^{[k-1]}(\alpha_{k-1})\right], \quad k = 1, 2, \ldots, m.$$

In this way, the orthonormal system $(\varphi_k)_{k=0}^m$ corresponding to Δ_π is uniquely determined. We call $(\varphi_k)_{k=0}^m$ the *canonical Szegő orthonormal system corresponding to* Δ_π. If $m \geq 1$ then we define in addition

$$\lambda_k := \eta_k\overline{\eta_{k-1}}\frac{\varphi_k(\alpha_{k-1})}{\varphi_k^{[k]}(\alpha_{k-1})}, \quad k = 1, 2, \ldots, m, \tag{4.3}$$

and call these numbers the *Szegő parameters* of the orthonormal system $(\varphi_k)_{k=0}^m$ or simply of the ordered data set Δ_π. Observe that, similar as in [18, Remark 4.4 and Theorem 4.5] (cf. [7, Theorem 5.4] as well), (4.3) implies $\lambda_k \in \mathbb{D}$, $k = 1, 2, \ldots, m$, and an application of the Christoffel-Darboux formulas for orthogonal rational functions (i.e. use [6, Theorem 3.1.3]) one can obtain the following recursions.

Theorem 4.2. *For each $k \in \{1, 2, \ldots, m\}$, the following recurrence relation holds:*

$$\varphi_k(v) = \sqrt{\frac{1-|\alpha_k|^2}{(1-|\alpha_{k-1}|^2)(1-|\lambda_k|^2)}}\frac{1-\overline{\alpha_{k-1}}v}{1-\overline{\alpha_k}v}\left(b_{\alpha_{k-1}}(v)\varphi_{k-1}(v) + \lambda_k\varphi_{k-1}^{[k-1]}(v)\right). \tag{4.4}$$

Note that Theorem 4.2 provides under the assumptions $w_{10} = 0$ and $\mathbf{P}_\Delta > 0$ that the Szegő parameters λ_k, $k = 1, 2, \ldots, m$, determine the orthonormal system $(\varphi_k)_{k=0}^m$ completely. These considerations imply in connection with Lemma 3.2 and Remark 3.3 the following characterization of the solution set $\mathcal{S}_\Delta$ for Problem (MNP). Here, for a given function $g \in \mathcal{S}$ we introduce the data set

$$\Delta^{[g]} := \left\{\left(z_j, l_j, \left(\frac{1}{s!}g^{(s)}(z_j)\right)_{s=0}^{l_j-1}\right)_{j=1}^n\right\}$$

and $\Delta_\pi^{[g]}$ is the previous data set ordered according to $(\alpha_k)_{k=0}^m$ given by (3.6) and (3.7) in virtue of a permutation π of $\{0, 1, \ldots, m\}$ with $\pi(0) = 0$.

Corollary 4.3. *Let Δ be a data set given as in* (2.1) *such that $w_{10} = 0$ and $\mathbf{P}_\Delta > 0$. Furthermore, let $(\varphi_k)_{k=0}^m$ be the canonical Szegő orthonormal system corresponding to Δ_π, let $(\lambda_k)_{k=1}^m$ be the sequence of Szegő parameters of the ordered data set Δ_π, and let $g \in \mathcal{S}$. The following statements are equivalent:*

(i) $g \in \mathcal{S}_\Delta$.

(ii) $g(z_1) = 0$ *and* $(\varphi_k)_{k=0}^m$ *is an orthonormal system corresponding to* $\Delta_\pi^{[g]}$.

(iii) $g(z_1) = 0$, $\mathbf{P}_{\Delta^{[g]}} > 0$, *and* $(\lambda_k)_{k=1}^m$ *is the sequence of Szegő parameters of* $\Delta_\pi^{[g]}$.

The sequence of Szegő parameters $(\lambda_k)_{k=1}^m$ of the ordered data set Δ_π can be computed via (4.3) through the canonical Szegő orthonormal system $(\varphi_k)_{k=0}^m$ corresponding to Δ_π. Using an analog argumentation as in [6, Theorem 4.1.2], it is not hard to accept that this sequence of Szegő parameters can be recovered also by

$$\lambda_k = \eta_{k-1} \frac{\left\langle \rho_k \varphi_{k-1}, \varphi_{k-1}^{[k-1]} \right\rangle}{\left\langle \varrho_k \varphi_{k-1}, \varphi_{k-1} \right\rangle}, \quad k = 1, 2, \ldots, m, \tag{4.5}$$

where

$$\rho_k(v) := \frac{v - \alpha_{k-1}}{1 - \overline{\alpha_k} v}, \quad \varrho_k(v) := \frac{1 - \overline{\alpha_{k-1}} v}{1 - \overline{\alpha_k} v}.$$

Note that, based on this formula and Theorem 4.2, the canonical Szegő orthonormal system $(\varphi_k)_{k=0}^m$ can be recursively calculated from the given data Δ_π as well.

If arbitrary parameters $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{D}$ and some sequence $(\alpha_k)_{k=0}^m$ fulfilling (3.6) and (3.7) for a permutation π of $\{0, 1, \ldots, m\}$ with $\pi(0) = 0$ are given, and we define a sequence $(\varphi_k)_{k=0}^m$ of rational functions through (4.2) and the recurrence relations (4.4), $k = 1, 2, \ldots, m$, then in view of a Favard type theorem (see [6, Section 8.1]) and the Cayley transform (i.e. use Remark 2.2 and Lemma 3.2) there exists a $g \in \mathcal{S}$ such that $(\varphi_k)_{k=0}^m$ is the canonical Szegő orthonormal system corresponding to $\Delta_\pi^{[g]}$. In particular, if a data set Δ with $w_{10} = 0$, $\mathbf{P}_\Delta > 0$, and associated sequence of Szegő parameters $(\lambda_k)_{k=1}^m$ corresponding to an ordering $(\alpha_k)_{k=0}^m$ is taken for granted and if the sequence $(\psi_k)_{k=0}^m$ is defined by

$$\psi_0(v) := \frac{\sqrt{1 - |\alpha_0|^2}}{1 - \overline{\alpha_0} v} \tag{4.6}$$

and, for each $k \in \{1, 2, \ldots, m\}$, recursively by

$$\psi_k(v) := \sqrt{\frac{1-|\alpha_k|^2}{(1-|\alpha_{k-1}|^2)(1-|\lambda_k|^2)}} \frac{1-\overline{\alpha_{k-1}} v}{1-\overline{\alpha_k} v} \Big(b_{\alpha_{k-1}}(v) \psi_{k-1}(v) - \lambda_k \psi_{k-1}^{[k-1]}(v) \Big), \tag{4.7}$$

then there exists an $h \in \mathcal{S}$ such that $(\psi_k)_{k=0}^m$ is the canonical Szegő orthonormal system corresponding to $\Delta_\pi^{[h]}$. This sequence $(\psi_k)_{k=0}^m$ is called the *dual canonical Szegő orthonormal system* of the canonical Szegő orthonormal system $(\varphi_k)_{k=0}^m$ corresponding to Δ_π. Observe that the difference between the equalities in (4.4) and (4.7) consists only in the different sign in front of the Szegő parameters λ_k, $k = 1, 2, \ldots, m$.

5. Description of $\mathcal{S}_\Delta$ in terms of orthogonal rational functions

In this section we present a description of the solution set $\mathcal{S}_\Delta$ of Problem (MNP) for the non-degenerate case in terms of the orthogonal rational functions introduced in the previous section. Thus, in the following we suppose always a given data set Δ as in (2.1) such that the conditions $w_{10} = 0$ and $\mathbf{P}_\Delta > 0$ are satisfied. Furthermore, we tacitly assume an ordering of the data defining a sequence $(\alpha_k)_{k=0}^m$ fulfilling (3.6) and (3.7) in virtue of a permutation π of $\{0, 1, \ldots, m\}$ with $\pi(0) = 0$. We denote the sequence of

Szegő parameters of Δ_π by $(\lambda_k)_{k=1}^m$, we write $(\varphi_k)_{k=0}^m$ to indicate the canonical Szegő orthonormal system corresponding to Δ_π, and $(\psi_k)_{k=0}^m$ stands for the dual canonical Szegő orthonormal system of $(\varphi_k)_{k=0}^m$. In fact, we will see that the linear fractional transformation (1.2) obtained already in [7] leads directly to such a description of $\mathcal{S}_\Delta$, where the orthogonal rational functions φ_m and ψ_m are involved.

We recall first the definition of Schur-Nevanlinna pairs of rational functions introduced in [7, Section 3]. If $\mathbb{I} := \{0, 1, \dots, m\}$ and if $(\kappa_k)_{k\in\mathbb{I}}$ is a sequence of points belonging to $\mathbb{D}$, then the pair $[(\gamma_k)_{k\in\mathbb{I}}, (\delta_k)_{k\in\mathbb{I}}]$ of sequences of rational functions defined by

$$\gamma_0(v) := \sqrt{\frac{1-|\alpha_0|^2}{1-|\kappa_0|^2}}\frac{1}{1-\overline{\alpha_0}v}, \quad \delta_0(v) := \overline{\kappa_0}\sqrt{\frac{1-|\alpha_0|^2}{1-|\kappa_0|^2}}\frac{1}{1-\overline{\alpha_0}v} \tag{5.1}$$

and, for $k \in \mathbb{I} \setminus \{0\}$, recursively by

$$\gamma_k(v) := \sqrt{\frac{1-|\alpha_k|^2}{(1-|\alpha_{k-1}|^2)(1-|\kappa_k|^2)}}\frac{1-\overline{\alpha_{k-1}}v}{1-\overline{\alpha_k}v}\left(b_{\alpha_{k-1}}(v)\gamma_{k-1}(v) + \overline{\kappa_k}\delta_{k-1}^{[k-1]}(v)\right), \tag{5.2}$$

$$\delta_k(v) := \sqrt{\frac{1-|\alpha_k|^2}{(1-|\alpha_{k-1}|^2)(1-|\kappa_k|^2)}}\frac{1-\overline{\alpha_{k-1}}v}{1-\overline{\alpha_k}v}\left(b_{\alpha_{k-1}}(v)\delta_{k-1}(v) + \overline{\kappa_k}\gamma_{k-1}^{[k-1]}(v)\right), \tag{5.3}$$

where $\delta_{k-1}^{[k-1]}$ and $\gamma_{k-1}^{[k-1]}$ are the adjoints of δ_{k-1} and γ_{k-1} as explained in (3.9) (adjoint with respect to the points $\alpha_0, \alpha_1, \dots, \alpha_{k-1}$), is called the *Schur-Nevanlinna pair of rational functions corresponding to* $(\alpha_k, \kappa_k)_{k\in\mathbb{I}}$.

Lemma 5.1. *Let Δ be a data set given as in* (2.1) *such that $w_{10} = 0$ and $\mathbf{P}_\Delta > 0$. Furthermore, let $[(\gamma_k)_{k\in\mathbb{I}}, (\delta_k)_{k\in\mathbb{I}}]$ be the Schur-Nevanlinna pair of rational functions corresponding to $(\alpha_k, \kappa_k)_{k\in\mathbb{I}}$, where $\mathbb{I} := \{0, 1, \dots, m\}$, where $\kappa_0 := 0$, and where $\kappa_k := \overline{\lambda_k}$ for $k = 1, 2, \dots, m$. For each $k \in \mathbb{I}$,*

$$\gamma_k(v) = \frac{1}{2}\big(\varphi_k(v) + \psi_k(v)\big) \quad \textit{and} \quad \delta_k(v) = \frac{1}{2}\big(\varphi_k(v) - \psi_k(v)\big)$$

$$\Big(\textit{resp.}, \ \varphi_k(v) = \gamma_k(v) + \delta_k(v) \quad \textit{and} \quad \psi_k(v) = \gamma_k(v) - \delta_k(v)\Big).$$

Proof. Because of $\kappa_0 := 0$, (5.1), (4.2), and (4.6), we have

$$\gamma_0(v) = \sqrt{\frac{1-|\alpha_0|^2}{1-|\kappa_0|^2}}\frac{1}{1-\overline{\alpha_0}v} = \frac{\sqrt{1-|\alpha_0|^2}}{1-\overline{\alpha_0}v} = \frac{1}{2}\big(\varphi_0(v) + \psi_0(v)\big)$$

and

$$\delta_0(v) = \overline{\kappa_0}\sqrt{\frac{1-|\alpha_0|^2}{1-|\kappa_0|^2}}\frac{1}{1-\overline{\alpha_0}v} = 0 = \frac{1}{2}\big(\varphi_0(v) - \psi_0(v)\big).$$

Consequently, for $k \in \{1, 2, \dots, m\}$, by succinctly setting

$$r(v) := \sqrt{\frac{1-|\alpha_k|^2}{(1-|\alpha_{k-1}|^2)(1-|\kappa_k|^2)}}\frac{1-\overline{\alpha_{k-1}}v}{1-\overline{\alpha_k}v}$$

one can inductively get in view of $\kappa_k := \overline{\lambda_k}$, (5.2), (5.3), (4.4), and (4.7) the relation

$$\begin{aligned}\gamma_k(v) &= r(v)\Big(b_{\alpha_{k-1}}(v)\gamma_{k-1}(v) + \overline{\kappa_k}\delta_{k-1}^{[k-1]}(v)\Big)\\ &= r(v)\left(b_{\alpha_{k-1}}(v)\frac{1}{2}\big(\varphi_{k-1}(v)+\psi_{k-1}(v)\big) + \overline{\kappa_k}\frac{1}{2}\big(\varphi_{k-1}^{[k-1]}(v) - \psi_{k-1}^{[k-1]}(v)\big)\right)\\ &= \frac{1}{2}r(v)\Big(\big(b_{\alpha_{k-1}}(v)\varphi_{k-1}(v) + \lambda_k\varphi_{k-1}^{[k-1]}(v)\big) + \big(b_{\alpha_{k-1}}(v)\psi_{k-1}(v) - \lambda_k\psi_{k-1}^{[k-1]}(v)\big)\Big)\\ &= \frac{1}{2}\big(\varphi_k(v)+\psi_k(v)\big)\end{aligned}$$

and similarly

$$\delta_k(v) = \frac{1}{2}\big(\varphi_k(v) - \psi_k(v)\big).$$

These equalities imply immediately the others. □

Lemma 5.1 shows a simple connection between the considerations in the previous section and in [7]. But in [7], the construction of the Schur-Nevanlinna pair of rational functions $[(\gamma_k)_{k\in\mathbb{I}}, (\delta_k)_{k\in\mathbb{I}}]$ which leads to the description (1.2) of $\mathcal{S}_\Delta$ is based on certain Schur parameters appearing in the following Schur-Nevanlinna algorithm instead of the Szegő parameters λ_k, $k = 1, 2, \ldots, m$.

Let a $g \in \mathcal{S}$ be given and set $g_0 := g$. Then, as long as

$$s_k := g_k(\alpha_k) \tag{5.4}$$

belongs to $\mathbb{D}$ we can define recursively

$$g_{k+1}(z) := \frac{1}{b_{\alpha_k}(z)}\frac{g_k(z) - s_k}{1 - \overline{s_k}g_k(z)}, \quad k = 0, 1, \ldots, m. \tag{5.5}$$

If $g \in \mathcal{S}$ such that the Schur-Nevanlinna algorithm can be carried out at least m times (that is after obtaining g_m and s_m) then $(s_k)_{k=0}^m$ from (5.4) is called the sequence of *Schur parameters* associated with the pair $[g, (\alpha_k)_{k=0}^m]$.

The connection between Szegő parameters and Schur parameters below is an essential tool for the following.

Lemma 5.2. *Let Δ be a data set given as in* (2.1) *such that $w_{10} = 0$ and $\mathbf{P}_\Delta > 0$. If $g \in \mathcal{S}_\Delta$ then the Schur-Nevanlinna algorithm can be carried out at least m times for g and the sequence $(s_k)_{k=0}^m$ of Schur parameters associated with the pair $[g, (\alpha_k)_{k=0}^m]$ is given by $s_0 = 0$ and*

$$s_k = -\overline{\eta_{k-1}\lambda_k}, \quad k = 1, 2, \ldots, m. \tag{5.6}$$

Proof. Let $g \in \mathcal{S}_\Delta$. From [7, Theorem 2.3] we already know that the Schur-Nevanlinna algorithm can be carried out at least m times for g (and arbitrary points belonging to $\mathbb{D}$). In particular, the sequence $(s_k)_{k=0}^m$ of Schur parameters associated with $[g, (\alpha_k)_{k=0}^m]$ is well defined. Now, let $[(\gamma_k)_{k\in\mathbb{I}}, (\delta_k)_{k\in\mathbb{I}}]$ be the Schur-Nevanlinna pair of rational functions corresponding to $(\alpha_k, \kappa_k)_{k\in\mathbb{I}}$, where $\mathbb{I} := \{0, 1, \ldots, m\}$, where $\kappa_0 := 0$, and where $\kappa_k := \overline{\lambda_k}$

for $k = 1, 2, \ldots, m$. Thus, [7, Corollary 3.6] implies that $\gamma_m^{[m]}(z) \neq 0$ for each $z \in \mathbb{D}$ and that the function g_0 defined by

$$g_0(z) := \frac{\delta_m^{[m]}(z)}{\gamma_m^{[m]}(z)}, \quad z \in \mathbb{D},$$

belongs to $\mathcal{S}$. We show now that the function g_0 actually belongs to $\mathcal{S}_\Delta$. An application of [7, Remark 3.9] leads $\delta_m^{[m]}(\alpha_0) = 0$ and hence

$$g_0(z_1) = g_0(\alpha_0) = 0. \tag{5.7}$$

From $g_0 \in \mathcal{S}$, (5.7), and Remark 2.2 it follows that the function Ω_0 given by

$$\Omega_0(z) := \frac{\gamma_m^{[m]}(z) - \delta_m^{[m]}(z)}{\gamma_m^{[m]}(z) + \delta_m^{[m]}(z)}, \quad z \in \mathbb{D},$$

belongs to $\mathcal{C}$. Because of Lemma 5.1 and (3.9) we can also write

$$\Omega_0(z) = \frac{\psi_m^{[m]}(z)}{\varphi_m^{[m]}(z)}, \quad z \in \mathbb{D}.$$

Moreover (see (3.5)), there is a Borel measure μ_0 on $\mathbb{T}$ such that

$$\Omega_0(z) = \mathrm{i}\, \mathfrak{Im}\, \Omega_0(0) + \int_{\mathbb{T}} \frac{u+z}{u-z}\, \mu_0(\mathrm{d}u), \quad z \in \mathbb{D}.$$

Similar as in [6, Theorem 4.2.6], one can see that

$$\mu_0(A) = \frac{1}{2\pi} \int_A \frac{1 - |\alpha_m|^2}{|u - \alpha_m|^2} \frac{1}{|\varphi_m(u)|^2}\, \lambda(\mathrm{d}u), \quad A \in \mathfrak{B},$$

where λ denotes the linear Lebesgue-Borel measure on $\mathbb{T}$. In view of $g_0 \in \mathcal{S}$, (5.7), and Corollary 4.3, to obtain $g_0 \in \mathcal{S}_\Delta$ it remains to verify that $(\varphi_k)_{k=0}^m$ is an orthonormal system corresponding to $\Delta_\pi^{[g_0]}$. Since, for each $x, y \in \mathfrak{H}_m$, the inner product of x and y (similarly defined as in (3.4), but by means of the entries of $\mathbf{P}_{\Delta^{[g_0]}}$) is obtained by integration of $\overline{y(u)}x(u)$, $u \in \mathbb{T}$, with respect to the associated measure μ_0 of g_0 (cf. Lemma 3.2), we have to check that

$$\int_{\mathbb{T}} \overline{\varphi_k(u)} \varphi_j(u)\, \mu_0(\mathrm{d}u) = \delta_{jk}, \quad j, k = 0, 1, \ldots, m.$$

This can be done as in [6, Lemma 8.1.3] (which includes an application of Poisson's integral formula). Furthermore, by using [7, Corollary 3.6 and Theorem 6.3] we see that $\widetilde{\gamma}_m^{[m]}(z) \neq 0$ for each $z \in \mathbb{D}$ and that the function $\widetilde{g}_0$ defined by

$$\widetilde{g}_0(z) := \frac{\widetilde{\delta}_m^{[m]}(z)}{\widetilde{\gamma}_m^{[m]}(z)}, \quad z \in \mathbb{D},$$

belongs to $\mathcal{S}_\Delta$ as well, if $[(\widetilde{\gamma}_k)_{k \in \mathbb{I}}, (\widetilde{\delta}_k)_{k \in \mathbb{I}}]$ is the Schur-Nevanlinna pair of rational functions corresponding to $(\alpha_k, \widetilde{\kappa}_k)_{k \in \mathbb{I}}$, where $\widetilde{\kappa}_0 := s_0$ and $\widetilde{\kappa}_k := -s_k \eta_{k-1}$ for $k = 1, 2, \ldots, m$. Finally, from [7, Lemma 6.2] one can conclude $s_0 = 0$ and (5.6). □

Now, we are able to prove the main result of this section.

Theorem 5.3. *Let Δ be a data set given as in* (2.1) *such that $w_{10} = 0$ and $\mathbf{P}_\Delta > 0$. Furthermore, let $(\varphi_k)_{k=0}^m$ be the canonical Szegő orthonormal system corresponding to Δ_π and let $(\psi_k)_{k=0}^m$ be the dual canonical Szegő orthonormal system of $(\varphi_k)_{k=0}^m$. Then for each $h \in \mathcal{S}$ and each $z \in \mathbb{D}$ the number $\big(\varphi_m^{[m]}(z)+\psi_m^{[m]}(z)\big)+b_{\alpha_m}(z)\big(\varphi_m(z)-\psi_m(z)\big)h(z)$ is not zero and the relation*

$$g(z) = \frac{\big(\varphi_m^{[m]}(z) - \psi_m^{[m]}(z)\big) + b_{\alpha_m}(z)\big(\varphi_m(z) + \psi_m(z)\big)h(z)}{\big(\varphi_m^{[m]}(z) + \psi_m^{[m]}(z)\big) + b_{\alpha_m}(z)\big(\varphi_m(z) - \psi_m(z)\big)h(z)}, \quad z \in \mathbb{D},$$

establishes a bijective correspondence between the set $\mathcal{S}_\Delta$ of all Schur functions g fulfilling (1.1) *and the class $\mathcal{S}$ of all Schur functions h.*

Proof. By virtue of Lemma 5.1 and Lemma 5.2, the assertion is an immediate consequence of [7, Equation (6.2) and Theorem 6.3]. □

From the presented connection between Szegő parameters and Schur parameters in Lemma 5.2 one can also obtain the following statement (cf. [6, Theorem 6.2.5] and [18, Proposition 7.1]) which completes the considerations at the end of Section 4.

Proposition 5.4. *Let Δ be a data set given as in* (2.1) *such that $w_{10} = 0$ and $\mathbf{P}_\Delta > 0$. If $g \in \mathcal{S}_\Delta$ then $-g \in \mathcal{S}$, $-g(0) = 0$, $\mathbf{P}_{\Delta^{[-g]}} = \mathbf{P}_\Delta > 0$, and the dual canonical Szegő orthonormal system $(\psi_k)_{k=0}^m$ of $(\varphi_k)_{k=0}^m$ for Δ_π is the canonical Szegő orthonormal system corresponding to $\Delta_\pi^{[-g]}$.*

Proof. Because of $g \in \mathcal{S}_\Delta$ we have $g \in \mathcal{S}$, $g(0) = 0$, and $\Delta = \Delta^{[g]}$. Hence, it follows immediately that $-g \in \mathcal{S}$ and $-g(0) = 0$ as well as an application of Remark 2.1 yields

$$\mathbf{P}_{\Delta^{[-g]}} = \mathbf{P}_{\Delta^{[g]}} = \mathbf{P}_\Delta > 0.$$

Thus, if $m = 0$ then $(\psi_k)_{k=0}^m$ is apparently the canonical Szegő orthonormal system corresponding to $\Delta_\pi^{[-g]}$. Now, let $m \geq 1$. In view of Lemma 5.2, let $(s_k)_{k=0}^m$ be the sequence of Schur parameters associated with $[g, (\alpha_k)_{k=0}^m]$. Obviously, $(-s_k)_{k=0}^m$ is then the sequence of Schur parameters associated with $[-g, (\alpha_k)_{k=0}^m]$. Therefore, from Lemma 5.2 we can conclude that, if $(\lambda_k)_{k=1}^m$ is the sequence of Szegő parameters of $\Delta_\pi^{[g]}$ then $(-\lambda_k)_{k=1}^m$ is the sequence of Szegő parameters of $\Delta_\pi^{[-g]}$. Because of the recurrence relations (4.4) and (4.7) the proof is complete. □

6. Description of $\mathcal{S}_\Delta$ in terms of $\mathbf{P}_\Delta$

Starting from the recurrence formulas (4.4) and (4.7), in the case $m \geq 1$ the representation of the solution set $\mathcal{S}_\Delta$ of Problem (MNP) in Theorem 5.3 contains the given data implicitly by the Szegő parameters (which can be computed via (4.3), (4.5), or (5.6)). As demonstrated in the following, by using (4.1) one can also express the relations more explicitly. Essentially, in this section we derive from Theorem 5.3 a similar description of $\mathcal{S}_\Delta$ for the non-degenerate case, but in terms of $\mathbf{P}_\Delta$ (without assuming $w_{10} = 0$).

In the present section we suppose always a given data set Δ as in (2.1) such that the condition $\mathbf{P}_\Delta > 0$ is satisfied and we assume some ordering in $(\alpha_k)_{k=0}^m$ such that it satisfies (3.6) and (3.7) for a permutation π of $\{0, 1, \dots, m\}$ with $\pi(0) = 0$. Furthermore, in the case of $m = 0$ let

$$x(v) := \frac{1}{1 - \overline{\alpha_0} v}, \quad y(v) := \frac{\overline{w_{10}}}{1 - \overline{\alpha_0} v}$$

and if $m \geq 1$ then we set

$$x(v) := \begin{vmatrix} \mathbf{P}_{(m-1,m)} \\ \mathbf{e}_m(v) \end{vmatrix}, \quad y(v) := \begin{vmatrix} \mathbf{P}_{(m-1,m)} \\ \mathbf{e}_m(v)\mathbf{W}_m^* \end{vmatrix},$$

where we use the notation

$$\mathbf{P}_{(r,s)} := \begin{pmatrix} p_{00} & \cdots & p_{0s} \\ \vdots & \ddots & \vdots \\ p_{r0} & \cdots & p_{rs} \end{pmatrix}, \quad r, s = 0, 1, \dots, m,$$

and

$$\mathbf{e}_m(v) := \begin{pmatrix} e_0(v) & e_1(v) & \cdots & e_m(v) \end{pmatrix}$$

with p_{jk} and $e_\ell(v)$, $j, k, \ell = 0, 1, \dots, m$, as in Section 3. Note that the functions x and y belong in each case to $\mathfrak{H}_m$, whereas (3.9) implies

$$x^{[m]}(v) = -\eta_0 \frac{1}{1 - \overline{\alpha_0} v}, \quad y^{[m]}(v) = -\eta_0 \frac{w_{10}}{1 - \overline{\alpha_0} v}$$

in the case of $m = 0$ and if $m \geq 1$ then

$$x^{[m]}(v) = \begin{vmatrix} \mathbf{P}_{(m,m-1)} & \mathbf{e}_m^{[m]}(v) \end{vmatrix}, \quad y^{[m]}(v) = \begin{vmatrix} \mathbf{P}_{(m,m-1)} & \mathbf{W}_m \mathbf{e}_m^{[m]}(v) \end{vmatrix},$$

where

$$\mathbf{e}_m^{[m]}(v) := \begin{pmatrix} e_0^{[m]}(v) & e_1^{[m]}(v) & \cdots & e_m^{[m]}(v) \end{pmatrix}^T.$$

Lemma 6.1. *Let Δ be a data set given as in* (2.1) *such that $w_{10} = 0$ and $\mathbf{P}_\Delta > 0$. Furthermore, let $(\varphi_k)_{k=0}^m$ be the canonical Szegő orthonormal system corresponding to Δ_π and let $(\psi_k)_{k=0}^m$ be the dual canonical Szegő orthonormal system of $(\varphi_k)_{k=0}^m$. Then there is a non-zero complex number c such that*

$$x(v) = c\big(\varphi_m(v) + \psi_m(v)\big), \quad y(v) = c\big(\varphi_m(v) - \psi_m(v)\big).$$

Proof. Because of $w_{10} = 0$, (4.2), and (4.6) the assertion is obvious if $m = 0$. Now, let $m \geq 1$. From Theorem 5.3 we already know that the function g_0 defined by

$$g_0(z) := \frac{\varphi_m^{[m]}(z) - \psi_m^{[m]}(z)}{\varphi_m^{[m]}(z) + \psi_m^{[m]}(z)}, \quad z \in \mathbb{D}, \tag{6.1}$$

belongs to $\mathcal{S}_\Delta$. In view of Proposition 5.4, it follows $\mathbf{P}_{\Delta^{[-g_0]}} = \mathbf{P}_\Delta$ and that $(\psi_k)_{k=0}^m$ is the canonical Szegő orthonormal system corresponding to $\Delta_\pi^{[-g_0]}$. Hence, by virtue of (4.1), (3.3), (2.4), and (3.8) there are some non-zero complex numbers d_1 and d_2 such that

$$\varphi_m(v) = d_1 \begin{vmatrix} \mathbf{P}_{(m-1,m)} \\ \mathbf{e}_m(v)(\mathbf{I} + \mathbf{W}_m^*) \end{vmatrix}, \quad \psi_m(v) = d_2 \begin{vmatrix} \mathbf{P}_{(m-1,m)} \\ \mathbf{e}_m(v)(\mathbf{I} - \mathbf{W}_m^*) \end{vmatrix}. \tag{6.2}$$

Since (note property (II) of adjoint rational functions, $g_0 \in \mathcal{S}_\Delta$, and again (3.8))

$$(\mathbf{I} + \mathbf{W}_m)\, \mathbf{e}_m^{[m]}(\alpha_m) = (\mathbf{I} + \mathbf{W}_m) \begin{pmatrix} 0 \\ e_m^{[m]}(\alpha_m) \end{pmatrix} = \begin{pmatrix} 0 \\ \big(1 + g_0(\alpha_m)\big) e_m^{[m]}(\alpha_m) \end{pmatrix},$$

$$(\mathbf{I} - \mathbf{W}_m)\, \mathbf{e}_m^{[m]}(\alpha_m) = (\mathbf{I} - \mathbf{W}_m) \begin{pmatrix} 0 \\ e_m^{[m]}(\alpha_m) \end{pmatrix} = \begin{pmatrix} 0 \\ \big(1 - g_0(\alpha_m)\big) e_m^{[m]}(\alpha_m) \end{pmatrix},$$

the relations (6.2) and (3.9) imply particularly

$$\varphi_m^{[m]}(\alpha_m) = \overline{d_1} \begin{vmatrix} \mathbf{P}_{(m,m-1)} & \begin{matrix} 0 \\ \big(1 + g_0(\alpha_m)\big) e_m^{[m]}(\alpha_m) \end{matrix} \end{vmatrix},$$

$$\psi_m^{[m]}(\alpha_m) = \overline{d_2} \begin{vmatrix} \mathbf{P}_{(m,m-1)} & \begin{matrix} 0 \\ \big(1 - g_0(\alpha_m)\big) e_m^{[m]}(\alpha_m) \end{matrix} \end{vmatrix}.$$

Therefore, from (6.1) one can conclude

$$\begin{aligned} g_0(\alpha_m) &= \frac{\varphi_m^{[m]}(\alpha_m) - \psi_m^{[m]}(\alpha_m)}{\varphi_m^{[m]}(\alpha_m) + \psi_m^{[m]}(\alpha_m)} \\ &= \frac{\overline{d_1}\big(1 + g_0(\alpha_m)\big) e_m^{[m]}(\alpha_m) \det P_{m-1} - \overline{d_2}\big(1 - g_0(\alpha_m)\big) e_m^{[m]}(\alpha_m) \det P_{m-1}}{\overline{d_1}\big(1 + g_0(\alpha_m)\big) e_m^{[m]}(\alpha_m) \det P_{m-1} + \overline{d_2}\big(1 - g_0(\alpha_m)\big) e_m^{[m]}(\alpha_m) \det P_{m-1}} \\ &= \frac{1 - \frac{\overline{d_2}\left(1 - g_0(\alpha_m)\right)}{\overline{d_1}\left(1 + g_0(\alpha_m)\right)}}{1 + \frac{\overline{d_2}\left(1 - g_0(\alpha_m)\right)}{\overline{d_1}\left(1 + g_0(\alpha_m)\right)}}. \end{aligned}$$

Consequently (cf. Remark 2.2), it follows

$$\frac{\overline{d_2}}{\overline{d_1}} \frac{1 - g_0(\alpha_m)}{1 + g_0(\alpha_m)} = \frac{1 - g_0(\alpha_m)}{1 + g_0(\alpha_m)}$$

which supplies $\overline{d_1} = \overline{d_2}$, i.e. $d_1 = d_2$. Finally, by setting $c := \frac{1}{2d_1}$ and using some calculation rules of determinants we get using (6.2) and $d_1 = d_2$ the equalities

$$x(v) = \frac{1}{2} \begin{vmatrix} \mathbf{P}_{(m-1,m)} \\ \mathbf{e}_m(v)\big((\mathbf{I} + \mathbf{W}_m^*) + (\mathbf{I} - \mathbf{W}_m^*)\big) \end{vmatrix} = c\big(\varphi_m(v) + \psi_m(v)\big),$$

$$y(v) = \frac{1}{2} \begin{vmatrix} \mathbf{P}_{(m-1,m)} \\ \mathbf{e}_m(v)\big((\mathbf{I} + \mathbf{W}_m^*) - (\mathbf{I} - \mathbf{W}_m^*)\big) \end{vmatrix} = c\big(\varphi_m(v) - \psi_m(v)\big). \qquad \square$$

Theorem 6.2. *Let Δ be a data set given as in* (2.1) *such that* $\mathbf{P}_\Delta > 0$. *Then for each* $h \in \mathcal{S}$ *and each* $z \in \mathbb{D}$ *the number* $x^{[m]}(z) + b_{\alpha_m}(z)y(z)h(z)$ *is not zero and the relation*

$$g(z) = \frac{y^{[m]}(z) + b_{\alpha_m}(z)x(z)h(z)}{x^{[m]}(z) + b_{\alpha_m}(z)y(z)h(z)}, \quad z \in \mathbb{D},$$

establishes a bijective correspondence between the set $\mathcal{S}_\Delta$ *of all Schur functions* g *fulfilling* (1.1) *and the class* $\mathcal{S}$ *of all Schur functions* h.

Proof. In the case of $m = 0$, the statement is an immediate consequence of (5.1) and (1.2) (i.e. the description of $\mathcal{S}_\Delta$ which is already proved in [7, Theorem 6.3]). Now, let $m \geq 1$. We first remark that one can restrict the proof to the special case of $\alpha_k = \beta_k$, $k = 0, 1, \ldots, m$, with β_k as in (3.6). Indeed this can be done, since only some rows and columns are simultaneously rearranged by the matrices behind the determinant formulas defining the functions x and y unlike the general case and this is not essential in view of the linear fractional transformation stated in Theorem 6.2. Because of this particular choice of $(\alpha_k)_{k=0}^m$, the relation (3.8) leads to

$$\mathbf{W}_m = \mathbf{W}. \tag{6.3}$$

Furthermore, note that $\mathbf{P}_\Delta > 0$ implicates $w_{10} \in \mathbb{D}$. If $w_{10} = 0$ then a combination of Lemma 6.1 and Theorem 5.3 yields the assertion. Based on this and the fact that Remark 2.4 and (2.6) imply a correspondence between the data set Δ and a data set

$$\widetilde{\Delta} := \left\{ \left(z_j, l_j, (\widetilde{w}_{js})_{s=0}^{l_j-1} \right)_{j=1}^n \right\}$$

with $\widetilde{w}_{10} = 0$, by a straightforward calculation one can see that for each $h \in \mathcal{S}$ and each $z \in \mathbb{D}$ the condition

$$\overline{w_{10}}\widetilde{y}^{[m]}(z) + \widetilde{x}^{[m]}(z) + b_{\alpha_m}(z)\big(\overline{w_{10}}\widetilde{x}(z) + \widetilde{y}(z)\big)h(z) \neq 0 \tag{6.4}$$

is satisfied and that the relation

$$g(z) = \frac{w_{10}\widetilde{x}^{[m]}(z) + \widetilde{y}^{[m]}(z) + b_{\alpha_m}(z)\big(\widetilde{x}(z) + w_{10}\widetilde{y}(z)\big)h(z)}{\overline{w_{10}}\widetilde{y}^{[m]}(z) + \widetilde{x}^{[m]}(z) + b_{\alpha_m}(z)\big(\overline{w_{10}}\widetilde{x}(z) + \widetilde{y}(z)\big)h(z)}, \quad z \in \mathbb{D}, \tag{6.5}$$

establishes a bijective correspondence between the set $\mathcal{S}_\Delta$ of all Schur functions g fulfilling (1.1) and the class $\mathcal{S}$ of all Schur functions h in general, where

$$\widetilde{x}(v) := \begin{vmatrix} \widetilde{\mathbf{P}}_{(m-1,m)} \\ \mathbf{e}_m(v) \end{vmatrix}, \quad \widetilde{y}(v) := \begin{vmatrix} \widetilde{\mathbf{P}}_{(m-1,m)} \\ \mathbf{e}_m(v)\widetilde{\mathbf{W}}_m^* \end{vmatrix},$$

defined with respect to $\widetilde{\Delta}_\pi$ similar as x, y are defined with respect to Δ_π. Moreover, we get in view of Remark 2.4, (2.6), (2.4), (3.8), and (6.3) at first

$$(\mathbf{I} + w_{10}\widetilde{\mathbf{W}}_m^*) = (1 - |w_{10}|^2)\big((\mathbf{I} - \overline{w_{10}}\,\mathbf{W}_m)^{-1}\big)^*,$$

$$\overline{w_{10}}\,\mathbf{I}+\widetilde{\mathbf{W}}_m^*=(1-|w_{10}|^2)\mathbf{W}_m^*\big((\mathbf{I}-\overline{w_{10}}\,\mathbf{W}_m)^{-1}\big)^*$$

and thus, by using some elementary calculation rules of determinants, the equalities

$$\begin{aligned}\widetilde{x}(v)+w_{10}\widetilde{y}(v) &= \begin{vmatrix}\widetilde{\mathbf{P}}_{(m-1,m)}\\ \mathbf{e}_m(v)(\mathbf{I}+w_{10}\widetilde{\mathbf{W}}_m^*)\end{vmatrix}\\ &= \begin{vmatrix}(1-|w_{10}|^2)(\mathbf{I}-\overline{w_{10}}\,\mathbf{W}_{m-1})^{-1}\mathbf{P}_{(m-1,m)}\big((\mathbf{I}-\overline{w_{10}}\,\mathbf{W}_m)^{-1}\big)^*\\ (1-|w_{10}|^2)\mathbf{e}_m(v)\big((\mathbf{I}-\overline{w_{10}}\,\mathbf{W}_m)^{-1}\big)^*\end{vmatrix}\\ &= \frac{(1-|w_{10}|^2)^{m+1}(1-\overline{w_{10}}w_{n0})}{|\det[\mathbf{I}-\overline{w_{10}}\,\mathbf{W}_m]|^2}x(v),\end{aligned}$$

$$\begin{aligned}\overline{w_{10}}\widetilde{x}(v)+\widetilde{y}(v) &= \begin{vmatrix}\widetilde{\mathbf{P}}_{(m-1,m)}\\ \mathbf{e}_m(v)(\overline{w_{10}}\,\mathbf{I}+\widetilde{\mathbf{W}}_m^*)\end{vmatrix}\\ &= \begin{vmatrix}(1-|w_{10}|^2)(\mathbf{I}-\overline{w_{10}}\,\mathbf{W}_{m-1})^{-1}\mathbf{P}_{(m-1,m)}\big((\mathbf{I}-\overline{w_{10}}\,\mathbf{W}_m)^{-1}\big)^*\\ (1-|w_{10}|^2)\mathbf{e}_m(v)\mathbf{W}_m^*\big((\mathbf{I}-\overline{w_{10}}\,\mathbf{W}_m)^{-1}\big)^*\end{vmatrix}\\ &= \frac{(1-|w_{10}|^2)^{m+1}(1-\overline{w_{10}}w_{n0})}{|\det[\mathbf{I}-\overline{w_{10}}\,\mathbf{W}_m]|^2}y(v),\end{aligned}$$

where $\mathbf{W}_{m-1}$ stands in this computation for the complex $(m\times m)$-matrix which is obtained from $\mathbf{W}_m$ (and hence from $\mathbf{W}$ due to (6.3)) by deleting the last row and column. Consequently, with a view to (3.9), (6.4), and (6.5) the proof is complete. □

If the point $z\in\mathbb{D}$ is fixed in the description of $\mathcal{S}_\Delta$ by the linear fractional transformation according to Theorem 6.2 then the set

$$\mathfrak{K}_\Delta(z):=\{g(z):\ g\in\mathcal{S}_\Delta\} \tag{6.6}$$

is a closed disk in $\mathbb{D}$, the boundary of which is sometimes called *Weyl circle*. Using some well-known properties of linear fractional transformations (cf. [27, Proposition 2]), it can easily be shown that the center c_z and the radius r_z of this Weyl circle are given by the rational functions x and y in terms of the given data as presented in the following.

Corollary 6.3. *Under the assumptions of* Theorem 6.2, *if $z\in\mathbb{D}$ is fixed then the set $\mathfrak{K}_\Delta(z)$ defined as in* (6.6) *can be described by*

$$\mathfrak{K}_\Delta(z)=\{w:\ |w-c_z|\le r_z\},$$

where the parameters c_z and r_z are given by

$$c_z=\frac{y^{[m]}(z)\overline{x^{[m]}(z)}-b_{\alpha_m}(z)x(z)\overline{b_{\alpha_m}(z)y(z)}}{|x^{[m]}(z)|^2-|b_{\alpha_m}(z)y(z)|^2},\quad r_z=\frac{|b_{\alpha_m}(z)|\,|x(z)x^{[m]}(z)-y(z)y^{[m]}(z)|}{|x^{[m]}(z)|^2-|b_{\alpha_m}(z)y(z)|^2}.$$

Remark 6.4. If we define the functions $g_\bullet$ and $h_\bullet$, for each $z \in \mathbb{D}$, by

$$g_\bullet(z) := c_z, \quad h_\bullet(z) := -\overline{b_{\alpha_m}(z)\frac{y(z)}{x^{[m]}(z)}},$$

then the function $g_\bullet$ is continuous, satisfies the condition $g_\bullet(z_j) = w_{j0}$ for $j = 1, 2, \dots, n$, and admits the representation

$$g_\bullet(z) = \frac{y^{[m]}(z) + b_{\alpha_m}(z)x(z)h_\bullet(z)}{x^{[m]}(z) + b_{\alpha_m}(z)y(z)h_\bullet(z)}, \quad z \in \mathbb{D}.$$

However, since the function $h_\bullet$ is not holomorphic in $\mathbb{D}$ (in particular $h_\bullet \notin \mathcal{S}$), in view of Theorem 6.2 it follows $g_\bullet \notin \mathcal{S}_\Delta$.

Following the geometrical considerations, one can also see that the Weyl circle with center c_z and radius r_z can be described as an Apollonius circle (cf. [17]).

Corollary 6.5. *Under the assumptions of* Theorem 6.2, *if $z \in \mathbb{D}$ is fixed then the set $\mathfrak{K}_\Delta(z)$ defined as in* (6.6) *can be described by*

$$\mathfrak{K}_\Delta(z) = \left\{ v : \left| \frac{v - a_{1,z}}{v - a_{2,z}} \right| \le |b_{\alpha_m}(z)| d_z \right\},$$

where

$$a_{1,z} := \frac{y^{[m]}(z)}{x^{[m]}(z)}, \quad a_{2,z} := \frac{x(z)}{y(z)}, \quad d_z := \left| \frac{y(z)}{x^{[m]}(z)} \right|.$$

Remark 6.6. Clearly, $\mathfrak{K}_\Delta(z_j)$, $j = 1, 2, \dots, n$, contains only the value w_{j0}. But, if we consider instead the set

$$\mathfrak{K}'_\Delta(z_j) := \left\{ \frac{1}{l_j!} g^{(l_j)}(z_j) : \ g \in \mathcal{S}_\Delta \right\}, \quad j = 1, 2, \dots, n,$$

and if we choose just in (3.7) a sequence $(\alpha_k)_{k=0}^m$ so that $\alpha_m = z_j$ then by a straightforward calculation (cf. [27, Section 6]) one can see that

$$\mathfrak{K}'_\Delta(z_j) = \{ w : \ |w - c'_{z_j}| \le r'_{z_j} \},$$

where the parameters of that circle are given by

$$c'_{z_j} = \frac{1}{l_j!} g_0^{(l_j)}(z_j), \quad r'_{z_j} = \frac{1}{l_j!} f^{(l_j)}(z_j) \frac{1}{x^{[m]}(z_j)}$$

and where the rational functions g_0 and f are defined, for each $z \in \mathbb{D}$, by

$$g_0(z) := \frac{y^{[m]}(z)}{x^{[m]}(z)}, \quad f(z) := b_{\alpha_m}(z)\big(x(z)x^{[m]}(z) - y(z)y^{[m]}(z)\big).$$

We remark finally that, based on the linear fractional transformation stated in Theorem 5.3, one can express the parameters of the circles $\mathfrak{K}_\Delta(z)$, $z \in \mathbb{D}$, and $\mathfrak{K}'_\Delta(z_j)$, $j = 1, 2, \dots, n$, in terms of the orthogonal rational functions φ_m and ψ_m introduced in Section 4 as well (see also [7, Section 6]).

7. The degenerate case

In this section we consider the case of exactly one solution in Problem (MNP).

Theorem 7.1. *If Δ is a given data set as in* (2.1) *such that* $\mathbf{P}_\Delta \geq 0$, *then there is exactly one Schur function g fulfilling* (1.1) *if and only if* $\det \mathbf{P}_\Delta = 0$.

Proof. With a view to Theorem 6.2 it is enough to prove that in the case of $\det \mathbf{P}_\Delta = 0$ there is exactly one Schur function g fulfilling (1.1). Firstly, we show that Problem (MNP) has at most one solution $g \in \mathcal{S}_\Delta$ when $\det \mathbf{P}_\Delta = 0$. If $|w_{j0}| = 1$ for some $j = 1, 2, \ldots, n$ then the maximum modulus principle for holomorphic functions implies that at most the constant function with value w_{j0} can belong to $\mathcal{S}_\Delta$. We suppose now (2.7). Furthermore, due to Remark 2.4 and (2.6) we can assume $w_{10} = 0$ without loss of generality. Moreover, let $g \in \mathcal{S}_\Delta$. In particular, the Cayley transform Ω of g according to Remark 2.2 and the associated measure μ of g which is uniquely determined by (3.5) are well defined. Taking into account $\det \mathbf{P}_\Delta = 0$, we consider then a non-zero $(m+1)$-vector $\mathbf{x}_0$ with the property

$$\mathbf{P}_\Delta \mathbf{x}_0 = 0$$

and the corresponding rational function x_0 defined by

$$x_0(v) := \big(f_{10}(v)\, f_{11}(v) \cdots f_{1,l_1-1}(v) \ \cdots \ f_{n0}(v)\, f_{n1}(v) \cdots f_{n,l_n-1}(v)\big)\mathbf{x}_0.$$

By virtue of (3.4) and Lemma 3.2 we get

$$0 = \mathbf{x}_0^* \mathbf{P}_\Delta \mathbf{x}_0 = \Big\langle x_0, x_0 \Big\rangle = \int_{\mathbb{T}} |x_0(u)|^2 \, \mu(\mathrm{d}u).$$

Therefore, the measure μ is concentrated (not exceeding) on the finite set of all (mutually different) zeros $u_1, u_2, \ldots, u_{m'} \in \mathbb{T}$ of the function x_0, where this set is independent of the solution g. Hence, with some non-negative numbers $a_1, a_2, \ldots, a_{m'}$ we have

$$\mu(A) = \sum_{r=1}^{m'} a_r \varepsilon_{u_r, \mathfrak{B}}(A), \quad A \in \mathfrak{B}, \tag{7.1}$$

where $\varepsilon_{u_r,\mathfrak{B}}$ stands for the Dirac measure with unit mass located at u_r, $r = 1, 2, \ldots, m'$. Choosing (note Remark 3.1) for each $r \in \{1, 2, \ldots, m'\}$ a function $x_r \in \mathfrak{H}_\Delta$ which is one at u_r and zero at the remaining points u_j, $0 \leq j \leq m'$ with $j \neq r$, it follows that

$$a_r = \Big\langle x_r, x_r \Big\rangle = \mathbf{x}_r^* \mathbf{P}_\Delta \mathbf{x}_r,$$

where the $(m+1)$-vector $\mathbf{x}_r$ is given by the relation

$$x_r(v) = \big(f_{10}(v)\, f_{11}(v) \cdots f_{1,l_1-1}(v) \ \cdots \ f_{n0}(v)\, f_{n1}(v) \cdots f_{n,l_n-1}(v)\big)\mathbf{x}_r,$$

so that the coefficients a_r in (7.1) are independent of g as well. Thus, Problem (MNP) has at most one solution. It stays to verify that there exists a Schur function g fulfilling (1.1). In order to show that we replace the given data Δ, for each $k \in \mathbb{N}$, by

$$\Delta^{\rangle k \langle} := \Big\{ \Big(z_j, l_j, (w_{js}^{\rangle k \langle})_{s=0}^{l_j - 1} \Big)_{j=1}^{n} \Big\},$$

where

$$w_{js}^{\rangle k\langle} := \begin{cases} w_{j0} + \frac{1}{k+1} & \text{if } s = 0, \\ w_{js} & \text{if } s \neq 0, \end{cases} \qquad s = 0, 1, \dots, l_j - 1,\ j = 1, 2, \dots, n.$$

Since $\mathbf{P}_\Delta \geq 0$ and since the constant function

$$h_k(u) = \frac{1}{k+1}, \quad u \in \mathbb{D},$$

is a Schur function with $\mathbf{P}_{\Delta^{[h_k]}} > 0$ for each $k \in \mathbb{N}$ (use, e.g., [19, Corollary 3.6]), we have

$$\mathbf{P}_{\Delta^{\rangle k\langle}} > 0, \quad k \in \mathbb{N},$$

and

$$\lim_{k\to\infty} \mathbf{P}_{\Delta^{\rangle k\langle}} = \mathbf{P}_\Delta.$$

From Theorem 6.2 we know that a solution g_k of Problem (MNP) concerning the data set $\Delta^{\rangle k\langle}$ exists. Consequently, an application of Montel's theorem yields that there is a Schur function g_0 and a subsequence of $(g_k)_{k\in\mathbb{N}}$ which converges to g_0. This function g_0 is a solution of Problem (MNP) for Δ by construction. □

Let Δ be a given data set as in (2.1) such that

$$\mathbf{P}_\Delta \geq 0 \quad \text{and} \quad \det \mathbf{P}_\Delta = 0.$$

An explicit form of the unique solution for that case according to Theorem 7.1 can be obtained as follows. Denote this solution by g_0. If $m = 0$ then it follows $|w_{10}| = 1$, so that g_0 is obviously the constant function with value w_{10}. Now, let $m \geq 1$ and let $(\alpha_k)_{k=0}^m$ be a sequence fulfilling (3.6) and (3.7) for a permutation π of $\{0, 1, \dots, m\}$ with $\pi(0) = 0$ and $\pi(m) = m$. (Note that $\pi(0) = 0$ and $\pi(m) = m$ is chosen to ensure the conditions $\alpha_0 = z_1$ and $\alpha_m = z_n$ only for technical reasons.) We assume at first that

$$\det \mathbf{P}_{m-1} \neq 0.$$

Let the rational functions x and y be given as in Section 6. Due to the determinant formulas defining x and y, $\det \mathbf{P}_{m-1} \neq 0$, and the Schur factorization, one can also write

$$x(v) = \det \mathbf{P}_{m-1} \left(e_m(v) - \big(e_0(v) \ \cdots \ e_{m-1}(v)\big) \mathbf{P}_{m-1}^{-1} \begin{pmatrix} p_{0m} \\ \vdots \\ p_{m-1,m} \end{pmatrix} \right)$$

and

$$y(v) = \det \mathbf{P}_{m-1} \left(\left(\sum_{s=0}^{l_n-1} \overline{w_{ns}} e_{n,l_n-1-s}(v) \right) - \big(e_0(v) \ \cdots \ e_{m-1}(v)\big) \mathbf{W}_{m-1}^* \mathbf{P}_{m-1}^{-1} \begin{pmatrix} p_{0m} \\ \vdots \\ p_{m-1,m} \end{pmatrix} \right),$$

where $\mathbf{W}_{m-1}$ stands again for the complex $(m \times m)$-matrix which is obtained from the matrix $\mathbf{W}_m$ defined as in (3.8) by deleting the last row and column. In particular, by using property (II) of adjoint rational functions, we have

$$x^{[m]}(\alpha_m) = e_m^{[m]}(\alpha_m) \det \mathbf{P}_{m-1} \neq 0$$

so that $x^{[m]}$ is not identically equal to zero. We follow now the argument in the proof of Theorem 7.1 (part of existence). For each $k \in \mathbb{N}$, let x_k and y_k be denote the rational functions defined as in Section 6 but corresponding to the data set $\Delta^{\rangle k \langle}$. As a special solution g_k in this proof above the function

$$g_k(z) = \frac{y_k^{[m]}(z)}{x_k^{[m]}(z)}, \quad z \in \mathbb{D},$$

can be chosen for each $k \in \mathbb{N}$ (see Theorem 6.2). Evidently, if k tends to ∞ then the functions $x_k^{[m]}$ and $y_k^{[m]}$ converge to $y^{[m]}$ and $x^{[m]}$, respectively, uniformly on $\mathbb{D}$. Consequently, for the solution g_0 of Problem (MNP) we get for each $z \in \mathbb{D}$ the representation

$$g_0(z) = \frac{y^{[m]}(z)}{x^{[m]}(z)} \tag{7.2}$$

which leads to

$$g_0(z) = \frac{\left(\sum_{s=0}^{l_n-1} w_{ns} e_{n,l_n-1-s}^{[m]}(z)\right) - \begin{pmatrix} p_{m0} & \cdots & p_{m,m-1}\end{pmatrix} \mathbf{P}_{m-1}^{-1} \mathbf{W}_{m-1} \begin{pmatrix} e_0^{[m]}(z) \\ \vdots \\ e_{m-1}^{[m]}(z)\end{pmatrix}}{e_m^{[m]}(z) - \begin{pmatrix} p_{m0} & \cdots & p_{m,m-1}\end{pmatrix} \mathbf{P}_{m-1}^{-1} \begin{pmatrix} e_0^{[m]}(z) \\ \vdots \\ e_{m-1}^{[m]}(z)\end{pmatrix}}, \tag{7.3}$$

because of (3.9) and the formulas for the rational functions x, y presented in this section. Besides, since $\det \mathbf{P}_\Delta = 0$ and $\det \mathbf{P}_{m-1} \neq 0$ it follows (see, e.g., [19, Corollary 3.6]) that g_0 has to be a Blaschke product of degree m, i.e. there are some $\varsigma_1, \varsigma_2, \ldots, \varsigma_m \in \mathbb{D}$ and a $\eta \in \mathbb{T}$ such that

$$g_0(z) = \eta \prod_{j=1}^{m} b_{\varsigma_j}(z), \quad z \in \mathbb{D}.$$

Thus, a comparison with (7.2) implies by virtue of (3.9) (cf. [6, Theorem 2.2.1 (3.)]) that $\varsigma_1, \varsigma_2, \ldots, \varsigma_m$ are exactly the zeros of $y^{[m]}$ (possible repeated according to its multiplicity), that $\frac{1}{\overline{\varsigma_1}}, \frac{1}{\overline{\varsigma_2}}, \ldots, \frac{1}{\overline{\varsigma_m}}$ are exactly the zeros of $x^{[m]}$ (possible repeated according to its multiplicity as well), and that there is a $u \in \mathbb{T}$ such that

$$ux = y^{[m]}. \tag{7.4}$$

Summing up, we have obtained the following.

Proposition 7.2. *If* $\mathbf{P}_\Delta \geq 0$, $\det \mathbf{P}_\Delta = 0$, *and* $\det \mathbf{P}_{m-1} \neq 0$ *then the unique solution* g_0 *of* Problem (MNP) *subject to* Theorem 7.1 *is given, for each* $z \in \mathbb{D}$, *by* (7.2) *or in other words by* (7.3). *Moreover, there is a* $u \in \mathbb{T}$ *such that* g_0 *admits the representation*

$$g_0(z) = u\frac{x(z)}{x^{[m]}(z)}, \quad i.e. \quad g_0(z) = u\frac{e_m(z) - \big(e_0(z) \ \cdots \ e_{m-1}(z)\big)\mathbf{P}_{m-1}^{-1}\begin{pmatrix} p_{0m} \\ \vdots \\ p_{m-1,m}\end{pmatrix}}{e_m^{[m]}(z) - \big(p_{m0} \ \cdots \ p_{m,m-1}\big)\mathbf{P}_{m-1}^{-1}\begin{pmatrix} e_0^{[m]}(z) \\ \vdots \\ e_{m-1}^{[m]}(z)\end{pmatrix}},$$

for each $z \in \mathbb{D}$, *where the constant* u *can be computed from* (1.1) *or* (7.4).

Finally, we consider the case that the additional condition $\det \mathbf{P}_{m-1} \neq 0$ is not satisfied. Clearly, if $\det \mathbf{P}_0 = 0$ then g_0 is the constant function with value w_{10}. Otherwise, if $m' < m$ is the smallest positive integer such that $\det \mathbf{P}_{m'} = 0$ but $\det \mathbf{P}_{m'-1} \neq 0$ then we can look at Problem (MNP) for the corresponding smaller data set Δ'. According to Proposition 7.2 the unique solution g_0' associated with Δ' can be computed. Furthermore, the solution g_0 of Problem (MNP) for the data set Δ is obviously a solution for the smaller data set Δ' as well. It follows that $g_0 = g_0'$.

References

1. N.I. Akhiezer, *The classical moment problem and some related questions in analysis* (Oliver and Boyd, London, 1965).
2. A. Almendral, Nevanlinna parametrization of the solutions to some rational moment problems, *Analysis* **23** (2003), 107–124.
3. J.A. Ball, I. Gohberg, and L. Rodman, Interpolation of rational matrix functions, in *Operator Theory: Adv. and Appl. 45* (Birkhäuser, Basel, 1990).
4. P. Bruinsma, *Interpolation for Schur and Nevanlinna pairs*, (Dissertation, University of Groningen, 1991).
5. A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad, A Szegő theory for rational functions, in *Report TW 131* (Dept. of Computer Science, K.U. Leuven, 1990).
6. A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad, Orthogonal rational functions, in *Cambridge Monographs on Applied and Comput. Math. 5* (Cambridge University Press, Cambridge, 1999).
7. A. Bultheel and A. Lasarow, Schur-Nevanlinna sequences of rational functions, *Proc. Edinburgh Math. Soc.*, to appear.
8. C. Carathéodory, Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annehmen, *Math. Ann.* **64** (1907), 95–115.
9. G. Chen and Y. Hu, On the multiple Nevanlinna-Pick matrix interpolation problem in the class $\mathcal{C}_p$ and the Carathéodory matrix coefficient problem, *Linear Algebra Appl.* **283** (1998), 179–203.
10. M.M. Djrbashian, Orthogonal systems of rational functions on the circle with given set of poles (Russian), *Dokl. SSSR* **147** (1961), 1278–1281.
11. P. Delsarte, Y. Genin, and Y. Kamp, Schur parametrization of positive definite block-Toeplitz systems, *SIAM J. Appl. Math.* **36** (1979), 34–46.

12. V.K. Dubovoj, B. Fritzsche, and B. Kirstein, Matricial version of the classical Schur problem, in *Teubner-Texte zur Mathematik 129* (Teubner, Leipzig, 1992).
13. H. Dym, J contractive matrix functions, reproducing kernel Hilbert spaces and interpolation, in *CBMS Regional Conf. Ser. Math. 71* (Amer. Math. Soc., Providence, RI, 1989).
14. I.P. Fedčina, Tangential Nevanlinna-Pick problem with multiple points (Russian), *Dokl. Akad. Nauk Arm. SSR Ser. Mat.* **60**, (1975), 37–42.
15. C. Foiaş and A.E. Frazho, The commutant lifting approach to interpolation problems, in *Operator Theory: Adv. and Appl. 44* (Birkhäuser, Basel, 1990).
16. C. Foiaş, A.E. Frazho, I. Gohberg, and M.A. Kaashoek, Metric constrained interpolation, commutant lifting and systems, in *Operator Theory: Adv. and Appl. 100* (Birkhäuser, Basel, 1998).
17. Ja.L. Geronimus, On polynomials orthogonal on the unit circle, on the trigonometric moment problem and on associated functions of classes of Carathéodory - Schur (Russian), *Mat. USSR-Sb.* **15** (1944), 99–130.
18. H. Langer and A. Lasarow, Solution of a multiple Nevanlinna-Pick Problem via orthogonal rational functions, *J. Math. Analysis Appl.* **293** (2004), 605–632.
19. A. Lasarow, On generalized Schwarz-Pick-Potapov block matrices of matricial Schur functions and of functions which belong to the Potapov class, *Analysis* **23** (2003), 371–393.
20. R. Nevanlinna, Über beschränkte analytische Funktionen, *Ann. Acad. Sci. Fenn.* **A 32** (1929), 1–75.
21. G. Pick, Über die Beschränkungen analytischer Funktionen, welche durch vorgegebene Funktionswerte bewirkt werden, *Math. Ann.* **77** (1916), 7–23.
22. M. Rosenblum and J. Rovnyak, Hardy classes and operator theory, in *Oxford Math. Monographs* (Oxford University Press, New York, 1985).
23. L.A. Sakhnovich, Interpolation theory and its applications, in *Math. and its Appl. 428* (Kluwer Academic Publishers, Dordrecht, 1997).
24. I. Schur, Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind, *J. reine u. angewandte Math.* Teil I: **147** (1917), 205–232; Teil II: **148** (1918), 122–145.
25. G. Szegő, Orthogonal polynomials, in *Amer. Math. Soc. Coll. Publ. 23* (Amer. Math. Soc., Providence, RI, 1939).
26. S. Takahashi, Extension of the theorems of Carathéodory-Toeplitz-Schur and Pick, *Pacific J. Math.* **138** (1989), 391–399.
27. S. Takahashi, Nevanlinna parametrizations for the extended interpolation problem, *Pacific J. Math.* **146** (1990), 115–129.
28. H. Woracek, Multiple point interpolation in Nevanlinna classes, *Integr. Equ. Oper. Theory* **28** (1997), 97–109.