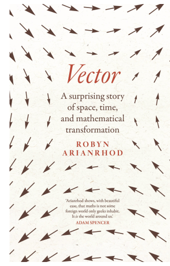
Robyn Arianrhod: **Vector. A surprisings story of space, time, and mathematical transformation**, U. of Chicago Press, 2024 (xxvii+445 p.) isbn: 97-0-226-82110-8



Robyn Arianrhod

This is basically an historical viewpoint of what made it possible to develop electromagnetism and relativity theory. The thesis of Arianrhod is that this is mainly due to the development of the concepts of vectors and tensors. Of course, once they are available, they are used for other applications that we are currently familiar with (robotics, GPS, search engines, machine learning algorithms, particle physics, etc.).

The formal *symbolic notation* in mathematics that is now standard have facilitated the development. These symbols (rather than verbal formulation of the problem and the solution method) stimulated a process of abstraction. This started with the development of algebra (under Arab influence) as a calculating tool to solve (algebraic) equations rather than the traditional Greek approach that was mainly geometric. But algebra was only introduced in Western Europe in the 16-18th century by pioneers like Cardano, Bombelli, Harriot, Descartes, Lagrange, Laplace somewhat simultaneously with the calculus of Newton and Leibniz. The main stimuli for these mathematical developments were the applications: solving equations that model (celestial) mechanical problems, given Newton's laws of gravity, the behaviour of light and heat, and the emerging electro-magnetic problems. Symbolic notation was somewhat promoted by Peacock and his student De Morgan, who inspired Hamilton but a complete formalization came only with Peano, who finally also defined a vector as an element of a vector space independent from the concept of the arrow with its length and direction.

The history of the vector starts with Rowan Hamilton who was trying to generalize complex numbers. A complex number $z = x + iy$ is basically a couple of real numbers (x, y), which are the coordinates in an orthogonal frame with basis $1 \sim (1, 0)$ and $i \sim (0, 1)$. So 1 and i are at the same time numbers one can compute with and (basis) vectors. An alternative is the polar representation of Argand, and this connection made it clear that a multiplication with $e^{i\theta}$ actually corresponds to a rotation. It was Hamilton's ambition to generalize this rotation in the plane to a rotation in 3D space. The crucial hurdle to take was that it was not sufficient to go from 2 to 3 variables, but that it took quadruples to solve the problem, which led him to develop quaternions $q = w + xi + yj + zk$ which required a basis satisfying $i^2 = j^2 = k^2 = ijk = -1$.

When Hamilton had this insight during a walk in Dundee on 10 October 1843, he was so excited that he carved these relations in the stones of Broom Bridge. These carvings are gone now, but there is a plaque to remember it.

He called w the scalar part and $xi + yj + zk$ the vector part. These i, j, k were again numbers that one can compute with, following some particular rules, while we now also see them as basis vectors, to represent the quaternion as a point in a 4D space with Cartesian coordinates (w, x, y, z). But it took a lot of controversy before that became clear because a vector as we know it today was not yet defined. There were two advantages in using quaternions: they contain a lot of information in one symbol, and it was possible to write down complex relations without the need to write out all the relations between the components. For example the product of two quaternions has a scalar part that is the product of the scalar parts, but the vector part needs the inner and outer product and a sum of scaled vector parts. But there was a lot of opposition because people like William Thomson (later lord Kelvin) considered quaternions as a gimmick that had no practical application.

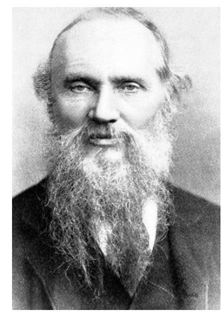
Although he did not use the term vector, the mathematical basis for vectors was paved by Grassmann, be developed a geometric theory to multiply *Strecke* or lines. He was however less well known and his papers were considered too abstract and difficult to read.

Meanwhile, Peter Tait, a student of Hamilton, and James Clerck Maxwell were schoolboy friends, and Maxwell was working on fluid dynamics and electromagnetic research. So Maxwell learned about electrical potentials, line and surface integrals, and force fields introduced by Faraday in analogy with fluid dynamics and Fourier's heat flow. That involved derivatives of vectors and vectors of (partial) derivative operators (divergence, curl, Laplace operator). The nabla operator, introduced by Hamilton, can be seen as $\nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k$, so that force (a vector) and potential (multivariate scalar function) are linked by $F = \nabla V$, and the Laplace operator was an inner product $\Delta = -\nabla \cdot \nabla$ and the inner product with a vectorfield v gives the divergence $\nabla \cdot v$ while the outer product $\nabla \times v$ gives the curl.


William Rowan Hamilton


James Clerk Maxwell


Peter Guthrie Tait


William Thomson (Lord Kelvin)

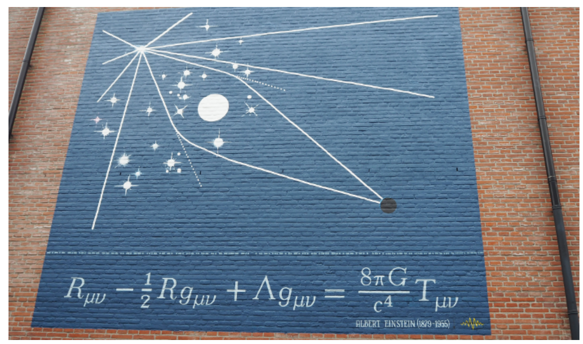
In fact Newton's equation $\mathbf{F} = m\mathbf{a}$ has 3 equations: $F_i = ma_i$, $i = x, y, z$, and in spacetime, this is 4 equations. But in general relativity each of the 4 dimensions influences the 4 components, which gives $4 \times 4 = 16$ equations, but because of symmetry this can be reduced to 10.

Maxwell died young at the age of 48 (1879), but his work was carried on by others: Lodge and Hertz, who proved the existence of electromagnetic waves while Heaviside and later Gibbs reformulated Maxwell's work. Maxwell had captured the equations in terms of potentials, which Heaviside considered unphysical, and so he came to the well known formulation that we are familiar with today: $\nabla \cdot E = 4\pi\rho$, $\nabla \cdot B = 0$, $\nabla \times E = -\frac{\partial B}{\partial t}$, $\nabla \times B = \frac{\partial E}{\partial t} + 4\pi J$, where (with appropriate normalizations) E and B are the electrical and magnetic field, ρ is the charge density and J is the current density. Heaviside got rid of potentials, but they are very much alive in current applications of gauge theory. Gibbs is the one who is rethinking this, not so much in terms of electromagnetic application or in terms of quaternions, but more in terms of vectors, and its possible generalization to dimension $n \geq 3$, although there is a problem to define the cross product for $n > 3$. Only later did Gibbs discover the work of Grassmann whom he acknowledges but he does not mention Maxwell. So after many controversies of quaternions versus vectors and component relations and abstract mathematics versus applied physicists, the electromagnetic results were accepted and vectors became a concept of their own.

Einstein's special relativity, linked space and time in a 4D Minkowski space under the condition of relative constant velocity. This was based on work by Poincaré and Lorentz. The Lorentz transforms between different reference frames are the ones that leave (gravitational) force invariant leading to the constant speed of light. This needed generalization to variable relative speed of the reference frames. Invariance is a key word here. The physics should be the same in whatever reference frame it is described in. This is translated into symmetry and groups describing the transformations. Moreover spacetime is curved. Computation of the length of a curve or the area of a curved patch had been discussed by Gauss before, but it was Riemann that introduced the manifolds that can be considered smooth enough so that they can be locally approximated by the tangent plane so that length $ds^2 = \sum_i dx_i^2$ and area $dS^2 = \sum_{i,j} a_{ij} dx_i dx_j$, which are quantities that should remain invariant. But when Einstein wanted to move to general relativity, the velocity is variable. It was here that he was asking his mathematics friend Grossmann to help him applying the differential (tensor) calculus of Ricci and Levi-Civita. The concept tensor was already noted by Maxwell when he was considering the stress of a volume submerged in a fluid. There are the 3 coordinate faces that are submitted to a 3D stress vector, so that it needed 9 coordinates to represent the stress. Hence the name tensor since it expresses the stress, a term previously introduced in the context of crystallography by Woldemar Voigt, inspired by Hamilton. But it was Ricci who integrated the work of many (Gauss, Riemann, Christoffel, Sophus Lie, Beltrami, Klein) to develop tensor analysis and how it relates to invariants, symmetry and groups.

Einstein took Maxwell's equation $\nabla \cdot E = 4\pi\rho$, re-introduced a scalar potential field V and wrote Newton's law of gravity as $\nabla \cdot \nabla V = \Delta^2 V = 4\pi G\rho$ where now ρ is mass density and G the gravitational constant. The problem was to make it hold in different frames. The invariant transformation in the 4D Minkowski space needed to replace the constant G by a 4D tensor representing geometry as well as gravity, which is subject to some symmetry conditions.

Arianrhod goes on to explain the covariant and contravariant index notation for tensors and how they are used to abbreviate summation. She also arrives at Einstein's field equations, briefly discusses the cosmological constant and the work of Emmy Noether about conservation laws.



Einstein's field equations on a wall of the Leiden university.

The author made her PhD about general relativity theory, which explains why she wants to include the derivation of Einstein's field equations. I liked in particular the entertaining way of how she introduces the history of the mathematical preliminaries. Its a detective story with a lot coincidences and emotions. However, as soon as she dives into the index notations of tensors, the contents becomes much more technical and I can imagine that the reader who is not sufficiently trained may loose interest.

Tensorlab is a matlab package for numerical tensor computations it has several demos of applications.

Her story is quite long and wide and she gives details about all the scientists involved: their character and background, and how they arrived at the result that is relevant in this context, their rivals and friends, their fame and reputation. Several of them were not even mathematicians, some were athletic, others wrote poems. Many were mentioned in this review, but many more play a role in Arianrhod's story. Therefore it is useful that there is a quite extensive list of names and subjects. Also the time line listing events is helping to place everything in historical order, because it is of course difficult to tell a coherent story strictly respecting the historical moments. Several findings rely on different ideas that are implicit is previous results. What is certainly interesting is to learn from this story that what we consider as pure mathematics are basically originating from different, often not directly related, applications, and that also the notation has played an important role leading to abstraction that eventually sublimated in a mathematical concept. And these abstract concepts have today many modern applications that are also briefly mentioned throughout the text. Adhemar Bultheel