Powell-Sabin spline wavelets for geometric modeling

for signals, images and surfaces

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18th October 2006

Survey

- 1D signals signal
  - wavelets
  - multiresolution
  - lifting
  - subdivision
- 3D signals (surfaces)
  - Powell-Sabin splines
  - subdivision
  - hierarchical basis, wavelet basis

1D signals



Fourier

$$
\begin{aligned}
x(t) &= \int_{-\infty}^{\infty} X(\omega) e^{i\omega t} \mathrm{d}\omega \\
&\approx \sum_{k=-\infty}^{\infty} X(\omega_k) e^{i\omega_k t} \\
&= \sum_{k=0}^{\infty} [a_k \cos(\omega_k t) + b_k \sin(\omega_k t)] \\
&= \sum_{k} \hat{x}_k \psi_k(t)
\end{aligned}
$$

The ψ_k are basis functions.

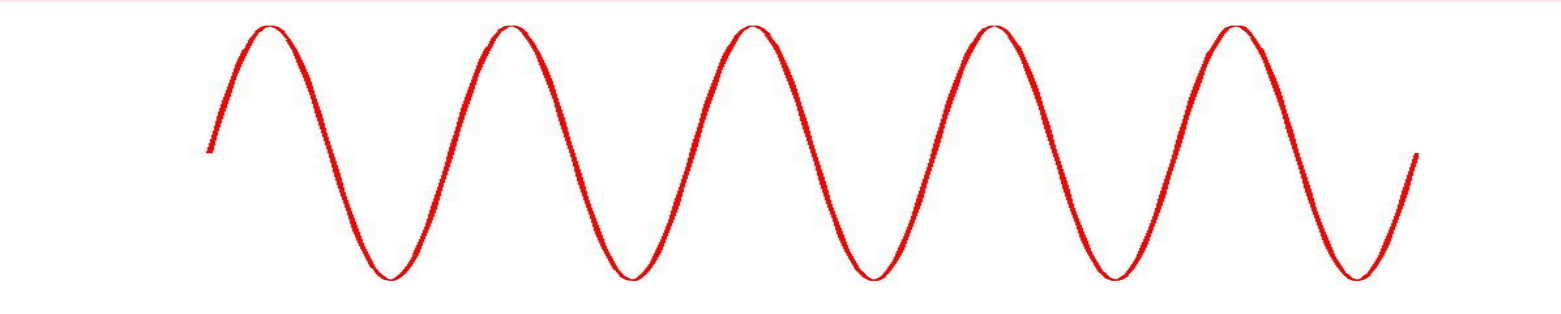
Fourier

Advantages

- every ψ_k corresponds to exactly 1 frequency
- orthogonal system
- generated by 1 "mother function"

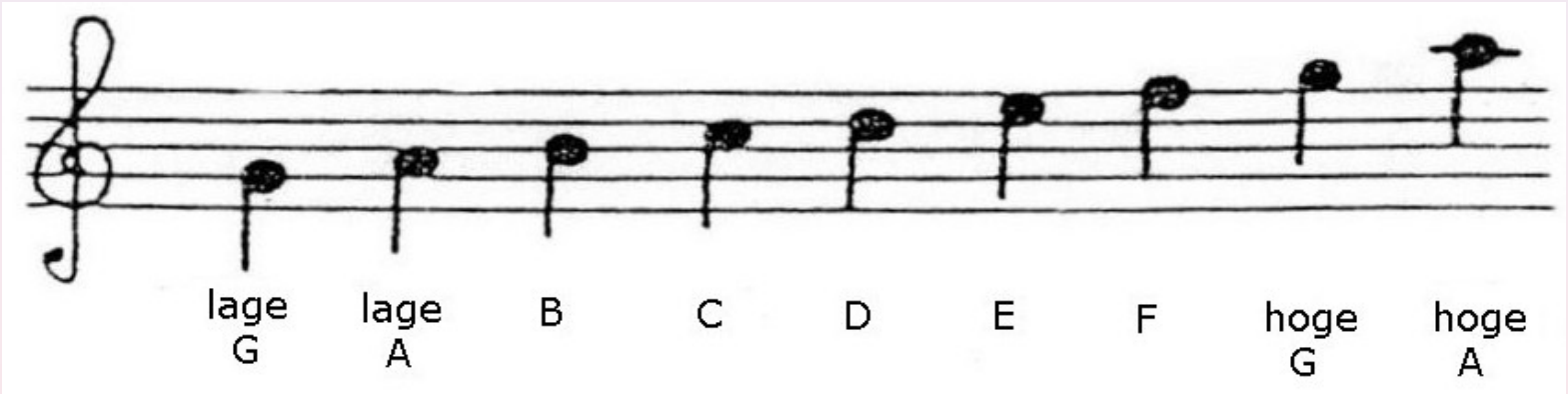
Disadvantage

- infinite support in time domain $\sum_k c_k \psi_k(t)$



Wavelets

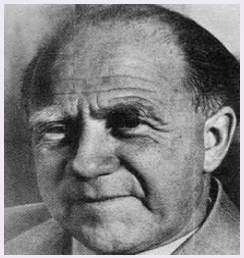
Look for basis functions that are local in time and frequency



- one specific frequency
- at one specific moment

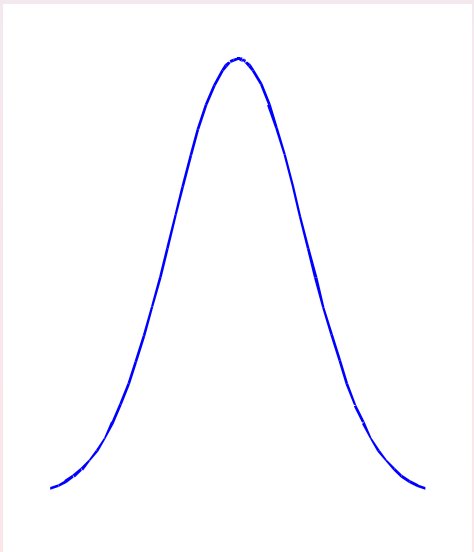
Wavelets

Heisenberg uncertainty principle

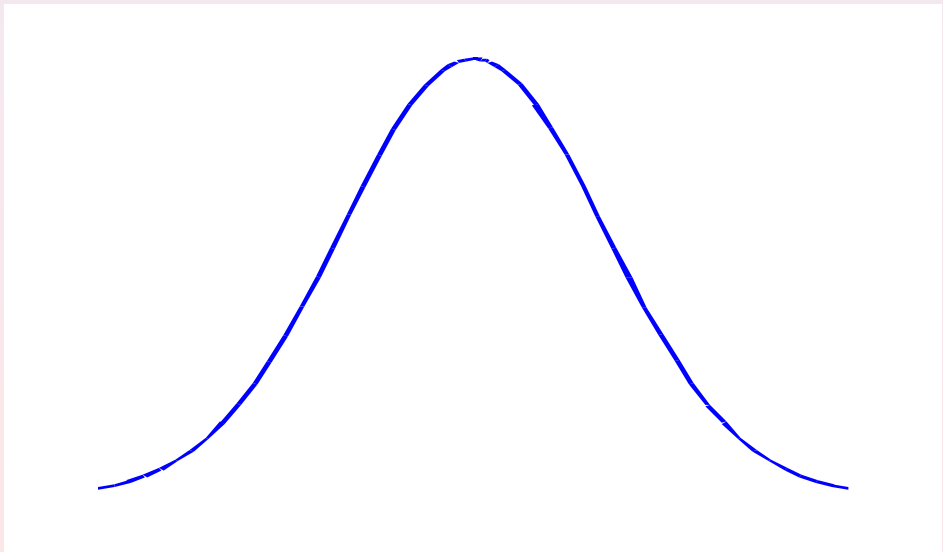


1901-1976, 1926

time domain



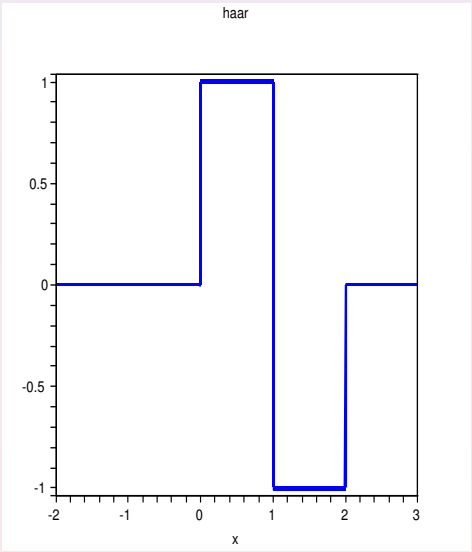
frequency domain

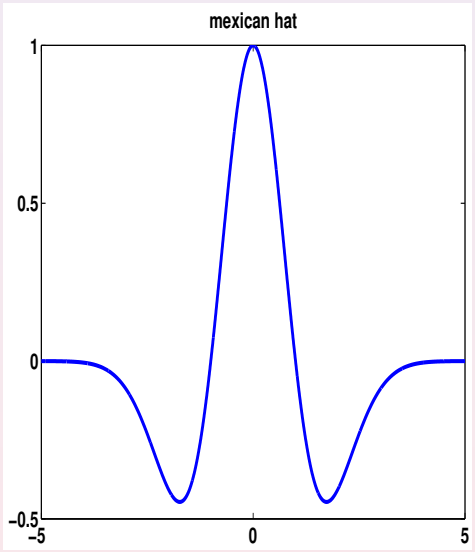


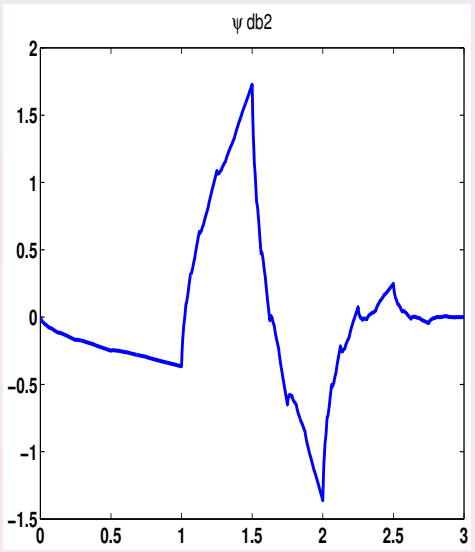
$$\sigma_t \sigma_\omega \geq \frac{1}{4}$$

Wavelets

- limited frequency content
- limited support in time









A. Haar
1885-1933
1909



I. Daubechies
1988

The integral is zero. Hence *wavelet*

Wavelets

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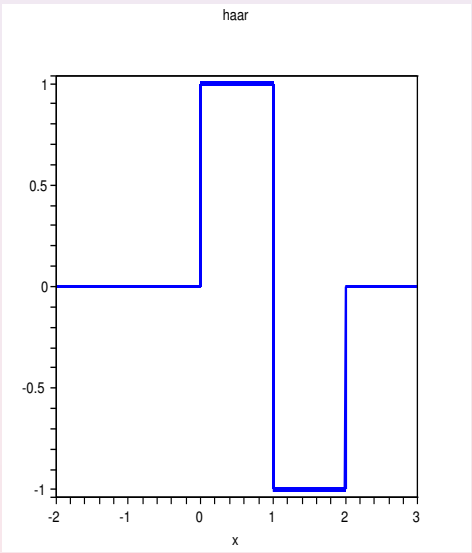
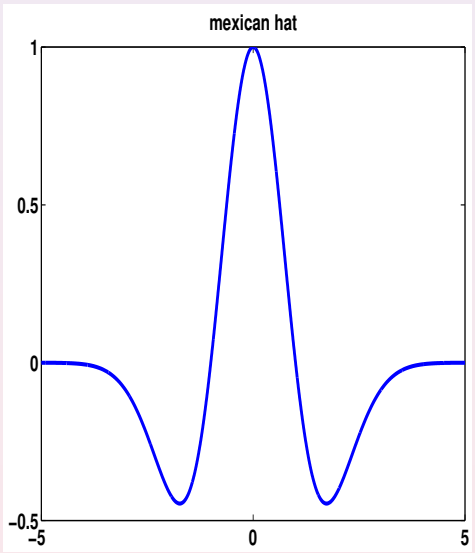
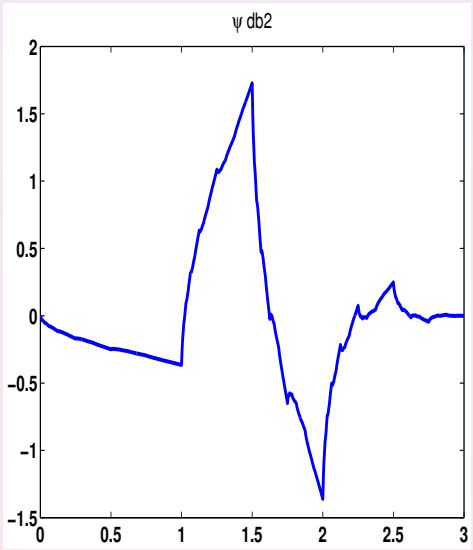
- limited support in time









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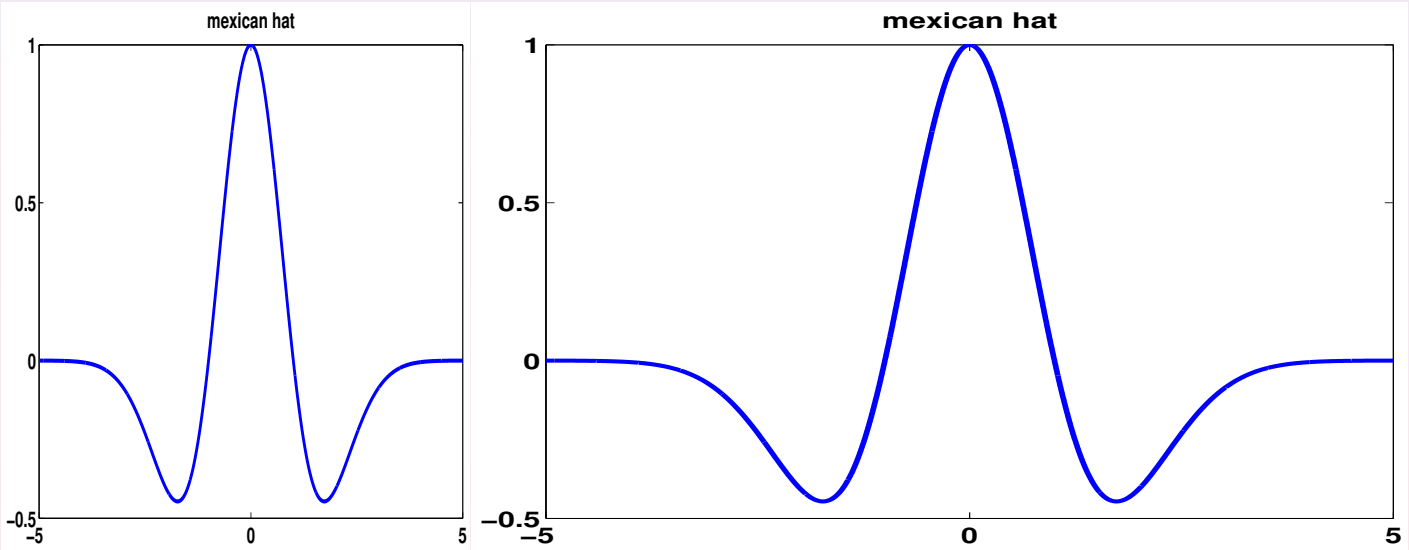
Wavelets

- compact support $\Rightarrow$ needs translates + dilates



- two parameters, 1 "mother function"

$$x(t) = \sum_{k,l} c_{k,l}\psi_{k,l}(t), \quad \psi_{k,l}(t) = 2^{k/2}\psi(2^k t - l)$$

Wavelets

Advantages

- local effect in both domains
- fast algorithm
- (bi)orthogonal basis
- additional properties (symmetry, smoothness,...)
- multiresolution/hierarchy

Disadvantage

- no explicit formula

Multiresolution

$$W_k = \operatorname{span}\{\psi_{k,l}(t)\}_{l=-\infty}^{\infty} = \{\psi_{k,0} \text{ and its translates}\}$$

$$x(t) = \underbrace{\underbrace{x_0(t)}_{\in V_0} + \underbrace{w_0(t)}_{\in W_0}}_{x_1(t) \in V_1} + \underbrace{w_1(t)}_{\in W_1} + \cdots$$

- Each W_k adds a layer of detail
- WT splits signal in layers of resolution
- $V_n = V_0 + W_0 + W_1 + \cdots + W_{n-1} =$ approximation space
- $V_n = \operatorname{span}\{\varphi_{n,l}\}_l = \{\varphi_{n,0} \text{ and translates }\}$

Multiresolution

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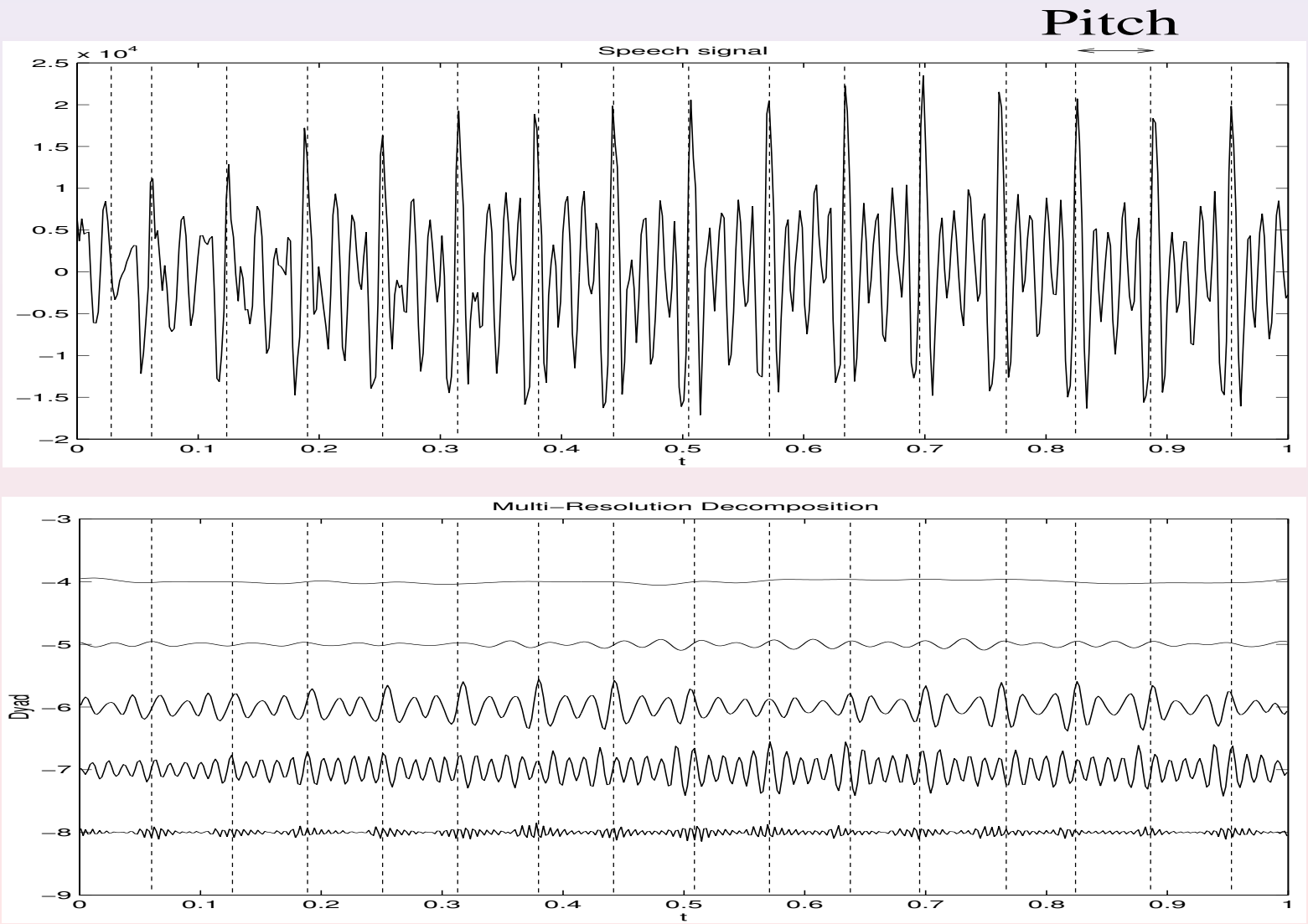
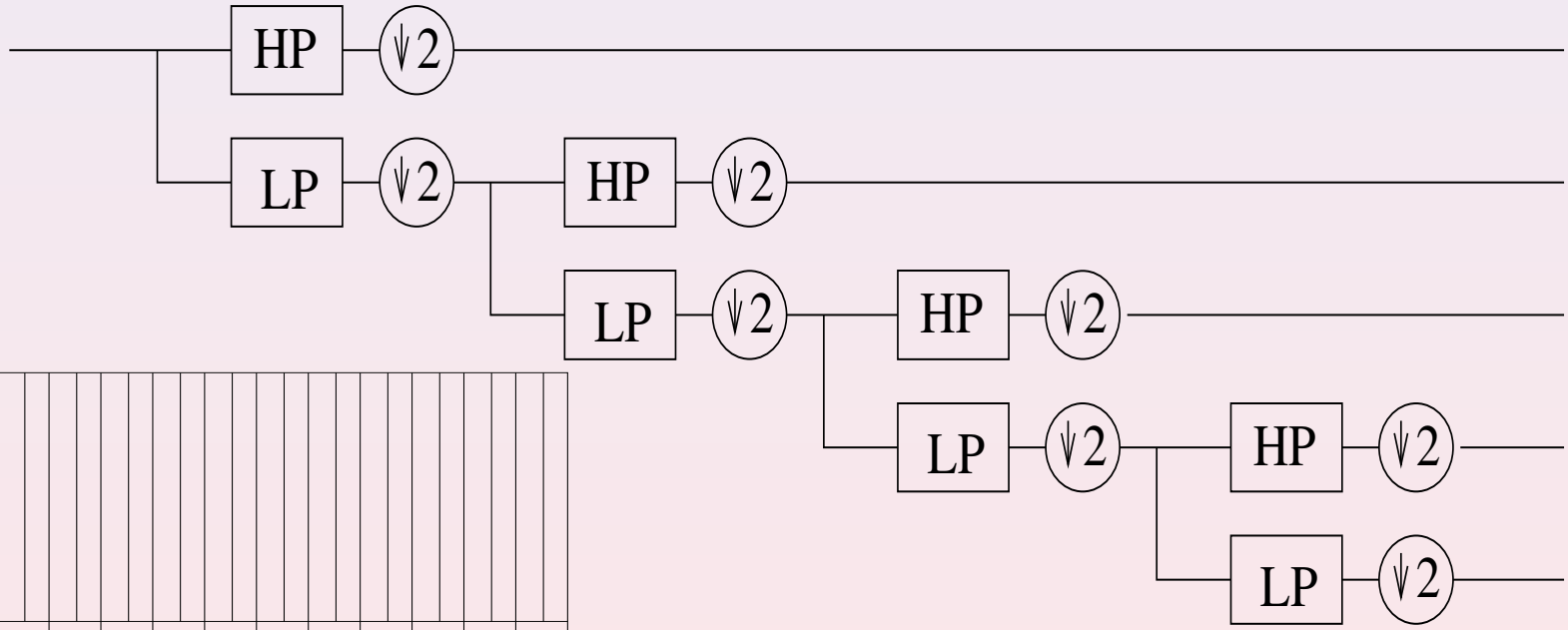
$$\begin{aligned} V_n &= V_{n-1} + W_{n-1} \quad \Rightarrow \quad x_n(t) = x_{n-1}(t) + w_{n-1}(t) \\ V_n &= \operatorname{span}\{\varphi_{n,l}(t)\}_l \\ &= \operatorname{span}\{\varphi_{n-1,l}(t)\}_l \cup \{\psi_{n-1,l}(t)\}_l \end{aligned}$$

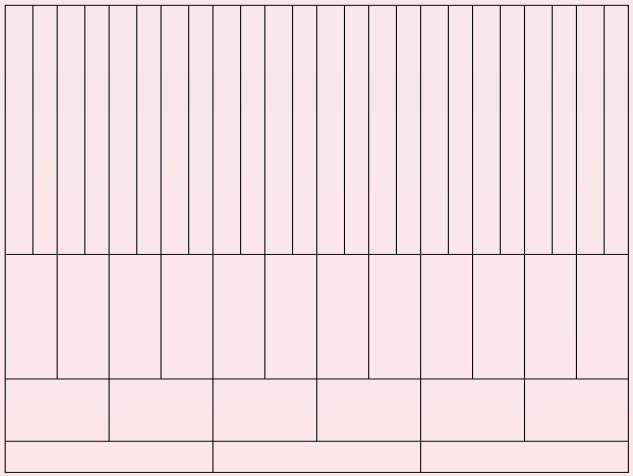
Multiresolution



Multiresolution

Recursive filterbank



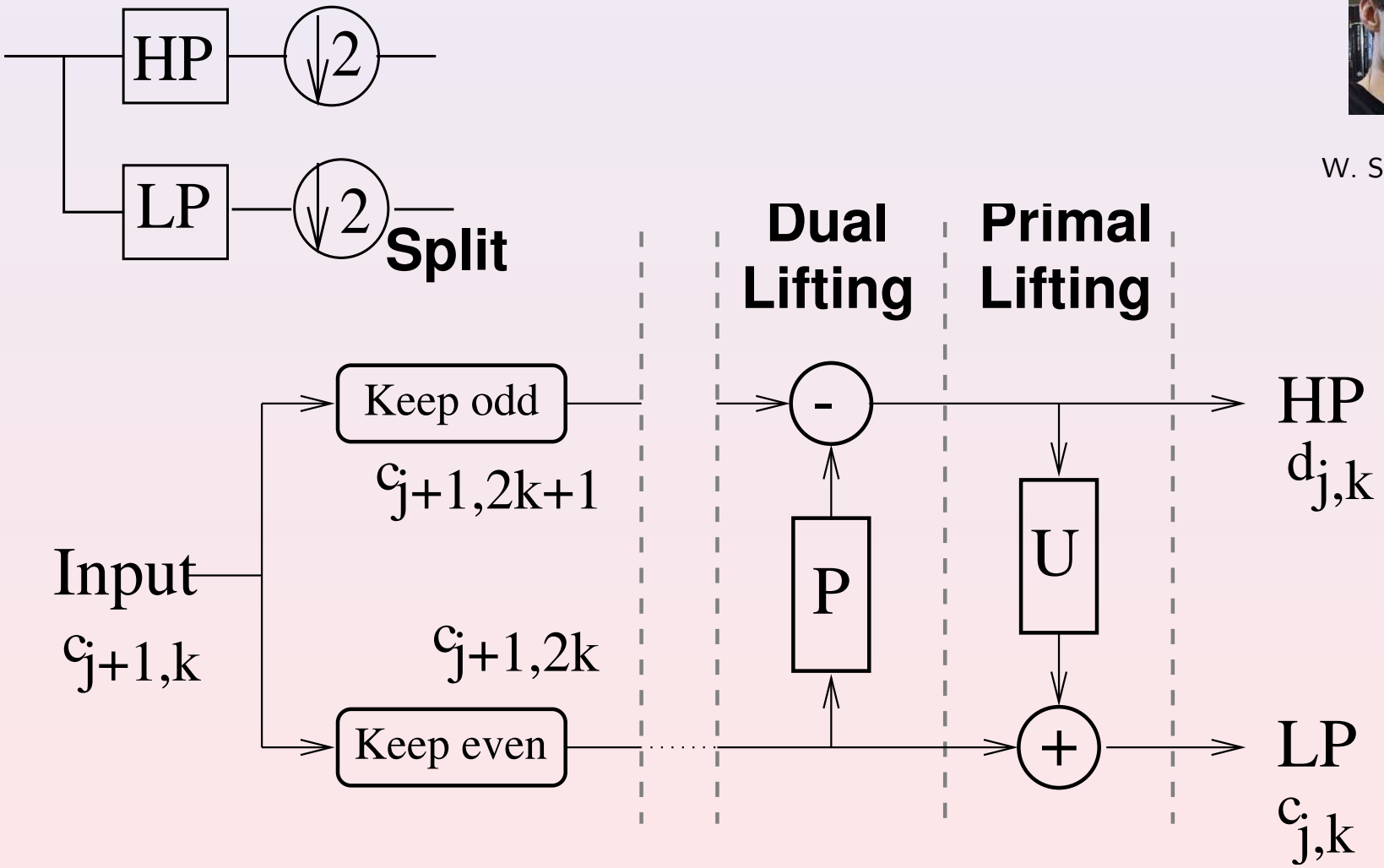


dyadic division of time-frequency plane

Lifting scheme

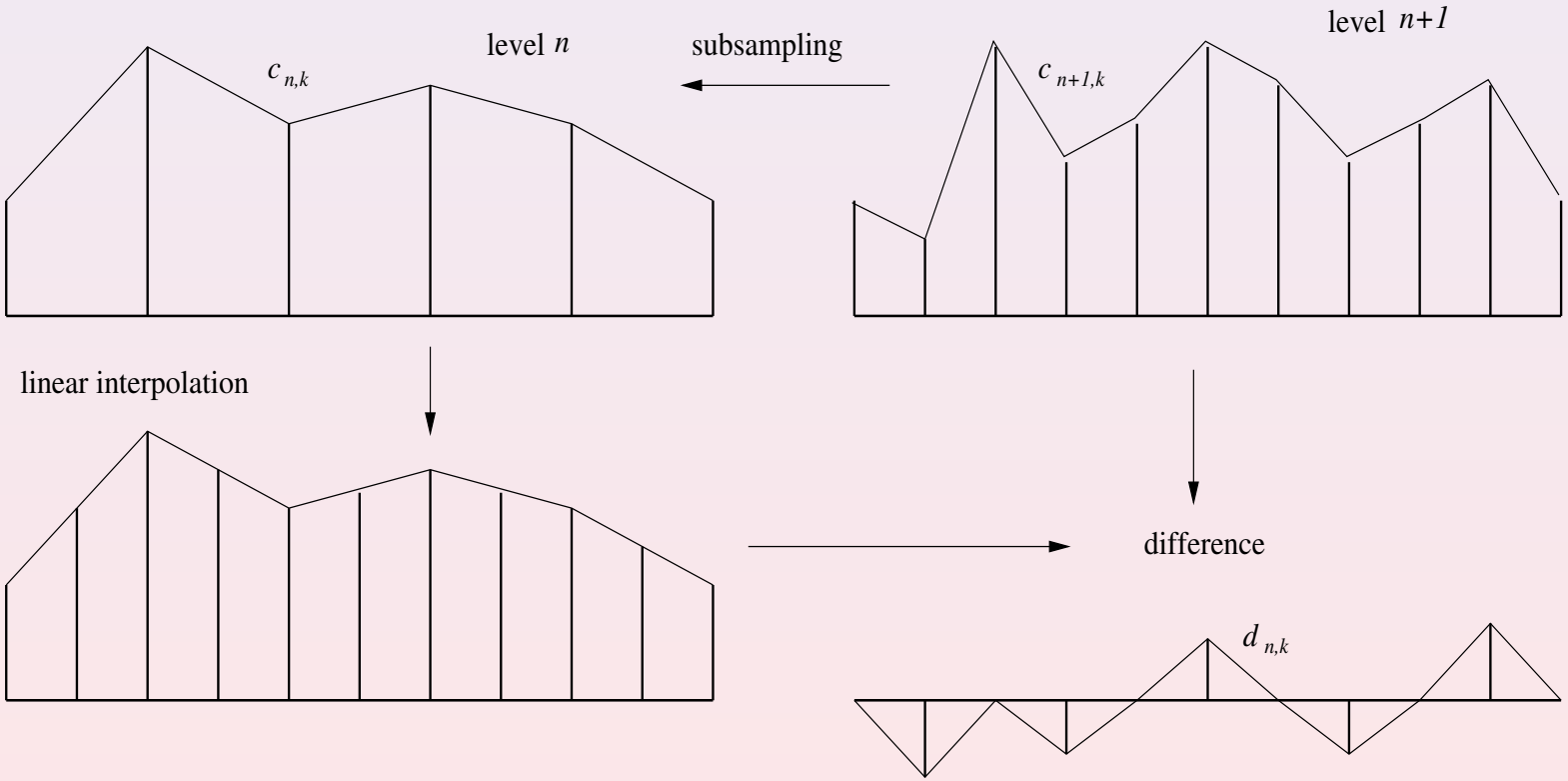


W. Sweldens 1994



Subdivison

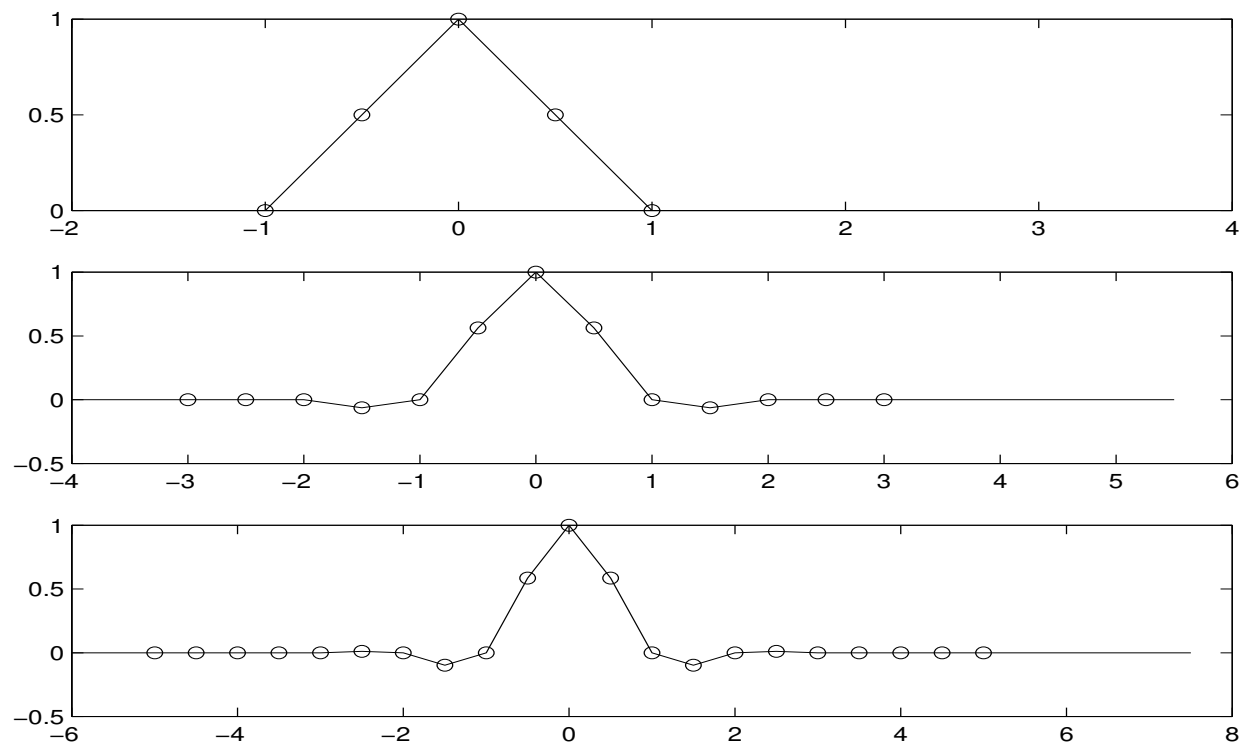
linear interpolation



Deslauriers-Dubuc subdivision

Subdivison

Scaling function φ by subdivision of $(\ldots, 0, 0, 1, 0, 0, \ldots)$, e.g. $n = 1, 3, 5$



All other φ_{kl} are dilates and translates of φ.

Summary 1D

- Wavelets
- Multiresolution
- Lifting scheme
- Subdivision

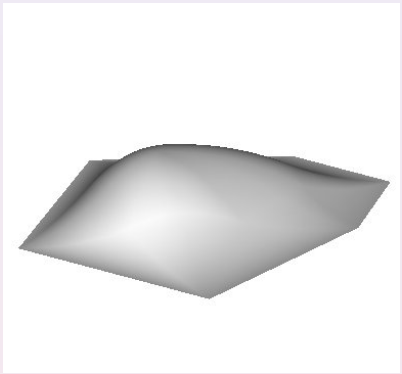
Surface representations

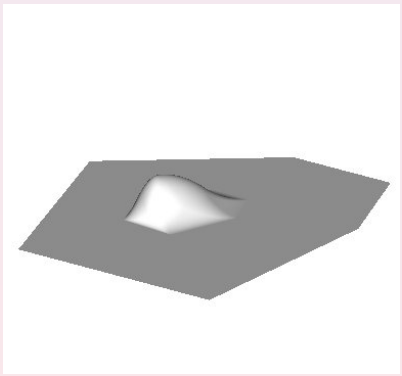


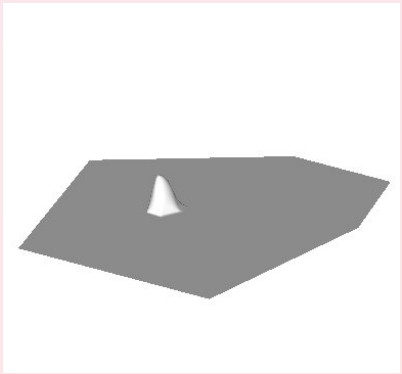
Different representations

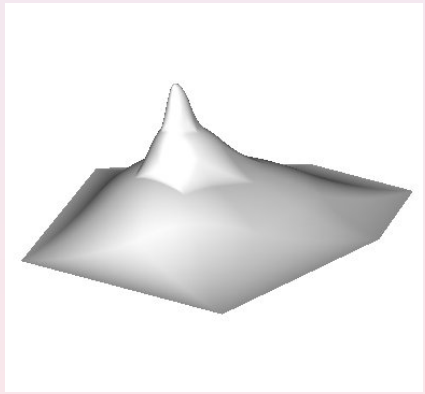
- Mathematical expression
- Polygon model
- Procedural definition

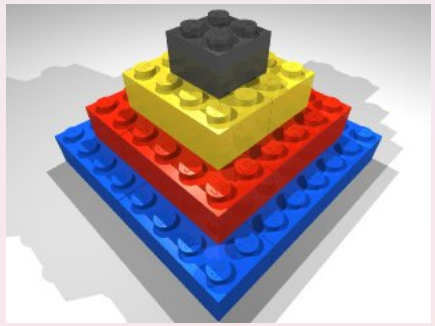
Splines as mathematical building blocks



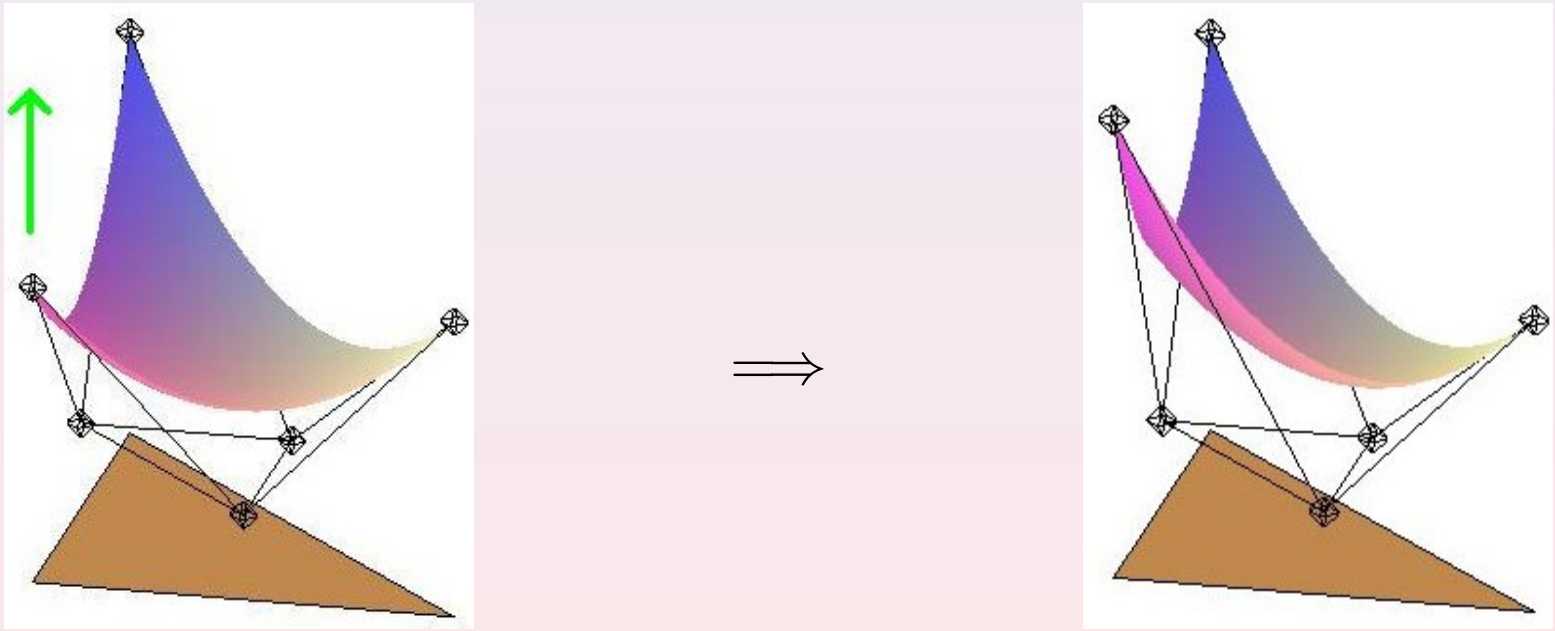




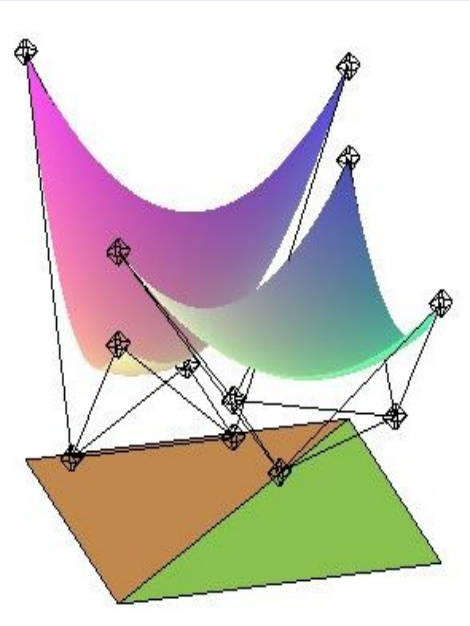




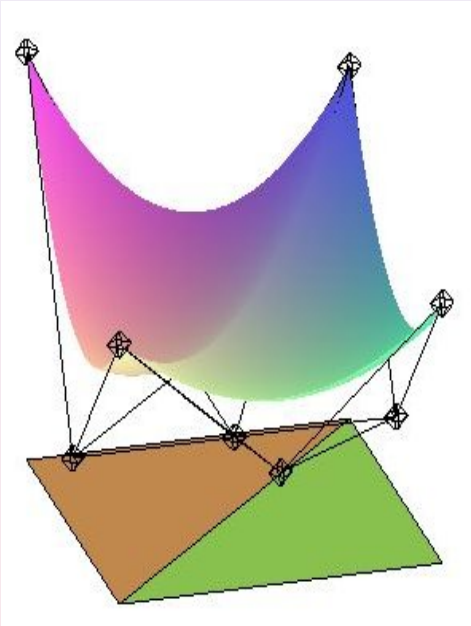
Bernstein–Bézier representation

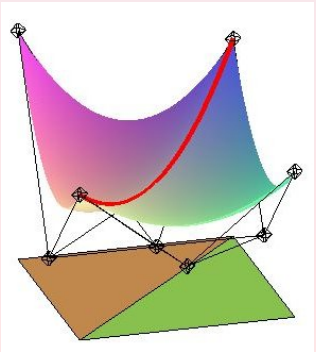


Stitching together Bézier triangles



$\Longrightarrow$

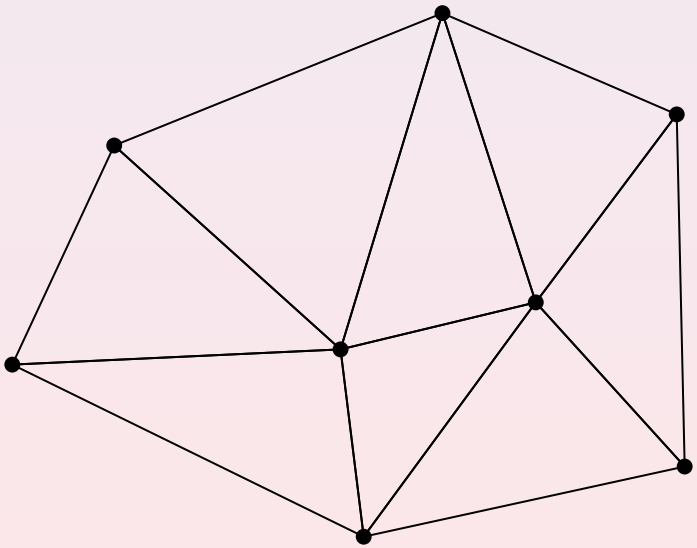




No C^1 continuity at the red curve

Powell-Sabin refinement

- Conformal triangulation Δ
- PS 6-split Δ^*.



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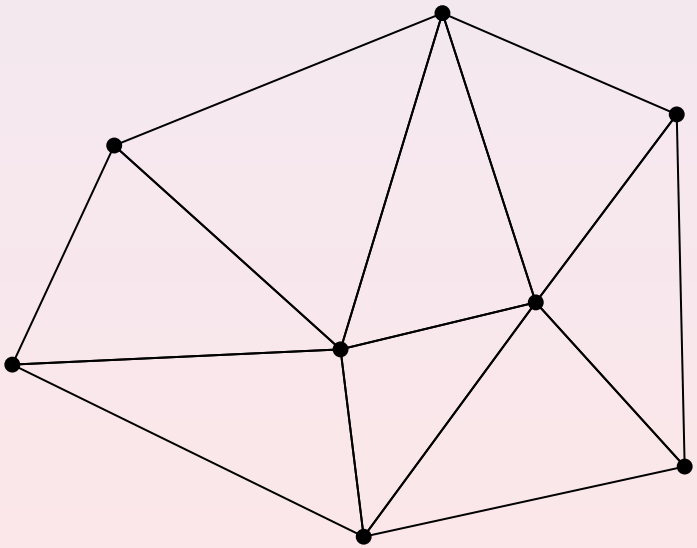
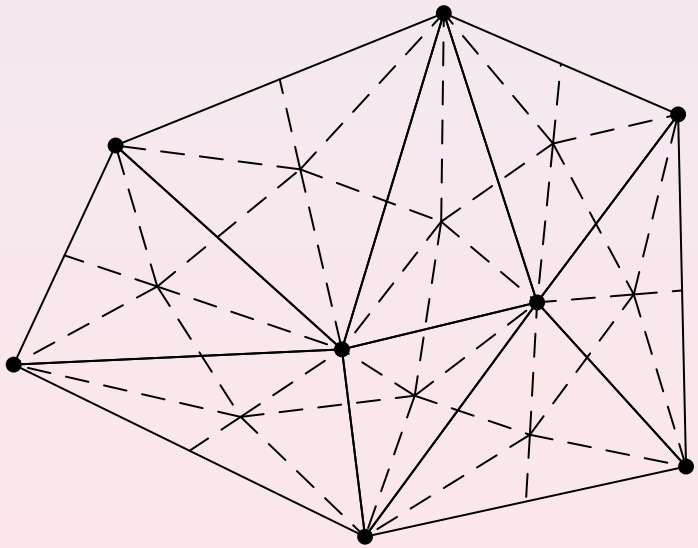
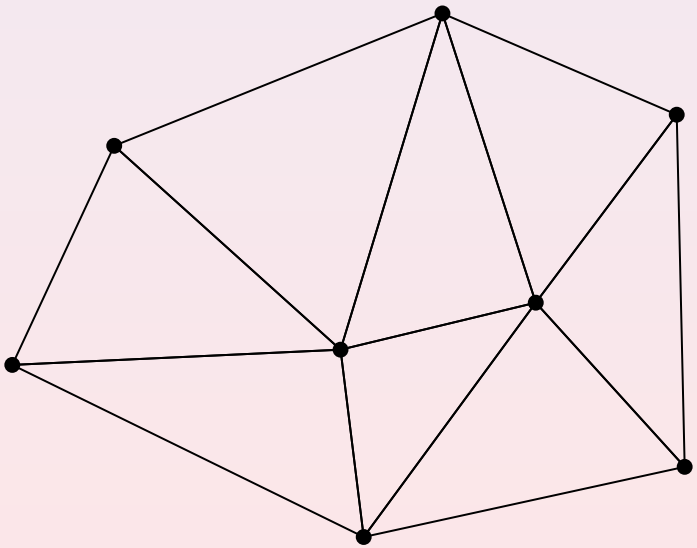


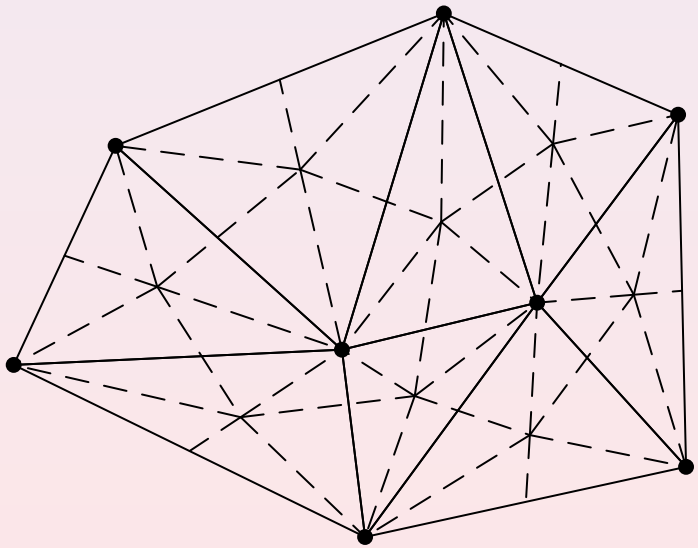


Powell-Sabin spline

$S_2^1(\Delta^*)$ = space of PS-splines

- Quadratic polynomial on triangles $t^* \in \Delta^*$
- C^1 continuity (function values and first derivative)





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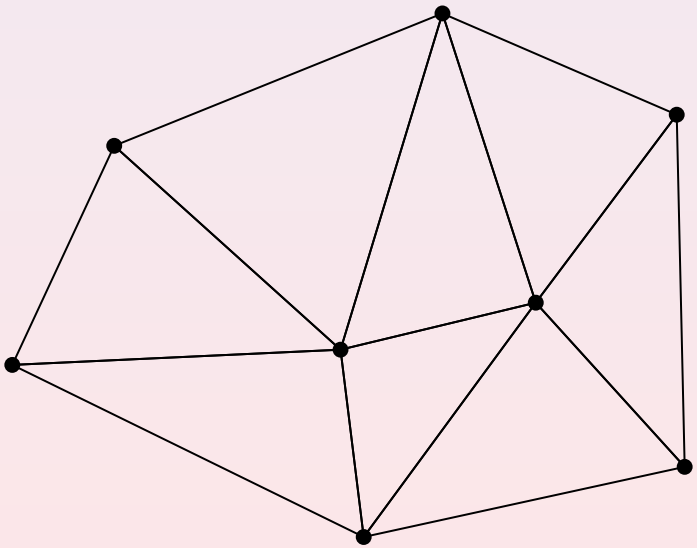

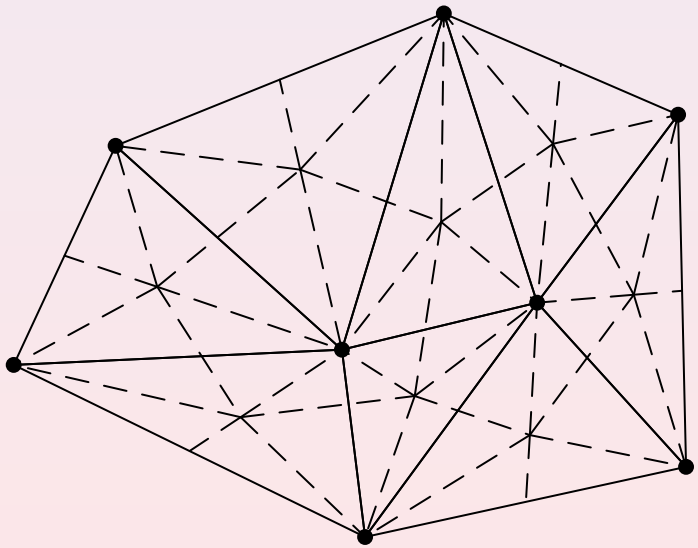

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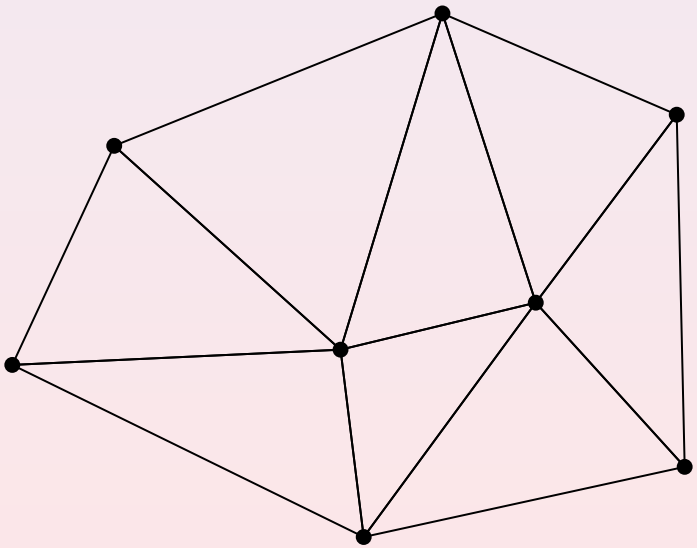

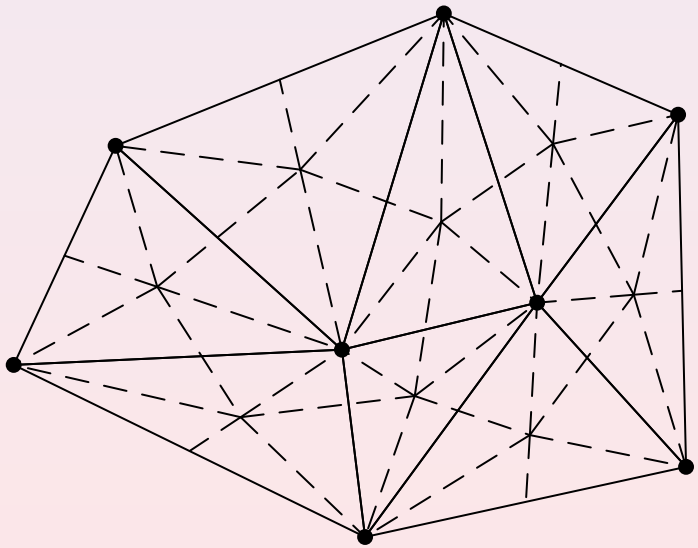

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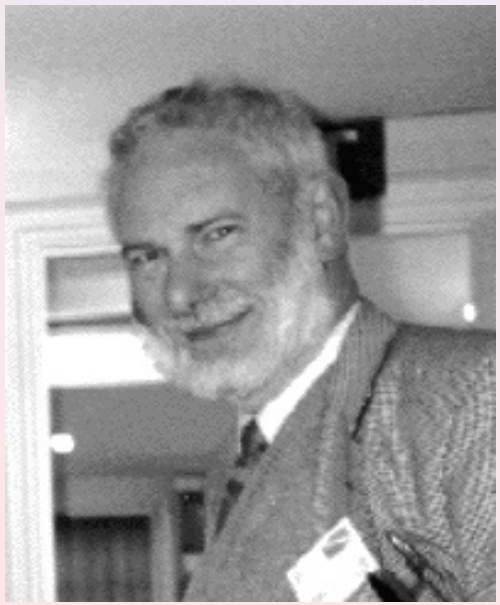

M.J.D. Powell & M.A. Sabin

Powell-Sabin spline Hermite interpolation

Dimension of $S_2^1(\Delta^*)$ is $3N$, $N = \#V$.
There is exactly one solution to the Hermite interpolation problem

- $s \in S_2^1(\Delta^*)$
- $s(V_i) = f_i$ *(same functionvalue)*
- $\nabla s(V_i) = \nabla f_i$ ($\nabla = [D_x, D_y]$) *(same normal)*

Thus $s(x, y)$ completely defined by the data

$$(s(V_i), s'_x(V_i), s'_y(V_i)) = (\alpha_i, \beta_i, \gamma_i), \quad i = 1, \ldots, N$$

Powell-Sabin spline, B-spline basis

$\forall V_i$: 3 basis functions B_{ij} for 3 independent $(\alpha_{ij}, \beta_{ij}, \gamma_{ij})$ values.

B_{ij}, $i = 1, \ldots, N$, $j = 1, 2, 3$

- $[B_{ij}(V_k), D_x B_{ij}(V_k), D_y B_{ij}(V_k)] = [0, 0, 0]$ for all $k \neq i$
- $[B_{ij}(V_i), D_x B_{ij}(V_i), D_y B_{ij}(V_i)] = [\alpha_{ij}, \beta_{ij}, \gamma_{ij}]$ for $j = 1, 2, 3$

Partition of unity:

- $\sum_{i=1}^{N} \sum_{j=1}^{3} B_{ij}(x, y) = 1,$
- $B_{ij}(x, y) \geq 0$

$$s(x, y) = \sum_{i=1}^{N} \sum_{j=1}^{3} c_{ij} B_{ij}(x, y)$$

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P. Dierckx 1997

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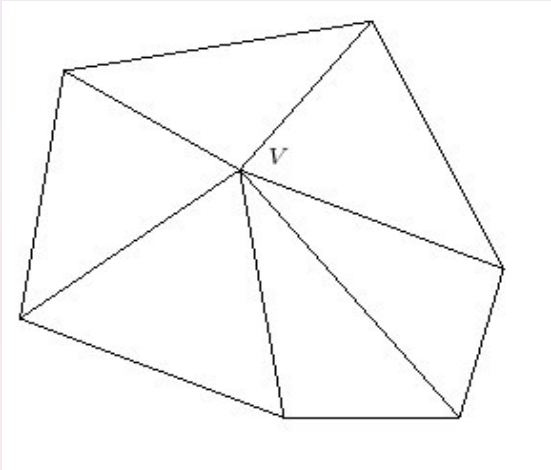
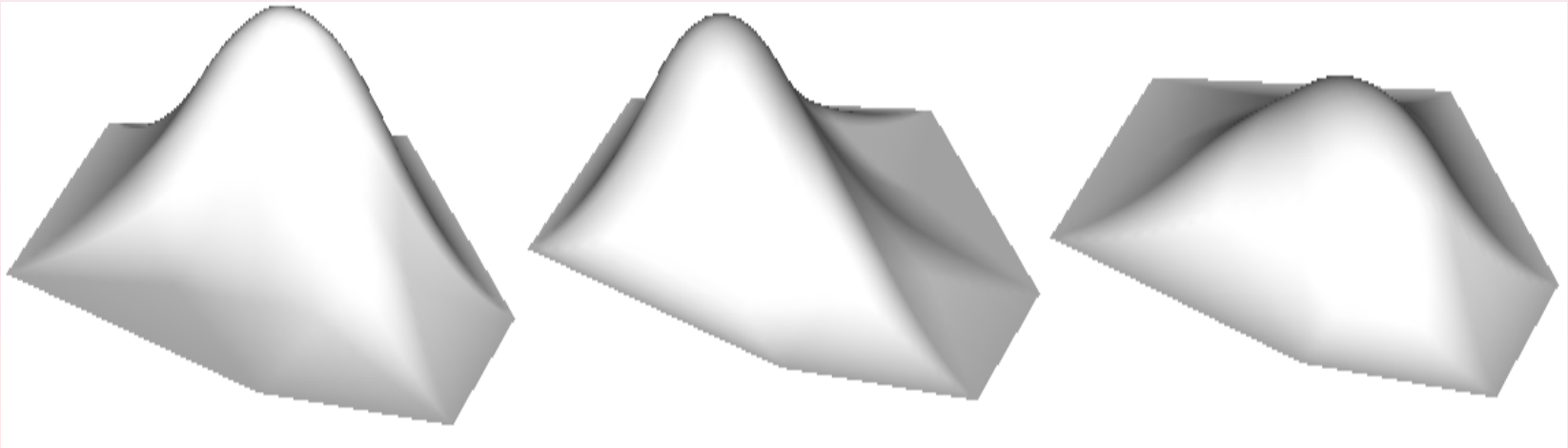
Powell-Sabin spline, B-spline basis

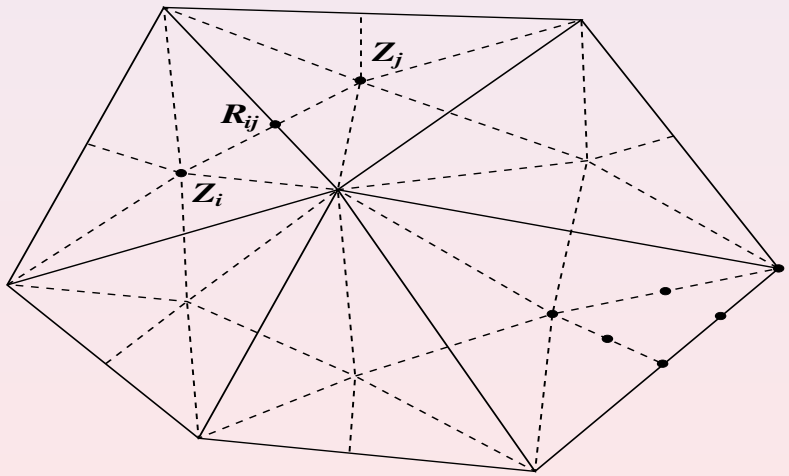
three B-splines





Powell-Sabin spline, B-spline basis

B-splines are local basis functions.
They differ from zero on a molecule (= 1-ring or star) M_i
I.e. all triangles of Δ having V_i as a vertex.



Powell-Sabin spline, control triangle

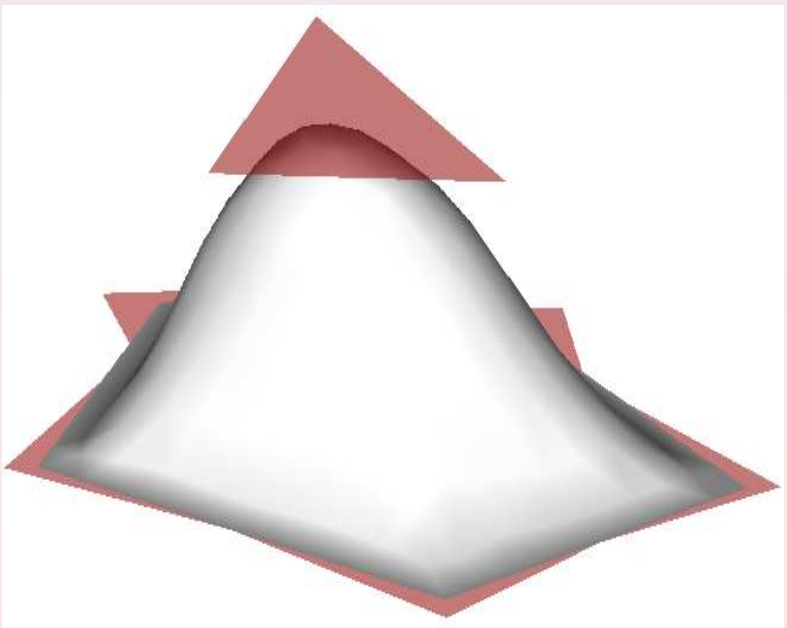
With B_{ij}, $i = 1, \ldots, N$, $j = 1, 2, 3$ one may represent any linear polynomial exactly

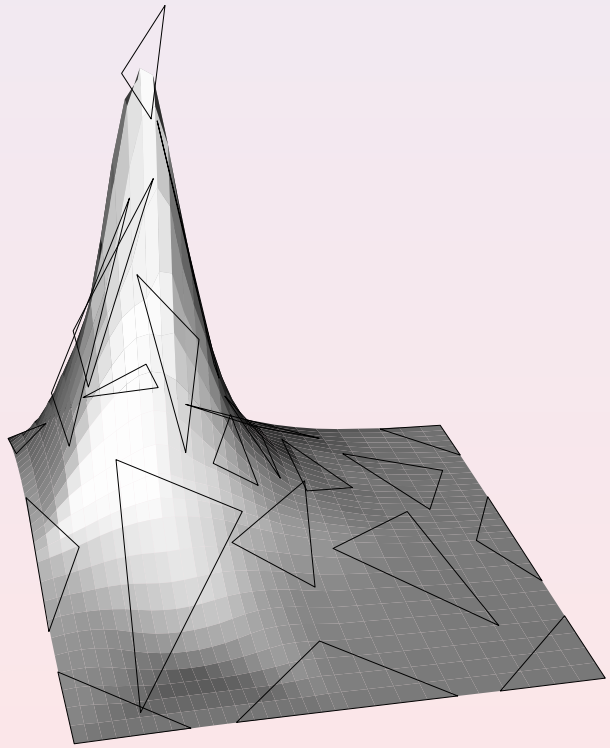
$$\begin{aligned} x &= \sum_{i=1}^{N} \sum_{j=1}^{3} X_{ij} B_{ij}(x, y) \\ y &= \sum_{i=1}^{N} \sum_{j=1}^{3} Y_{ij} B_{ij}(x, y) \\ s(x, y) &= \sum_{i=1}^{N} \sum_{j=1}^{3} c_{ij} B_{ij}(x, y) \end{aligned}$$

The (X_{ij}, Y_{ij}, c_{ij}), $j = 1, 2, 3$ are the vertices of a control triangle that is tangent to the surface in V_i for $i = 1, \ldots, N$.
It 'controls' the local shape of the spline surface.

Powell-Sabin spline, control triangle

PS surfaces and control triangles





Powell-Sabin spline, control triangle

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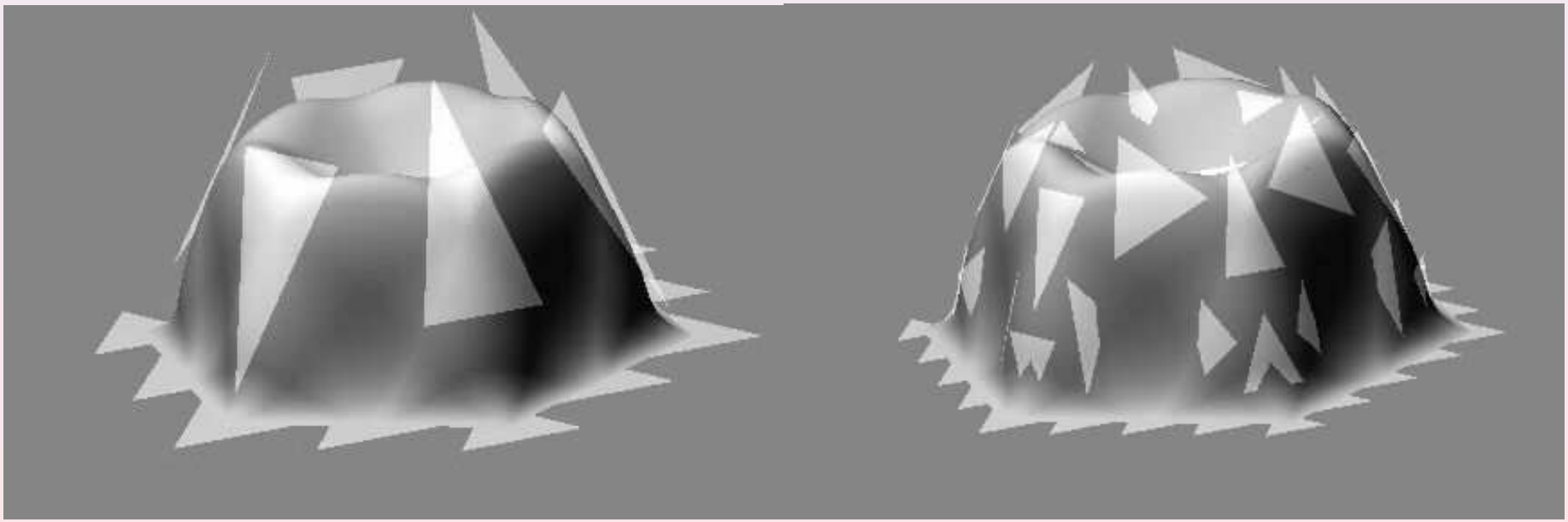


Powell-Sabin spline, evaluation

- Easy transformation of control triangles to Bézier control net
- Evaluation of Bézier polynomials and derivatives is fast de Casteljau algorithm





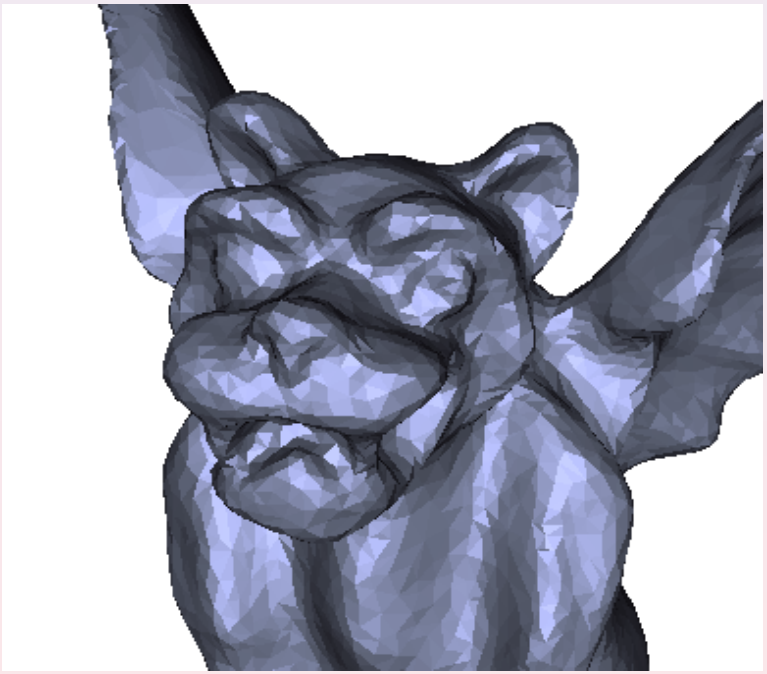
Pierre Étienne Bézier (1910-1999)

Paul de Faget de Casteljau

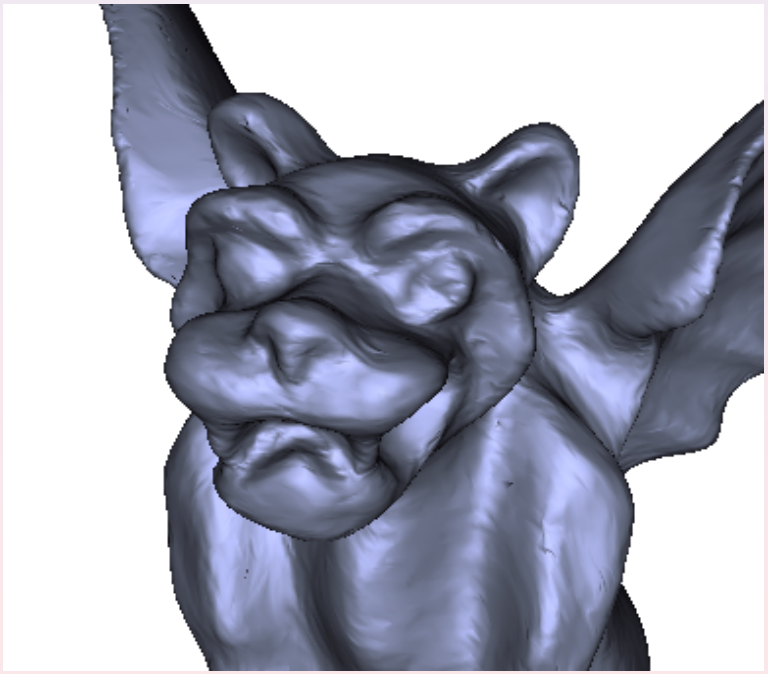
(1910-1999)

Application

Smoothing



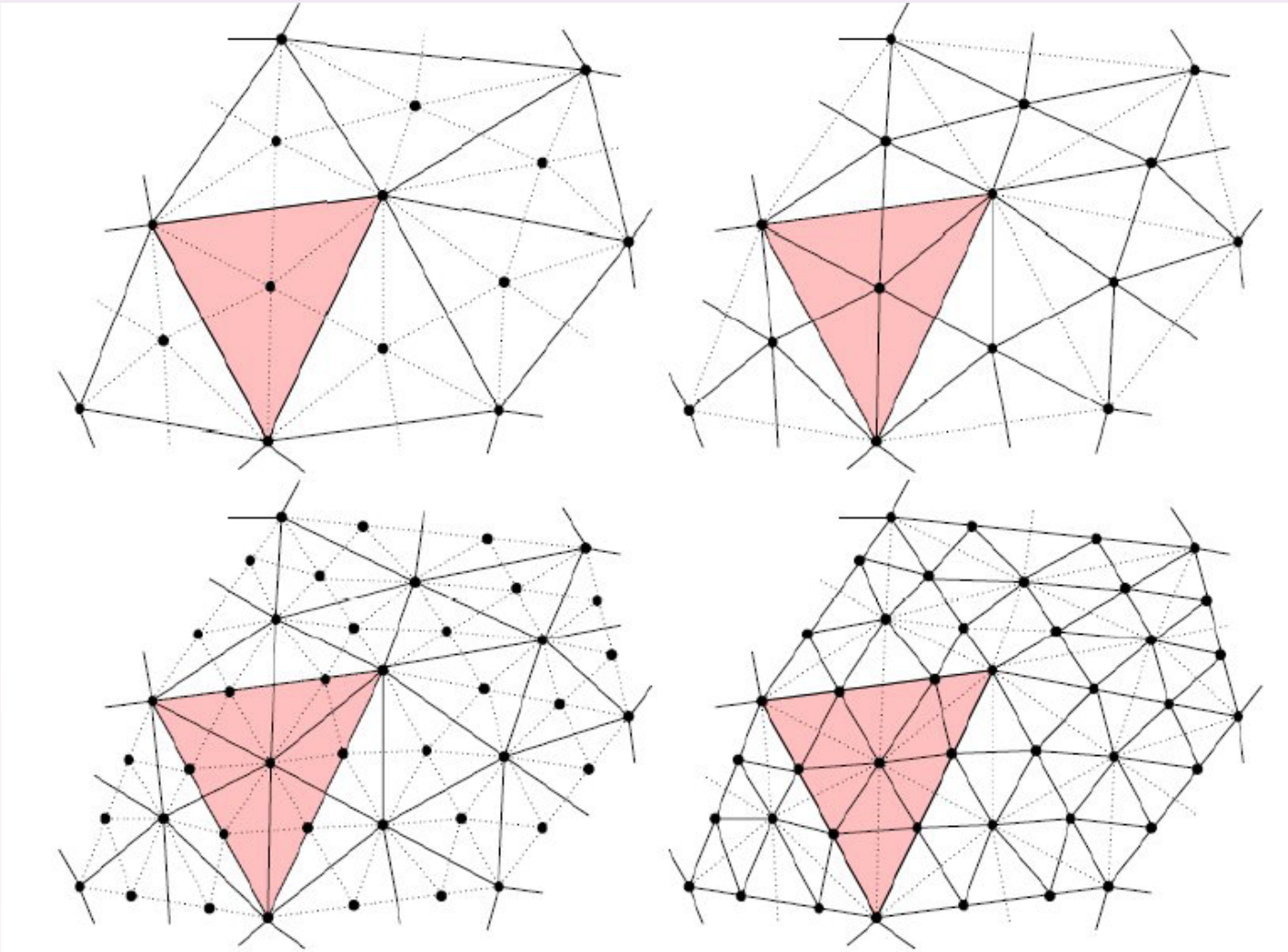
triangular mesh



PS splines

Subdivision

The $\sqrt{3}$ scheme results in a hierarchical basis



Hierarchical basis

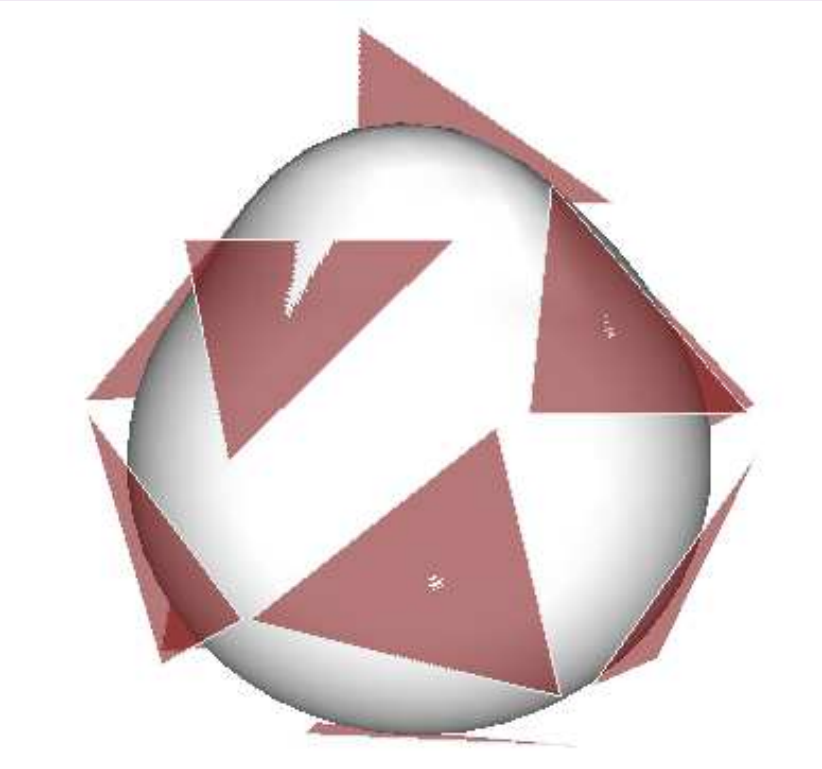
- triangulations $\Delta_0 \subset \Delta_1 \subset \Delta_2 \subset \cdots \subset \Delta_l \subset \cdots$
- PS 6-splits $\Delta_0^* \subset \Delta_1^* \subset \Delta_2^* \subset \cdots \subset \Delta_l^* \subset \cdots$
- PS splines $S_\ell = S_2^1(\Delta_\ell^*), \qquad S_\ell \subset S_{\ell+1}$
- B-spline bases: $S_\ell = \text{span}\{\phi_{\ell,j} : j = 1, \ldots, 3N_\ell\}$

$$s_\ell(x,y) = \phi_\ell c_\ell = \sum_{i=1}^{N_\ell} \sum_{j=1}^{3} B_{ij\ell}(x,y) c_{ij\ell}$$

$$\phi_{\ell+1} = [\phi_{\ell+1}^o \quad \phi_{\ell+1}^n],$$

$\phi_{\ell+1}^o$ correspond to old reused vertices of level ℓ
$\phi_{\ell+1}^n$ correspond to the new vertices of level $\ell+1$

The hierarchical basis



click image to start animation

- $\{V_i \in \Delta_\ell\} \subset \{V_i \in \Delta_{\ell+1}\}$
- $\Delta_\ell^* \subset \Delta_{\ell+1}^*$
- $S_\ell := S_2^1(\Delta_\ell^*), \quad S_\ell \subset S_{\ell+1}$

$$S_2 = S_0 \oplus W_0 \oplus W_1$$

- Largest triangles control S_0
- Medium triangles control W_0
- Smallest triangles control W_1

Hierarchical basis

- $S_\ell \subset S_{\ell+1}$ hence $\phi_\ell = \phi_{\ell+1} A_\ell$

$$[\phi_\ell \ \phi^n_{\ell+1}] = [\phi^o_{\ell+1} \ \ \phi^n_{\ell+1}] \begin{bmatrix} O_\ell & 0 \\ N_\ell & I \end{bmatrix}.$$

- ϕ_ℓ are like scaling functions (coarse info)
- $\phi^n_{\ell+1}$ are like wavelet functions (detail info)
- $S_{\ell+1} = S_\ell \oplus W_\ell$

Hierarchical basis

Problems

- What kind of functions can be represented?
  (In which function spaces will this converge?)
- How to represent edges?
- How about stability?
  (small $\delta c_\ell \Rightarrow$ small $\delta s_\ell = \phi_\ell \cdot \delta c_\ell$?)
  Important e.g. for thresholding, compression!
- No (bi)orthogonality
- $\phi^n_{\ell+1}$ not really wavelets $\Leftarrow$ no vanishing moment
- We are missing the UPDATE STEP of the lifting scheme

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Wavelet basis

Lifting scheme to construct wavelets.



W. Sweldens 1994

$$[\phi_\ell \ \phi^n_{\ell+1}] = [\phi^o_{\ell+1} \ \ \phi^n_{\ell+1}] \begin{bmatrix} O_\ell & 0 \\ N_\ell & I \end{bmatrix}$$

$$[\phi_\ell \ \ \psi_\ell] = \phi_{\ell+1}[\, P_\ell \ \ Q_\ell \,],$$

$$P_\ell = \begin{bmatrix} O_\ell \\ N_\ell \end{bmatrix}, \quad Q_\ell = \begin{bmatrix} 0 \\ I \end{bmatrix} - \begin{bmatrix} O_\ell \\ N_\ell \end{bmatrix} C_\ell$$

P_ℓ projects onto V_ℓ (coarse info)

Q_ℓ projects onto W_ℓ (detail info)

Wavelet basis

Lifting scheme to construct wavelets.



W. Sweldens 1994

Do an extra UPDATE

$$[\phi_\ell \ \psi_\ell] = [\phi^o_{\ell+1} \quad \phi^n_{\ell+1}] \begin{bmatrix} O_\ell & 0 \\ N_\ell & I \end{bmatrix} \begin{bmatrix} I & -C_\ell \\ 0 & I \end{bmatrix}$$

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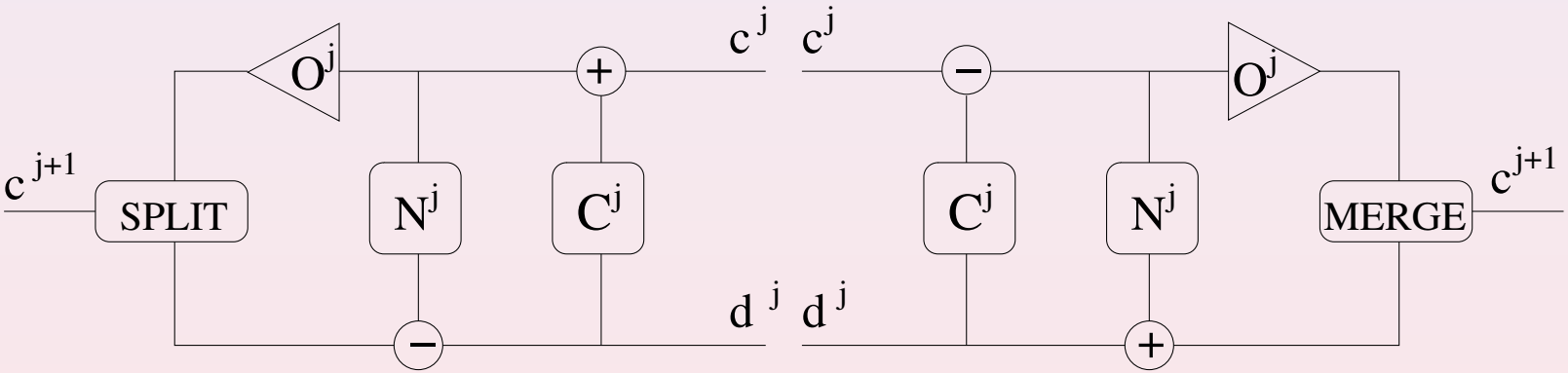
Q_ℓ projects onto W_ℓ (detail info)

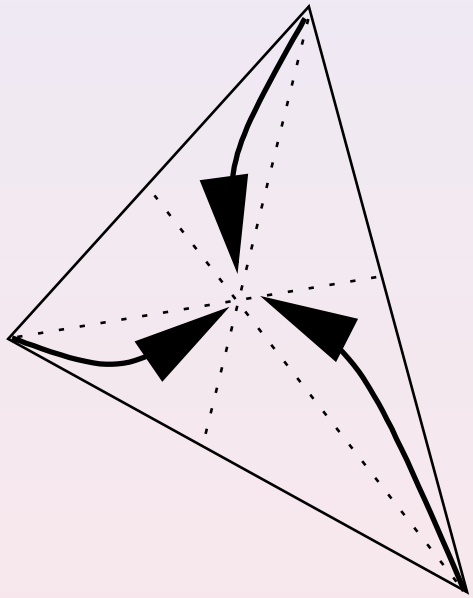
Lifting

Lifting with update C_j.



Wavelet basis update problems

$$\psi_\ell = \phi^n_{\ell+1} - \phi_\ell C_\ell$$



- (semi-)Orthogonality $\Rightarrow$ C_ℓ not sparse
- Fix C_ℓ sparse $\Rightarrow$ ψ_ℓ local support
- i.e. should be able to reproduce constants
- Remaining orthogonality conditions approximated by least squares

Wavelet basis update problems

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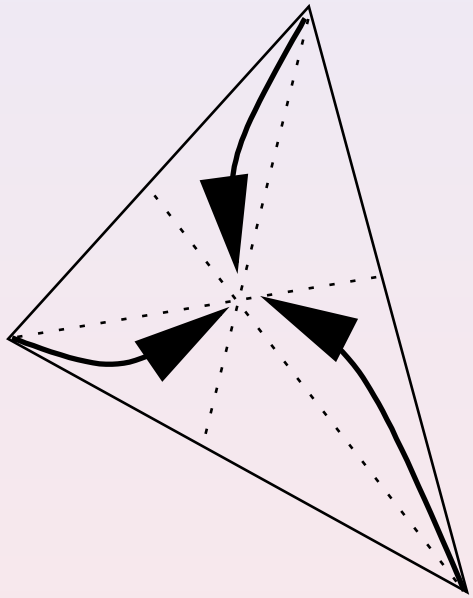

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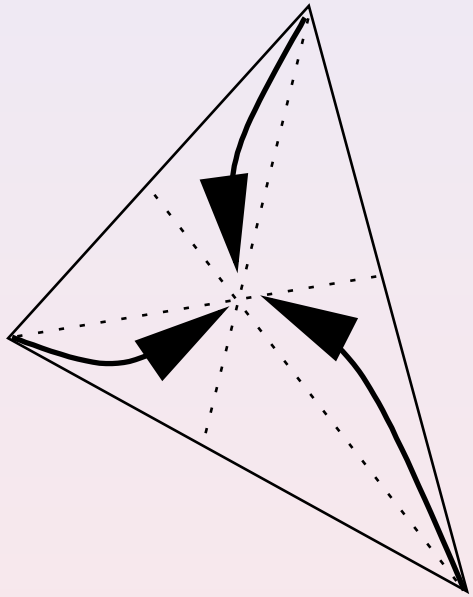

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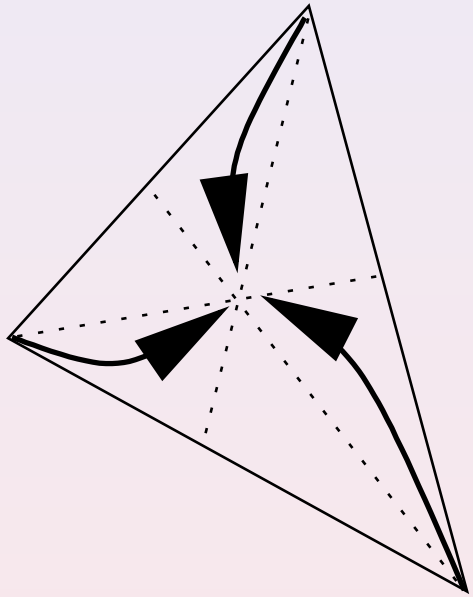

- Want local support $\Rightarrow$ C_ℓ sparse
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Wavelet basis update problems

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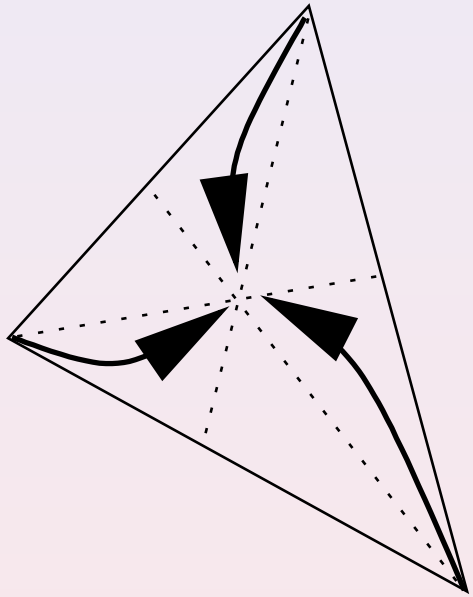

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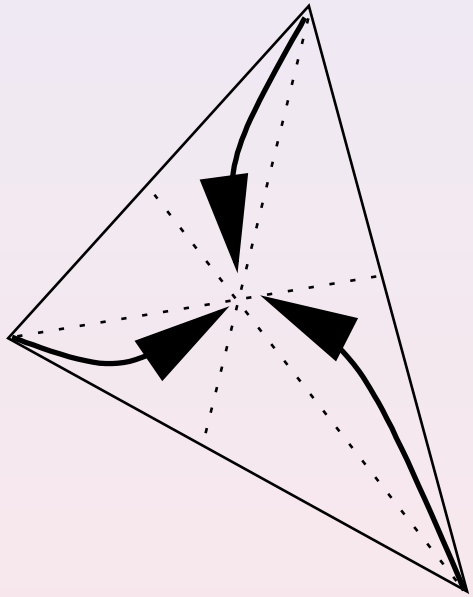
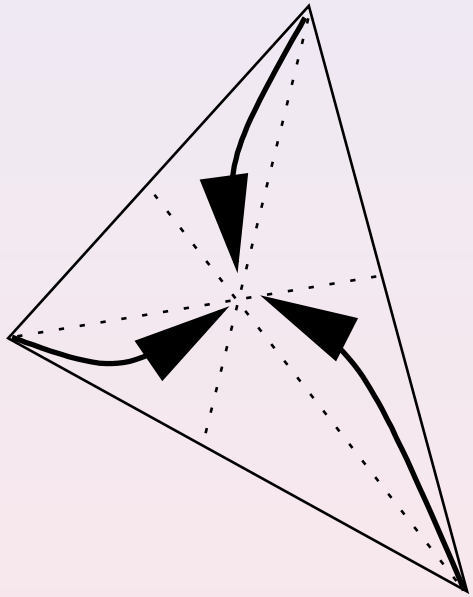


- Want local support $\Rightarrow$ C_ℓ sparse
- Fix C_ℓ sparse $\Rightarrow$ ψ_ℓ local support, but basis unstable
- i.e. should be able to reproduce constants
- Remaining orthogonality conditions approximated by least squares

Wavelet basis update problems

$$\psi_\ell = \phi^n_{\ell+1} - \phi_\ell C_\ell$$



- Want local support $\Rightarrow$ C_ℓ sparse
- Want stability $\Rightarrow$ need 1 vanishing moment for ψ_ℓ
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- Remaining orthogonality conditions approximated by least squares

Wavelet basis update problems

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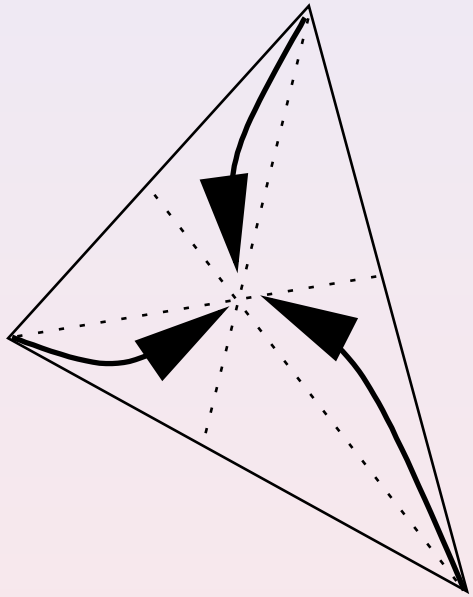

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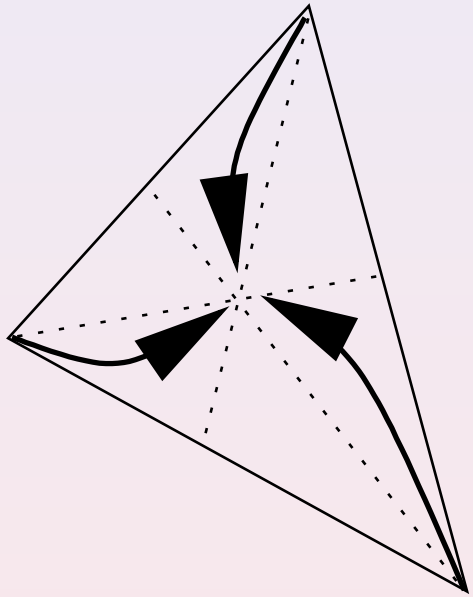

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Wavelet basis update problems

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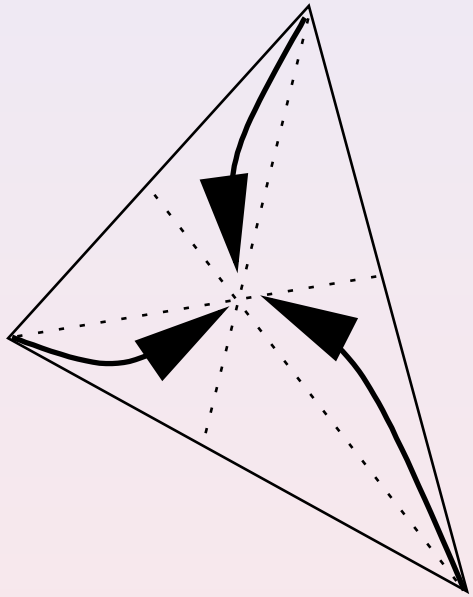

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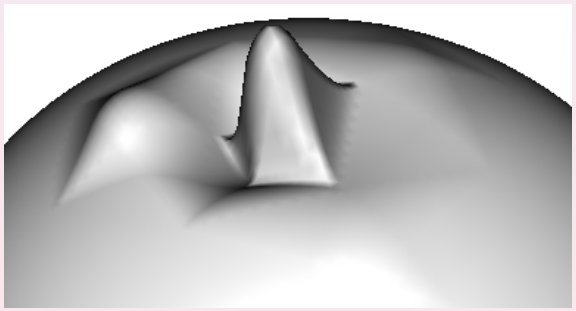
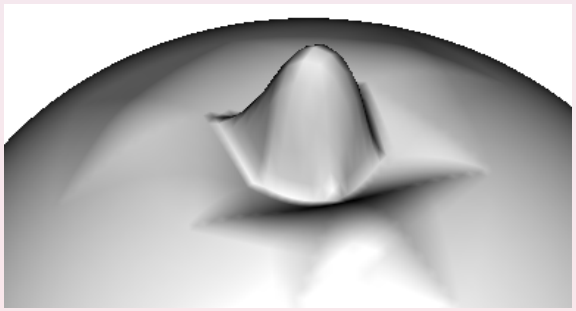
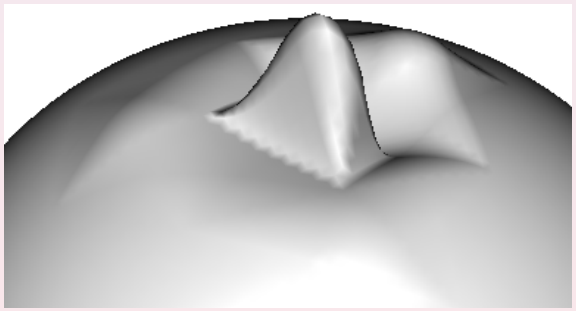
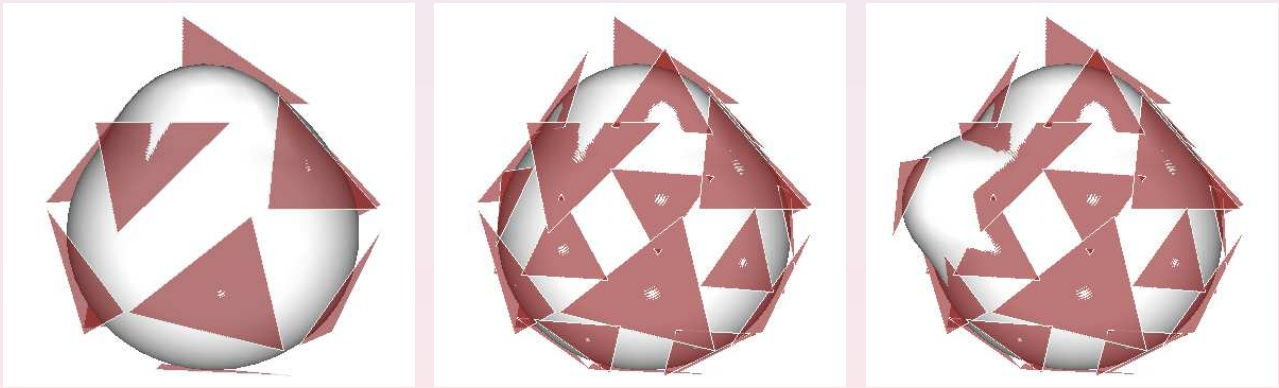
Spherical wavelets

A similar procedure can be followed to design spherical wavelets
= wavelets on a sphere
Project onto tangent plane 3 wavelets associated with one vertex

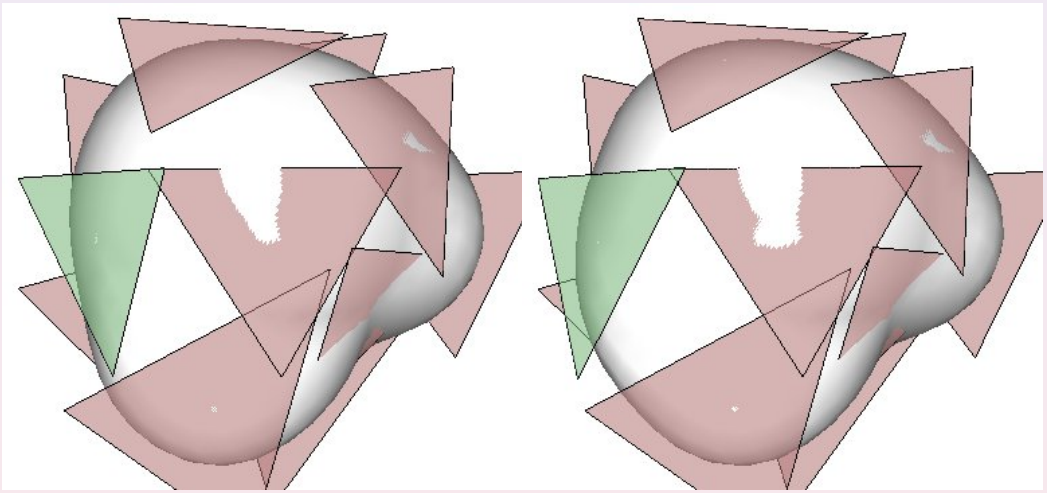
Spherical wavelets

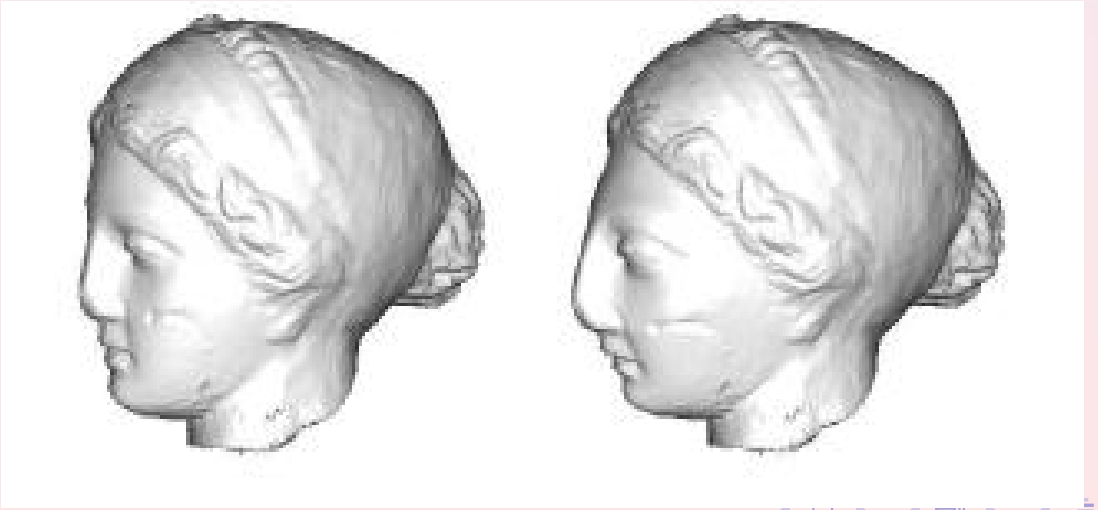
Spherical PS spline wavelets give local control



Spherical wavelets

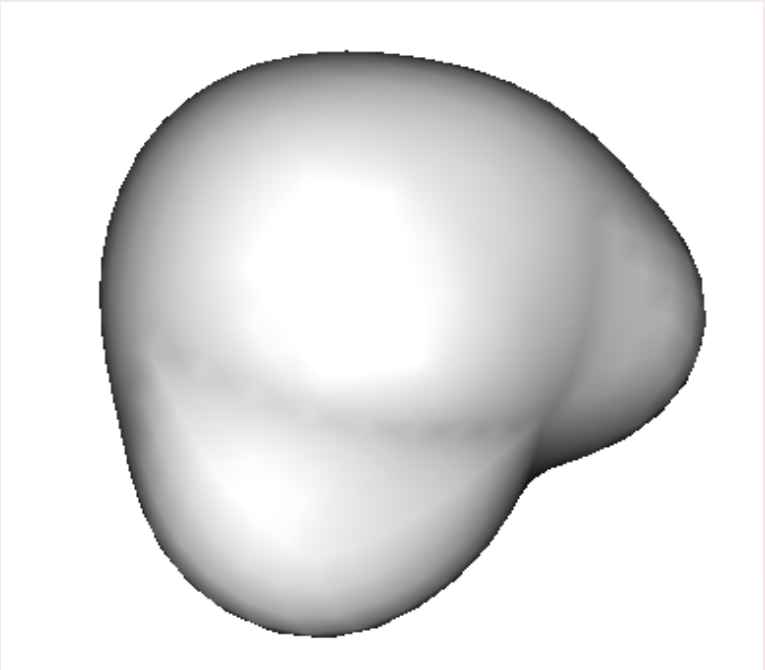
Effect of modifyng a local control triangle





Application: multiresolution structure

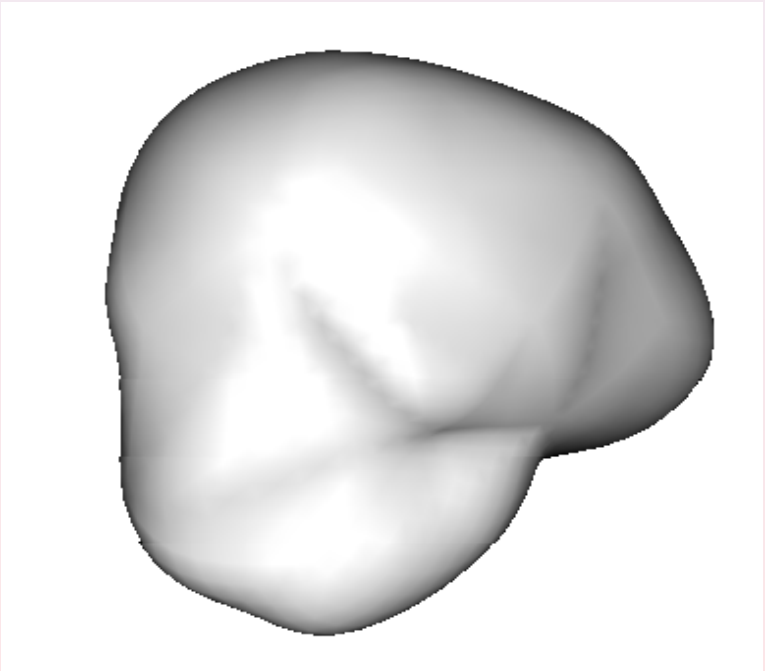
Scattered data $\Rightarrow$ PS-spline multiresolution structure
Different levels with $\sqrt{3}$ subdivision
starting from an icosaeder



level 0

Application: multiresolution structure

Scattered data $\Rightarrow$ PS-spline multiresolution structure
Different levels with $\sqrt{3}$ subdivision
starting from an icosaeder



level 1

Application: multiresolution structure

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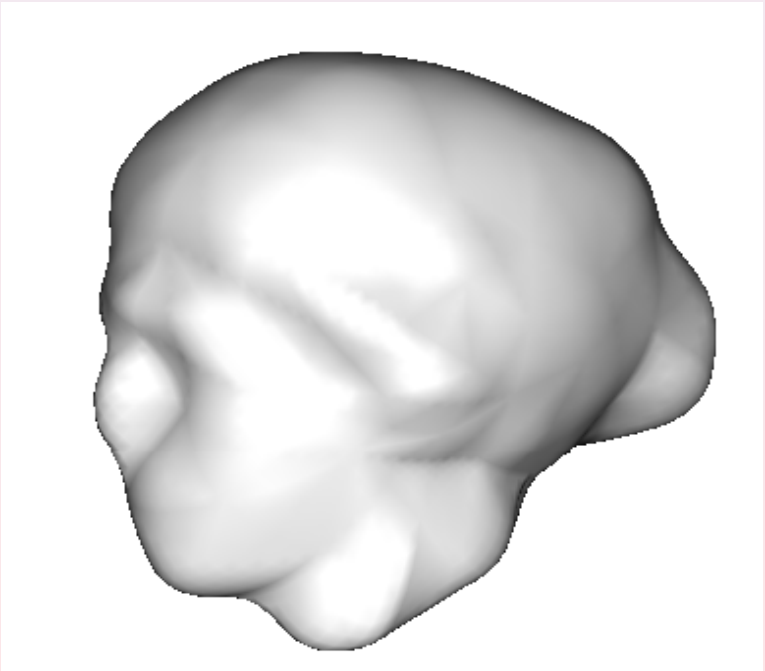
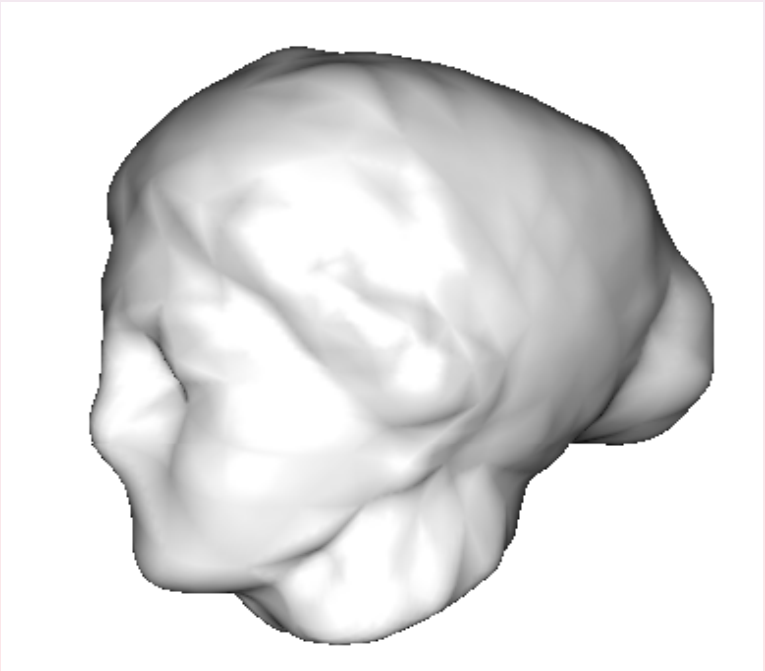


level 2

Application: multiresolution structure

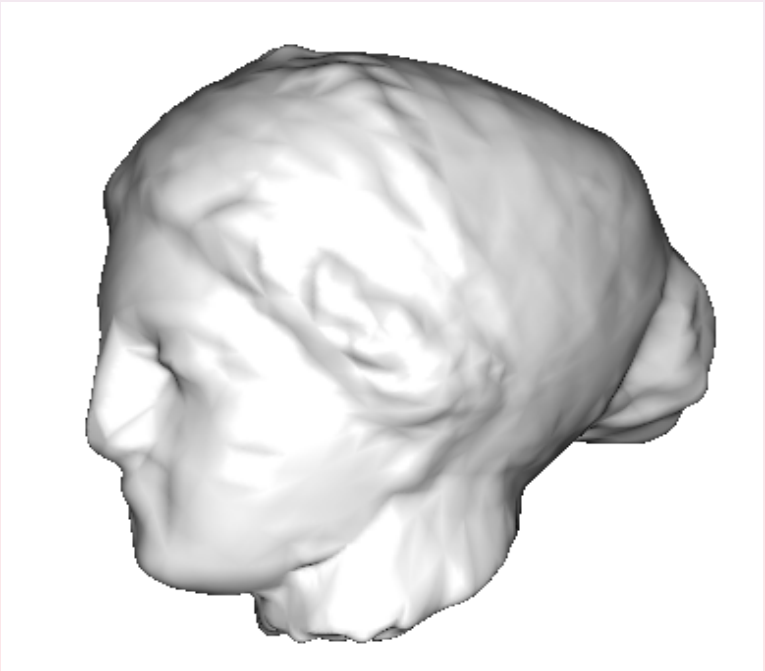
Scattered data $\Rightarrow$ PS-spline multiresolution structure
Different levels with $\sqrt{3}$ subdivision
starting from an icosaeder



level 3

Application: multiresolution structure

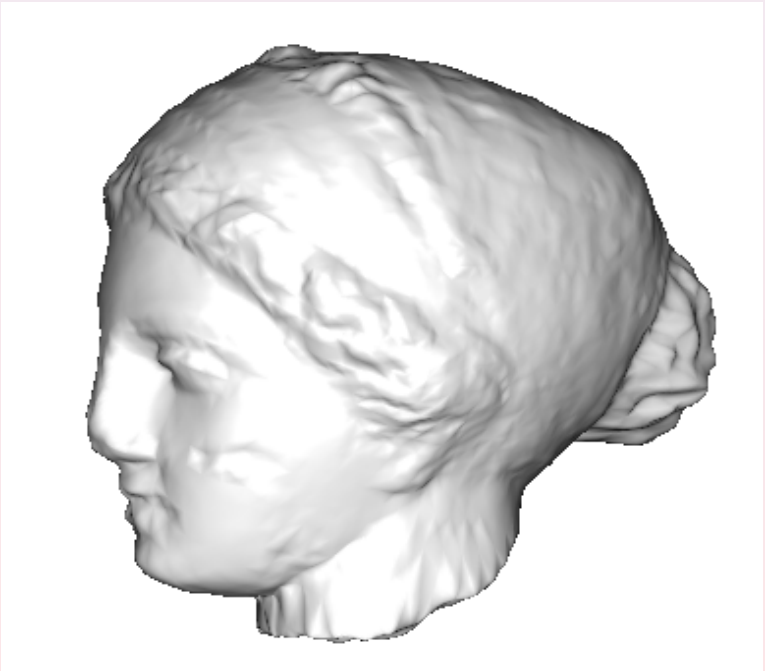
Scattered data $\Rightarrow$ PS-spline multiresolution structure
Different levels with $\sqrt{3}$ subdivision
starting from an icosaeder



level 4

Application: multiresolution structure

Scattered data $\Rightarrow$ PS-spline multiresolution structure
Different levels with $\sqrt{3}$ subdivision
starting from an icosaeder



level 5

Application: multiresolution structure

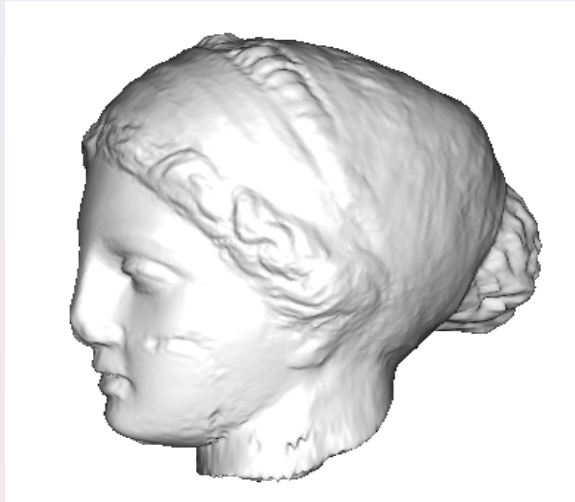
Scattered data $\Rightarrow$ PS-spline multiresolution structure
Different levels with $\sqrt{3}$ subdivision
starting from an icosaeder

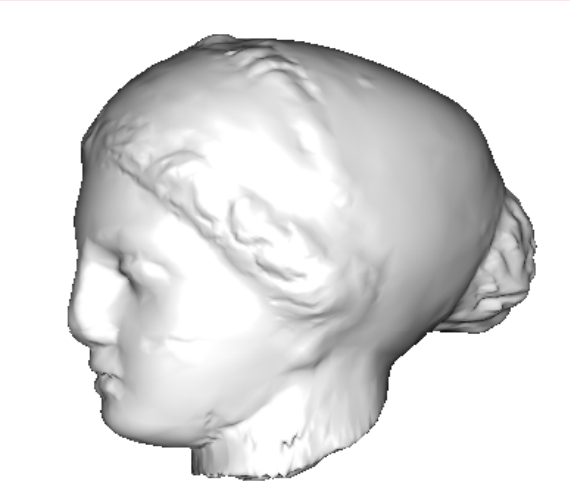


level 6

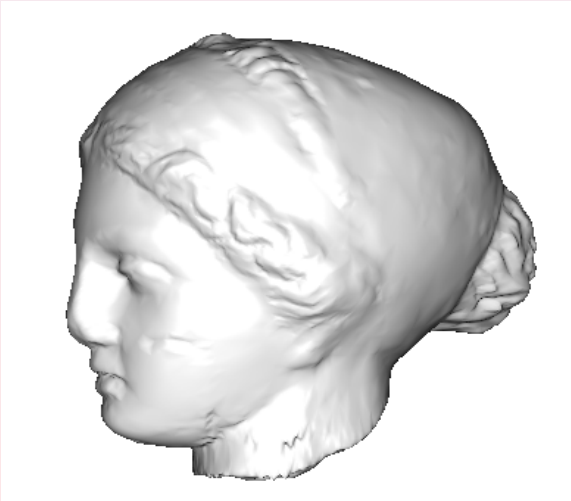
Application

Data reduction





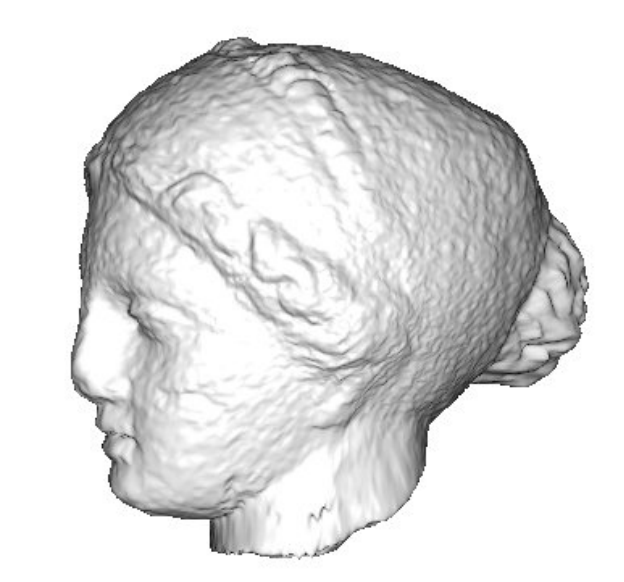
keep 14%



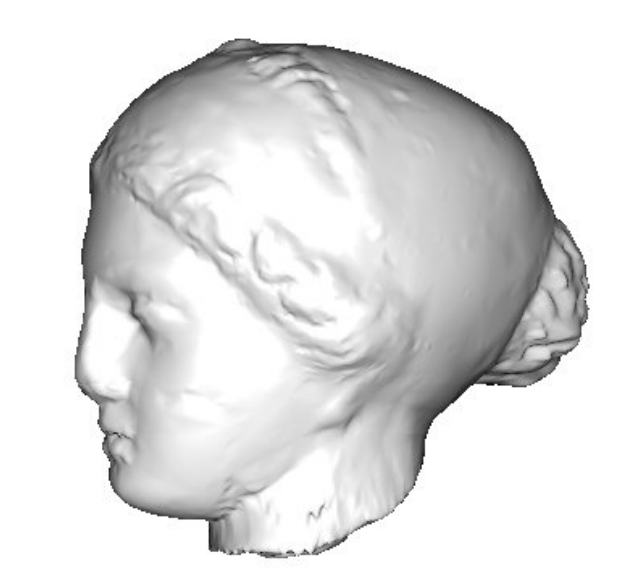
keep 26%

Application

Denoising



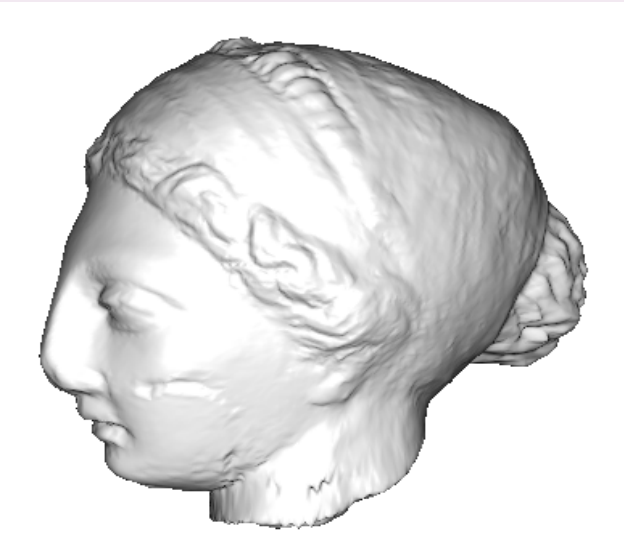
With noise



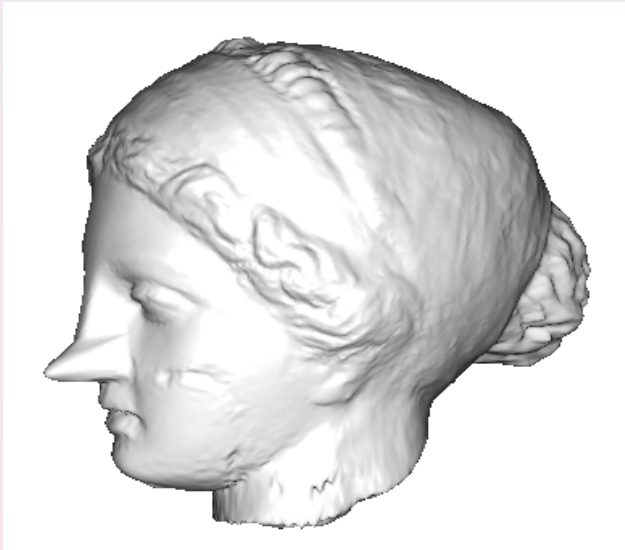
Denoised

Application

Multiresolution editing



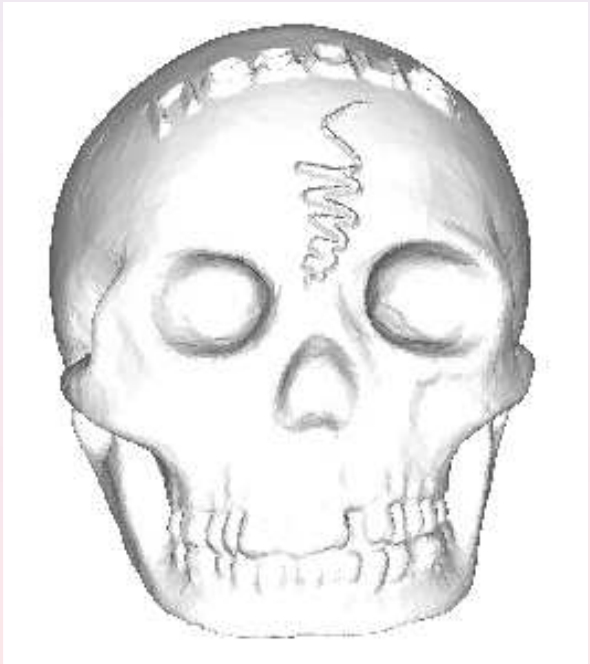
modify large
control triangle



modify small
control triangle

Application

Smoothing with less data



triangular mesh

(40000 triangles)

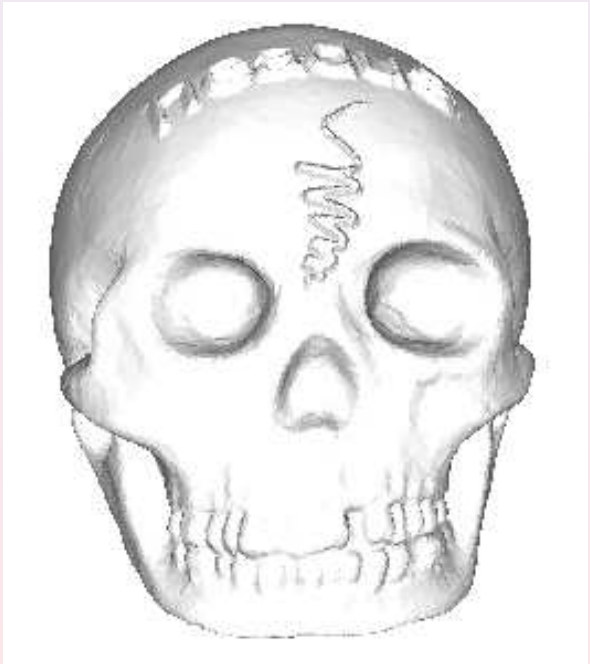


reduced mesh

(5000 triangles)

Application

Smoothing with less data



triangular mesh

(40000 triangles)

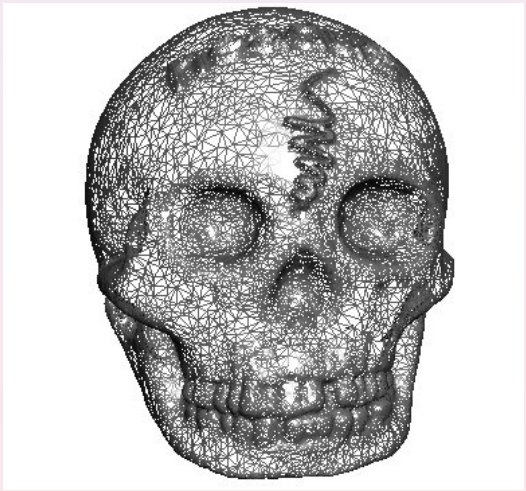


Powell–Sabin spline

(5000 control triangles)

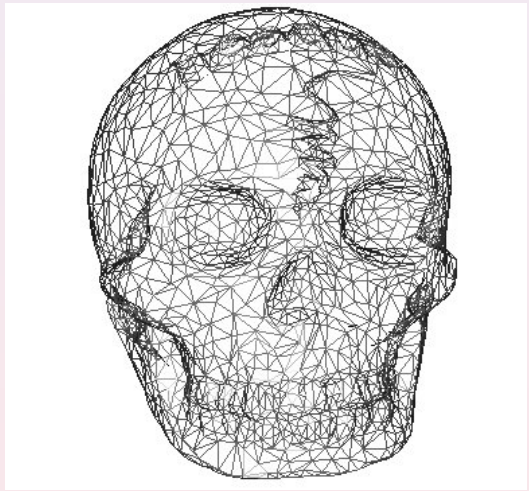
Application

Smoothing with less data



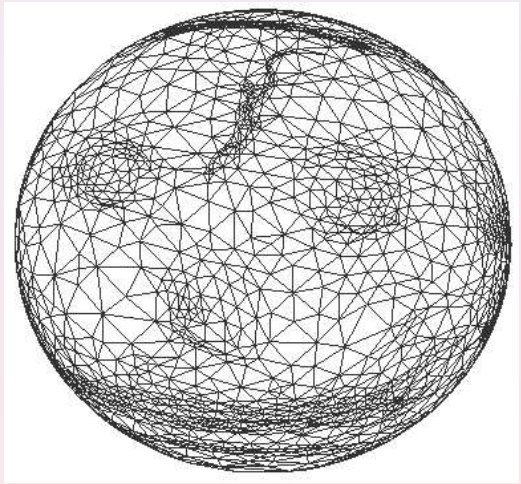
triangular mesh

(40000 triangles)



decimated mesh

(5000 control triangles)



spherical

parameterization

References

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