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# QUADRATURE FORMULAS ON THE UNIT CIRCLE BASED ON RATIONAL FUNCTIONS

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**Abstract**

Quadrature formulas on the unit circle were introduced by Jones et als. in 1989. On the other hand, Bultheel et als. also considered such quadratures by giving results concerning with error and convergence. In other recent papers, a more general situation was studied by the above four authors involving orthogonal rational functions on the unit circle which generalize the well known Szegö polynomials. In this paper, these quadratures are again analyzed and results about convergence given. Furthermore, an application to the Poisson integral is also made.

## 1. Preliminaries

In this paper we shall use the notation $\mathbf{T} = \{z : |z| = 1\}$, $\mathbf{D} = \{z : |z| < 1\}$, and $\mathbf{E} = \{z : |z| > 1\}$ for the unit circle, the open unit disc and the exterior of the unit circle. Let $\mu$ be be a finite Borel measure on $[-\pi,\pi]$. In order to estimate the integral,

$$I_\mu\{f\} = \int_{-\pi}^{\pi} f(e^{i\theta})d\mu(\theta) = \int_{\mathbf{T}} f(z)d\mu(z) = \int f(z)d\mu(z)$$

(we have taken the freedom to write the previous integral in different forms but having in mind that integration will always be over the unit circle in one form or another), the so-called Szegö quadrature formulas were introduced in [1]. (See, also [2] for a different approach). Such quadratures are of the form,

$$I_n\{f\} = \sum_{j=1}^{n} A_j^{(n)} f(x_j^{(n)}),\ x_i^{(n)} \neq x_j^{(n)},\ x_j^{(n)} \in \mathbf{T} \text{ and } A_j^{(n)} > 0,\ j = 1,\dots,n \tag{1.1}$$

so that, $I_\mu\{f\} = I_n\{f\}$ for all $f \in \Lambda_{-(n-1),(n-1)}$.(For every pair (p,q) of integers, $p \le q$, $\Lambda_{p,q}$ will denote the linear space of all Laurent polynomials of the form,

$$\sum_{j=1}^{n} c_j z^j,\ c_j \in \mathbb{C}$$

and $\Lambda$ the space of all Laurent polynomials ([3], [4]). It is known that such quadrature formulas (1.1) are of great interest to solve the trigonometric moment problem or equivalently the Schur coefficient problem (see, [1]). On the other hand, Waadeland [5] recently studied such quadratures for the Poisson integral, that is, when the measure $\mu$ is given by,

$$d\mu(\theta) = \frac{1}{2\pi}\,\frac{1 - r^2}{1-2r\cos\theta + r^2},\ r \in (0,1)$$

or more generally,

$$d\mu(\theta) = \frac{1-|r|^2}{|z-r|^2}\,\frac{d\theta}{2\pi},\ r \in \mathbf{D},\ z = e^{i\theta} \tag{1.2}$$

Observe that by taking r=0, we have the normalized Lebesgue measure $d\lambda(\theta)= d\theta/2\pi$. Szegö quadratures for such situation were also studied by Camacho and González-Vera in [6]. Finally, in [7] aspects concerning ~~with~~ error and convergence were analyzed.

In this paper, formulas (1.1) will be again considered, but instead of Laurent polynomilas, more general rational functions with prescribed poles not on **T** will be used, giving arise to the Rational Szegö formulas which were earlier introduced by ourselves in [8] and [9] and wherethe so-called Rational Szegö functions play a fundamental role.

For completeness, let $\{\alpha_i\}_1^\infty \subset \mathbf{D}$ be a given sequence and consider for n = 0,1,.. the nested spaces $\mathcal{L}_n$ of rational functions of degree n at most which are spanned by the basis of Partial Blaschke products $\{B_k\}_0^n$ where $B_0 = 1$, $B_n = \xi_n B_{n-1}$ for n = 1, 2,... and the Blaschke factors are defined as

$$\xi_n(z) = \frac{\bar{\alpha}_n}{|\alpha_n|}\,\frac{\alpha_n - z}{1 - \bar{\alpha}_n z}$$

By convention, we set $\bar{\alpha}_n/|\alpha_n| = -1$ for $\alpha_n = 0$. Sometimes, we shall also write

$$B_n(z) = \eta_n\ \omega_n(z)/\pi_n(z);\ \eta_n = (-1)^n \prod_{j=1}^{n} \bar{\alpha}_j/|\alpha_j|,\ \omega_n = \prod_1^n (z-\alpha_j) \text{ and } \pi_n = \prod_j^n (1-\bar{\alpha}_j z)$$

Note that if all the $\alpha_i$ are equal to zero, the spaces $\mathcal{L}_n$ collapse to the space

$\Pi_n$ of polynomials of degree n.

We also introduce the following transformation $f_*(z) = \overline{f(1/\bar{z})}$, which allows to define for $f_n \in \mathcal{L}_n$ the superstar conjugate as

$$f_n^*(z) = B_n(z) f_{n*}(z)$$

Let now the sequence $\{\phi_n : n = 0,1,...\}$ be obtained by orthonormalization of the sequence $\{B_n : n = 0,1,...\}$ with respect to the inner product induced by the measure $\mu$, namely

$$<f,g>_\mu = \int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}d\mu(\theta)$$

These functions are uniquely determined by the requirement that the leading coefficient $k_n$ in

$$\phi_n(z) = \sum_{j=1}^{n} k_j B_j(z)$$

is positive. We then have $k_n = \phi_n^*(\alpha_n)$.

Finally, in order to summarize the main result given in [9] for the Rational Szegö formulas, let us introduce the function spaces of the form

$\mathcal{R}_{p,q} = \mathcal{L}_{p*} + \mathcal{L}_q = \{ P/\omega_p\pi_q ; P \in \Pi_{p+q} \}$, p and q being nonnegative integers

Observe that $\mathcal{L}_{n*} = \text{span} \{1, B_{1*},...,B_{n*}\} = \text{span} \{1, 1/B_1,..., 1/B_n\}$. Therefore,

$$\mathcal{R}_{p,q} = \text{span} \{1/B_p, 1/B_{p-1},...,1/B_1, 1, B_1, ..., B_q\}.$$

($\mathcal{R}_{0,n} = \mathcal{L}_n$). When all the $\alpha_i$ are equal to zero, then $B_k(z) = z^k$ and one has $\mathcal{R}_{p,q} = \text{span} \{ z^k: k = -p, -p+1, ...,q\} = \Lambda_{-p,q}$. Furthermore, for $w \in \mathbf{T}$, set $\chi_n(z,w) = \phi_n(z) + w\phi_n^*(z)$, so that the following holds (see, [9]),

**Theorem 1**

i) $\chi_n(z,w)$ has n simple zeros which lie on the unit circle.

ii) Let $x_1, ..., x_n$ be the zeros of $f_n(z,w)$. Then, there exist positive numbers $A_1, ..., A_n$ such that the formula $I_n\{f\} = \sum_{j=1}^{n} A_j f(x_j)$ is exact, that is, $I_n\{f\} = I_\mu\{f\}$ for all $f \in \mathcal{R}_{n-1,n-1}$.

In this case, $\mathcal{R}_{n-1,n-1}$ is said to be a maximal domain of validity. Moreover, it was also proved in [9] that the only quadrature formulas with such a maximal domain of validity are just those ones given in Theorem 1, where the weights $A_j$'s are given by

$$A_j = \int L_j(z)d\mu(z) \tag{1.3}$$

$L_j(z) \in \mathcal{L}_{n-1} = \mathcal{R}_{0,n-1}$ defined by the interpolation conditions $L_j(x_i) = \delta_{ij}$. (Actually, a more general interpolating function space $\mathcal{R}_{p,q}$, p and q being nonnegative integers such that $p+q = n-1$, can be considered, so that the resulting quadrature formula does not depend on p and q. See [9]).

## 2. An alternative approach

In this section we shall give an alternative approach to get the above quadrature formulas using Hermite interpolation in the space $\mathcal{R}_{n-1,n-1}$ (compare with the approach given by Markov for the classical Gauss formulas [10] and with the one given in [7] for the Szegö formulas on the unit circle in the polynomial case). Writing $A = \{\alpha_i\}_1^\infty$ and $\hat{A} = \{1 / \bar{\alpha}_i : \alpha_i \in A\}$, it is easily seen that $\mathcal{R}_{p,q}$ represents a Chebyshev system on any set $X \subset \mathbb{C} - (A \cup \hat{A})$, so that given the distinct nodes $\{x_j : j=1,\dots,n\} \subset \mathbb{C} - (A \cup \hat{A})$, there exists a unique function $Q \in \mathcal{R}_{n-1,n-1}$ holding, such that

$$Q(x_i) = f(x_i),\ i = 1,2,\dots,n;\ Q'(x_i) = f'(x_i),\ i = 1,2,\dots,n-1$$

These are 2n-1 constraints, which corresponds to the dimension of $\mathcal{R}_{n-1,n-1}$. In order to determine such Hermite rational interpolant, we can write

$$Q(z) = \sum_{j=1}^{n} H_{j,o}(z)f(x_j) + \sum_{j=1}^{n-1} H_{j,1}(z)f'(x_j)$$

where $H_{j,0}$ and $H_{j,1}$ belong to $\mathcal{R}_{n-1,n-1}$ and satisfy the interpolation conditions

$$H_{i,0}(x_j) = \delta_{ij},\ 1 \le i,j \le n;\quad H'_{i,o}(x_j) = 0,\ 1 \le i \le n,\ 1 \le j \le n-1 \tag{2.1}$$

$$H_{i,1}(x_j) = 0,\ 1 \le i \le n-1,\ 1 \le j \le n;\ H'_{i,1}(x_j) = \delta_{ij},\ 1 \le i,j \le n-1 \tag{2.2}$$

Denoting $N_n(z) = \prod_1^n (z-x_j)$, we set for $i = 1,2,\dots,n$

$$L_i^2(z) = \left| \frac{N_n(z)}{z-x_i} \right|^2 \frac{\pi_{2n-2}(x_i)}{\pi_{2n-2}(z)[N_n'(x_i)]^2} \in \mathcal{L}_{2n-2}$$

which satisfies $L_i^2(x_j) = 0,\ 1 \le i \ne j \le n$. Define $L_{n,0}(z) = L_n^2(z)$ and

$$L_{i,0}(z) = L_i^2(z) + \lambda_i \frac{z - x_i}{z - x_n} L_i^2(z) \in \mathcal{L}_{n-1,n-1},\ \text{for } i = 1,2,\dots,n-1$$

with $\lambda_i \in \mathbb{C}$ chosen such that $L'_{i,0}(x_i) = 1$. Furthermore we set

$$L_{i,1}(z) = (x_i - x_n) \frac{z-x_i}{z-x_n} L_i^2(z) \in \mathcal{L}_{2n-2},\ i = 1,2,\dots,n-1$$

It is simple to check that

$$L_{i,0}(x_j) = \delta_{i,j},\ 1\le i,j\le n;\ L'_{i,j}(x_j) = 0,\ 1\le j\le n-1,\ 1\le i\le n \tag{2.3}$$

and

$$L_{i,1}(x_j) = 0,\ 1\le i\le n-1,\ 1\le j\le n;\ L'_{i,1}(x_j) = \delta_{i,j},\ 1\le i,j\le n-1 \tag{2.4}$$

From (2.3) and (2.4), we can set for $i = 1,2,\dots,n-1$

$$H_{i,o}(z) = \frac{\Pi_n^{2n-2}(1-\bar{\alpha}_j z)}{\Pi_n^{2n-2}(1-\bar{\alpha}_j x_i)} \frac{\omega_{n-1}(x_n)}{\omega_{n-1}(z)} [L_{i,0}(z) + \mu_i L_{i,1}(z)] \in \mathcal{R}_{n-1,n-1}$$

where $\mu_i$ is uniquely determined by the condition $H'_{i,o}(x_i) = 0$ and

$$H_{j,o}(z) = \frac{\Pi_n^{2n-2}(1-\bar{\alpha}_j z)}{\Pi_n^{2n-2}(1-\bar{\alpha}_j x_i)} \frac{\omega_{n-1}(x_n)}{\omega_{n-1}(z)} L_{n,0}(z) \in \mathcal{R}_{n-1,n-1}$$

satisfies the requirements (2.1). Similarly, the functions

$$H_{i,1}(z) = \frac{\Pi_n^{2n-2}(1-\bar{\alpha}_j z)}{\Pi_n^{2n-2}(1-\bar{\alpha}_j x_i)} \frac{\omega_{n-1}(x_n)}{\omega_{n-1}(z)} L_{i,0}(z) \in \mathcal{R}_{n-1,n-1} \tag{2.5}$$

satisfy the conditions (2.2).

Once the interpolating function $Q(z) \in \mathcal{R}_{n-1,n-1}$ has been characterized, one gets

$$\hat{I}_n\{f\} = \int Q(z)d\mu(z) = \sum_{j=1}^{n} \lambda_j^{(n)} f(x_j) + \sum_{j=1}^{n-1} \hat{\lambda}_j^{(n)} f'(x_j) \tag{2.6}$$

where $\lambda_j^{(n)} = \int H_{j,o}(z)d\mu$ and $\hat{\lambda}_j^{(n)} = \int H_{j,1}(z)d\mu$. Therefore $\hat{I}_n\{f\}$ can be considered as a quadrature formula which makes use of values of the function f and its derivative. Clearly $\hat{I}_n\{f\}$ has a domain of validity $\mathcal{R}_{n-1,n-1}$. However, an adequate choice of the nodes $\{x_j\}$ can greatly simplify formulas (2.6). Indeed, when $\{x_j\}$ are the zeros of $\phi_n + w\phi_n^*$ ($|w| = 1$), one has

**Theorem 2**

The quadrature formula $\hat{I}_n\{f\}$ given by (2.6) reduces to an n-point Rational Szegö (or an R-Szegö, for short) formula when the nodes are the zero of

$$\phi_n + w\phi_n^* \ (|w| = 1)$$

**Proof.** We write $\chi_n = \phi_n + w\phi_n^* = N_n(z)/\pi_n(z)$, $N_n \in \Pi_n$ and $N_n(x_j) = 0$, $j = 1,\dots,n$. Note that $N_n$ is not necessarily monic. By the characterization theorem for R-Szegö formulas (Theorem 1), it suffices to show that $\hat{\lambda}_i^{(n)} = 0$, for $i = 1, \dots, n-1$. But $\hat{\lambda}_i^{(n)} = \int H_{i,1}(z)d\mu$, where $H_{i,1}$ is given by (2.5).

Hence, we have to prove

$$\int \frac{\Pi_n^{2n-2}(1-\bar{\alpha}_j z)}{\omega_{n-1}(z)} \frac{z-x_i}{z-x_n} L_i^2(z)d\mu = 0$$

From the definition of $L_i^2$, this integral can be written as (up to a constant factor)

$$\int \frac{N_n(z)}{\pi_{n-1}(z)} \frac{N_{n-1}(z)}{(z-x_i)\omega_{n-1}(z)} d\mu = \int \frac{N_n(z}{\pi_n(z)} \frac{(1-\bar{\alpha}_n z)\ N_{n-1}(z)}{(z-x_i)\omega_{n-1}(z)} d\mu =$$

$$= \int \chi_n(z)h_*(z)d\mu = <\chi_n,h>_\mu$$

where $h(z) = \dfrac{(1-\bar{x}_1 z)\ \ldots\ (1-\bar{x}_{n-1}z)(z-\alpha_n)}{(1-\bar{x}_i z)\pi_{n-1}\ (z)} \in \mathcal{L}_{n-1}$ and $h(\alpha_n) = 0$. Thus, h belongs to $\mathcal{L}_{n-1} \cap \mathcal{A}_n(\alpha_n)$ ($\mathcal{A}_n(\alpha_n) = \{ f \in \mathcal{L}_{n-1}: f(\alpha_n) = 0 \}$). Now, by the orthogonality properties for $\chi_n$, it follows that $<\chi_n,h>_\mu = 0$.□

## Remark 1

Note that the same result can be obtained if the following interpolation problem is considered. Find $Q_i \in \mathcal{R}_{n-1,n-1}$ $(i = 1,2,...,n)$ such that

$$Q_i(x_j) = f(x_j),\ j = 1,2,...,n;\ Q_i'(x_j) = f'(x_j),\ 1 \le j \le n\ (i \ne j) \qquad (2.7)$$

We can conclude that an n-point R-Szegö formula $I_n\{f\}$ is given by $I_n\{f\} = \int Q_i(z)d\mu$, where $Q_i$ is the unique solution to the interpolation problem (2.7) and $\{x_j\}$ are the zeros of $\chi_n(z) = \phi_n + w\phi_n^*$ $(|w| = 1)$. Certainly, the given approach could be useful in order to give an expression for the error $E_n\{f\} = I_\mu\{f\} - I_n\{f\} = I_\mu\{f - Q_i\}$.

## 3. An application to the Poisson integral

We shall now characterize the n-point R-Szegö formulas for the measure $\mu$ induced by the Poisson integral kernel given by (1.2). In this sense, the first step is obtaining the orthonormal system $\{\phi_n\}$, $n = 0,1,...$ We know $\phi_0=1$ and for $n = 1,2,...,$ $\phi_n$ has to verify the conditions (i) $\phi_n \in \mathcal{L}_n - \mathcal{L}_{n-1}$; $<\phi_n,\phi_n>_\mu = 1$ and (ii) $<\phi_n,B_k>_\mu = 0$, $k = 0,1,...,n-1$.

From (i) one finds

$$<\phi_n,B_k>_\mu = \int \phi_n(z)\ \overline{B_k(z)}\ d\mu = \int \frac{\phi_n(z)(1-|r|^2)}{B_k(z)(z-r)(1-\bar{r}z)} d\lambda$$

$$= \frac{1-|r|^2}{2\pi i} \int_{\mathbf{T}} \frac{\phi_n(z)}{B_k(z)(z-r)(1-\bar{r}z)} dz$$

The denominator $B_k(z)(z-r)(1-\bar{r}z)$ vanishes at $z = \alpha_i$, $i = 1,2,\ldots,k$ and $z = r$, which are all inside the unit disc **D** and the other zero $1/\bar{r}$ is in **E**. If this integral has to vanish, $\phi_n$ should be zero at $z = \alpha_i$, $i = 1,2,\ldots,k$ and $z = r$. This gives in combination with condition (i) that $\phi_n$ should have the following form

$$\phi_n(z) = k_n \frac{(z-r)B_n(z)}{z - \alpha_n} \in \mathscr{L}_n \text{ and } k_n \neq 0 \tag{3.1}$$

For $0 \leq k \leq n-1$, it is easily seen that

$$<\phi_n,B_k>_\mu = \frac{k_n(1-|r|^2)}{2\pi i} \int_{\mathbf{T}} \frac{B_{n/k}(z)}{z - \alpha_n} \frac{dz}{1 - \bar{r}z} = 0 \ (k_n \neq 0)$$

where $B_{n/k} = B_n / B_k$, and thus $B_{n/k}(\alpha_n) = 0$. The constant $k_n$ is determined by

$$1 = <\phi_n,\phi_n>_\mu = |k_n|^2 \frac{1-|r|^2}{1-|\alpha_n|^2} \quad \text{so that } |k_n| = \left[\frac{1-|\alpha_n|^2}{1-|r|^2}\right]^{1/2}$$

The leading coefficient is found as follows

$$\phi_{n*}(z) = \bar{k}_n \frac{1-\bar{r}z}{1-\bar{\alpha}_n z} \frac{1}{B_n(z)} \tag{3.2}$$

So that $\phi_n^*(z) = \bar{k}_n \dfrac{1-\bar{r}z}{1-\bar{\alpha}_n z}$ and thus $\overline{\phi_n^*(\alpha_n)} = k_n \dfrac{1-\bar{\alpha}_n r}{1-|\alpha_n|^2}$. Since the leading coefficient has to be positive,

$$k_n = \left[\frac{1-|\alpha_n|^2}{1-|r|^2}\right]^{1/2} \exp(i\gamma_n),\ \gamma_n = -\arg(1 - \bar{\alpha}_n r)$$

We can check some particular cases.

(1) $r = 0$, which delivers the Lebesgue measure. Then

$$\gamma_n = -\arg(1 - \bar{\alpha}_n r) = -\arg(1) = 0 \text{ and } \phi_n(z) = \sqrt{1-|\alpha_n|^2} \frac{zB_n(z)}{z - \alpha_n}$$

This corresponds with the result in [11]. When all the $\alpha_i$ are equal to zero, we recover the well known result that $\phi_n(z) = z^n$.

(2) $r = \alpha_n$. One then has

$$\gamma_n = -\arg(1 - |\alpha_n|^2) = 0 \text{ and } \phi_n(z) = B_n(z)$$

(3) $r \neq 0$, $\alpha_k = 0$, $k = 1,2,\ldots$. Then

$$\phi_n(z) = \frac{(z-r)B_n(z)}{z} = (z-r)z^{n-1} = z^n + rz^{n-1}$$

This was obtained by Waadelandt in a recent paper [5].

The equation $\chi_n(z) = \phi_n(z) + \hat{w}\phi_n^*(z)$ which provides the nodes takes the form

$$\chi_n(z) = k_n[(z-r)B_n(z)/(z-\alpha_n)] + \hat{w}\bar{k}_n(1 - \bar{r}z)/(1 - \bar{\alpha}_n z) = 0$$

or equivalently

$[(z-r)B_n(z)/(z-\alpha_n)] + w\,(1- \bar{r}z)/(1 - \bar{\alpha}_n z) = 0$, with $w = \hat{w}\,\bar{k}_n/k_n \in \mathbf{T}$.

Using $B_n(z) = \eta_n\pi_n(z)/\omega_n(z)$, we get

$$\chi_n(z) = [(z-r)\eta_n\omega_{n-1}(z) + w\pi_{n-1}(z)(1-\bar{r}z)]/\pi_n(z) = N_n(z)/\pi_n(z),\ N_n \in \Pi_n \qquad (3.3)$$

The nodes $x_j$ satisfy $N_n(x_j) = 0$. When $r = 0$, one obtains

$$zB_{n-1}(z) = |\alpha_n|\,w\,/\,\bar{\alpha}_n \qquad (3.4)$$

Note that when $\alpha_i = 0$, this reduces to $z^n = -w$ and the nodes $x_j$ are uniformly distributed on $\mathbf{T}$, see [6]. Let us suppose that $n = 2$, $\alpha_1 = \alpha_2 = 1/2$, then (3.4) gives

$$2z^2 - (1+w)z + 2w = 0$$

and the interpolation nodes $x_1$, $x_2$ will be:

For $w = -1$: $x_1 = 1$, $x_2 = -1$

For $w = 1$: $x_1 = (1+\sqrt{3}i)/2$, $x_2 = (1-\sqrt{3}i)/2$

For $w = -i$: $x_1 = [(1+\sqrt{7}) - (1-\sqrt{7})i]/4$, $x_2 = [(1-\sqrt{7}) - (1+\sqrt{7})i]/4$

For $w = i$: $x_1 = [(1-\sqrt{7}) + (1+\sqrt{7})i]/4$, $x_2 = [(1+\sqrt{7}) + (1-\sqrt{7})i]/4$

For the weights, one has in the general case

$$A_j = \int L_j(z)d\mu(z) = \frac{1}{2\pi i}\int_{\mathbf{T}} \frac{\chi_n(z)}{z-x_j}\,\frac{1-\alpha_n z}{1-\bar{\alpha}_n x_j}\,\frac{1 - |r|^2}{\chi_n'(x_j)(z-r)(1-\bar{r}z)}\,dz$$

$$= \frac{1}{\chi_n'(x_j)}\,\frac{\chi_n(r)}{r-x_j}\,\frac{1-\bar{\alpha}_n r}{1-\bar{\alpha}_n x_j}$$

where $\chi_n(z) = N_n(z)/\pi_n(z)$ is given by (3.3). Since $\chi_n(r) = w(1-|r|^2)/(1-\bar{\alpha}_n r)$, we find

$$A_j = \frac{w(1-|r|^2)}{\chi_n'(x_j)(r-x_j)(1-\bar{\alpha}_n x_j)},\ j = 1,2,\ldots,n$$

On the other hand, since $\chi_n(x_j) = 0$, we get $\chi_n'(x_j) = N_n'(x_j)/\pi_n(x_j)$ where

$$N_n(z) = (z-r)\eta_n\omega_{n-1}(z) + w\pi_{n-1}(z)(1-\bar{r}z) \tag{3.5}$$

Use $N_n(x_j) = 0$, to find from (3.5) that

$$\pi_n(x_j) = -(1-\bar{\alpha}_n x_j)(x_j-r)\eta_n\omega_{n-1}(x_j)/w(1-\bar{r}x_j)$$

which implies

$$A_j = \eta_n \frac{1-|r|^2}{1-\bar{r}x_j} \frac{\omega_{n-1}(x_j)}{N_n'(x_j)} \tag{3.6}$$

Now use

$$\pi_k'(x_j) = -\pi_k(x_j)\sum_{j=1}^{k} \frac{\bar{\alpha}_i}{1-\bar{\alpha}_i x_j} \quad \text{and} \quad \omega_k'(x_j) = \omega_k(x_j)\sum_{j=1}^{k}\frac{1}{x_j-\alpha_j}$$

to obtain from (3.5) and (3.6)

$$A_j = (1-|r|^2)/(1-\bar{r}x_j)\left[1 + (x_j-r)\sum_{k=1}^{n-1}\left(\frac{1}{x_j-\alpha_k} - \frac{\bar{\alpha}_k}{1-\bar{\alpha}_k x_j}\right) - \frac{\bar{r}(r-x_j)}{1-\bar{r}x_j}\right]$$

so that for j=1,2,...,n

$$A_j = (1-|r|^2)\Big/\left[1 - |r|^2 + |x_j-r|^2\sum_{k=1}^{n-1}\frac{1-|\alpha_k|^2}{|x_j-\alpha_k|^2}\right] \tag{3.7}$$

where the positivity of the weights is clearly exposed. Again, when all the $\alpha_i$ are equal to zero one gets

$$A_j = \frac{1-|r|^2}{1-|r|^2 + (n-1)|x_j-r|^2}, \quad j = 1,2,\ldots,n$$

If we set $x_j = \exp(i\theta_j)$ and $r \in (0,1)$, then

$$|x_j - r|^2 = |\exp(i\theta_j) - r|^2 = 1 + r^2 - 2r\cos\theta_j$$

Therefore

$$A_j = \frac{1-|r|^2}{1-|r|^2 + (n-1)(1 + r^2 - 2r\cos\theta_j)}, \quad j = 1,2,\ldots,n \tag{3.8}$$

The same expression was obtained by Waadelandt [5] for the polynomial case

If $r = 0$ (Lebesgue measure), it follows from (3.8) that $A_j = n^{-1}$ for $j = 1,\ldots,n$. That means that the corresponding Szegö formula for the polynomial case has all its nodes equally spaced on the unit circle and all its weights are equal to $n^{-1}$. (Compare with the results given in [6]).

In the special case $r = \alpha_n$, (3.8) yields

$$A_j = (1-|\alpha_n|^2) \;/\; \left[ \; |x_j - \alpha_n|^2 \sum_{k=1}^{n} \frac{1-|\alpha_k|^2}{|x_j-\alpha_k|^2} \; \right] \tag{3.9}$$

Some concluding example: assume that $\alpha_k = \alpha$ for $k = 1,2,...,n$ and take also $r = \alpha$. Then

$$\phi_n(z) = B_n(z),\ \phi_n^*(z) = 1 \text{ and } \chi_n = \phi_n + w\phi_n^* = B_n(z) + w$$

Because now $B_n = \zeta^n$ with $\zeta(z) = (\bar{\alpha}/|\alpha|)(\alpha - z)/(1 - \bar{\alpha}z)$, the nodes $x_j$ are solutions of

$$\left( \frac{\alpha - z}{1 - \bar{\alpha}z} \right)^n = - w|\alpha|^n \,/\, \bar{\alpha}^n = \hat{w} \in \mathbf{T}$$

Setting $r_j = (\hat{w})^{1/n}$, $j = 1,2,...,n$, we get $x_j = (\alpha - r_j)/(1 - \bar{\alpha}r_j)$. As for the weights, one gets from (3.9)

$$A_j = \frac{1}{n},\ j = 1, 2,..., n.$$

## 4. Convergence

Let $I_n\{f\}$ , $n = 1,2,...,$ be a sequence of R-Szego formulas (take into account that for each n $I_n\{f\}$ represents an one-parameter family of quadrature formulas), that is

$$I_n\{f\} = \sum_{j=1}^{n} A_j^{(n)} f(x_j^{(n)}),\ x_i^{(n)} \neq x_j^{(n)} (i \neq j),\ x_j^{(n)} \in \mathbf{T} \text{ and } A_j^{(n)} > 0,\ j=1,2,...,n$$

where the weights $A_j^{(n)}$ are given by (1.3). In this section, we shall study the convergence of such quadratures for any function f in the class $R_\mu(\mathbf{T})$ of the integrable functions on $\mathbf{T}$ with respect to the measure $\mu$. For this purpose a first result we shall need is

**Lemma 1**

Let us define $\mathcal{R}_n = \mathcal{R}_{n,n} = \mathcal{L}_n + \mathcal{L}_{n*}$ and $\mathcal{R} = \mathcal{R}_\infty$, then $\mathcal{R}$ is dense in the class $C(\mathbf{T})$ of continuous functions on $\mathbf{T}$, iff $\sum (1-|\alpha_n|) = \infty$.

**Proof.** This is a direct consequence of the *"closure criterion"* discussed in Addendum A.2 of [12 p. 244].□

Wew are now ready to prove a first result asserting the convergence in the class $C(\mathbf{T})$. Indeed, one has

**Theorem 3**

Let f be a continuous function on $\mathbf{T}$, then

$$\lim_{n\to\infty} I_n\{f\} = I_\mu\{f\} = \int f(z)d\mu(z)$$

if $\sum (1-|\alpha_n|) = \infty$.

**Proof.** Let $\varepsilon$ be a given real positive number. Take

$$\varepsilon' = \varepsilon / 2\mu_0 \text{ where } \mu_0 = \int d\mu(z)$$

By Lemma 1, there exists $R_N \in \mathscr{R}_N$ such that

$$|f(z) - R_N(z)| < \epsilon', \ \forall z \in \mathbf{T}$$

Assume $n > N$ and write $I_n\{f\} = \sum_{j=1}^{n} A_j^{(n)} f(x_j^{(n)})$, then

$$I_\mu\{f\} - I_n\{f\} = I_\mu\{f-R_N\} + I_n\{R_N-f\}.$$

Hence,

$$|I_\mu\{f\} - I_n\{f\}| \le \int |f(z)-R_N(z)|\,d\mu(z) + \sum_{j=1}^{n} A_j^{(n)} |f(x_j^{(n)}) - R_N(x_j^{(n)})| \le 2\mu_0\varepsilon' = \varepsilon$$

(Recall that $A_j^{(n)} > 0$ , $j = 1,2,...n$ and $\sum_{j=1}^{n} A_j^{(n)} = \mu_0$).□

Assume now f a complex function defined on the unit circle **T** which is integrable with respect to the measure $\mu$. We can write

$$f(z) = f_1(\theta) + if_2(\theta), \ z = \exp(i\theta) \qquad (4.1)$$

where $f_j(\theta)$, $j = 1,2$ are real valued functions defined on the interval $[-\pi,\pi]$. Let us first suppose f is a continuous function, or equivalently $f_j$ ($j = 1,2$) are continuous functions. From Theorem 3, we can write

$$\lim_{n\to\infty} I_n\{f_j\} = I_\mu\{f_j\}, \ j = 1,2 \qquad (4.2)$$

Now parallelling rather closely the arguments given in [13 pp.127-129] it can be seen that (4.2) is also valid for integrable functions because of the fact that any sequence of integrable rules with positive weights which converges for all continuous functions, convergres for all integrable functions with respect to a finite Borel measure $\mu$ on $[-\pi,\pi]$.

Let now $f \in R_\mu(\mathbf{T})$ be. From (4.1), one can write

$$I_\mu\{f\} = I_\mu\{f_1\} + iI_\mu\{f_2\} \text{ and } I_n\{f\} = I_n\{f_1\} + iI_n\{f_2\} \qquad (4.3)$$

Thus, by (4.2) and (4.3) the next corollay immediately follows

**Corollary 1**

Under the same hypothesis as Theorem 1, one has

$$\lim_{n\to\infty} I_n\{f\} = I_\mu\{f\} \text{ for any } f \in R_\mu(\mathbf{T})$$

**Remark 2**

When all the $\alpha_i$ are equal to zero,then the Blaschke condition $\sum(1-|\alpha_n|)=\infty$ holds trivally, and the convergence of the Szegö quadrature formulas introduced in [1] is guaranteed in the class $R_\mu(\mathbf{T})$ (see also [7] for a direct proof). On the other hand, the special case when the sequence $\{\alpha_n\}$ consists of a finite number p of points cyclically repeated (see, e.g. [14]) the Blaschke condition also holds and therefore the convergence of the corresponding quadrature process is assured.

## References


[1] **W.B. Jones, O. Njastad and W.J. Thron.** Moment theory, orthogonal polynomials and continued fractions associated with the unit circle. *Bull London Math. Soc.,* 21:113-152 (1989)

[2] **G. Freud.** Orthogonal polynomials, Pergamon Press (1971)

[3] **W.B. Jones, O. Njastad and W.J. Thron.** Orthogonal Laurent polynomials and the strong Hamburguer moment problem. *J. Math. Anal. Appl.* 98, 528-554 (1984)

[4] **O. Njastad and W.J. Thron.** The theory of sequences of orthogonal L-polynomials. *Det Kongelige Norske Videnskabers Selskab*, 1: 54-91 (1983)

[5] **H. Waadeland.** A Segö quadrature formula for the Poisson integral. In *Proceedings IMACS'91* (Dublin) 12-13 (1991)

[6] **P. González-Vera and M. Camacho.** A note on para-orthogonality and biorthogonality. *Det Kongelige Norske Videnskabers Selskab.* To appear

[7] **A. Bultheel, P. González-Vera, E. Hendriksen and O. Njastad.** Orthogonality and quadrature on the unit circle. *IMACS annals on computing and applied mathematics.* Vol. 9, 205-210 (1991)

[8] **A. Bultheel, P. González-Vera, E. Hendriksen and O. Njastad.** A Segö theory for rational functions. Technical Report TW-131, K.U. Leuven, Dept. of Computer Science (1990)

[9] **A. Bultheel, P. González-Vera, E. Hendriksen and O. Njastad.** Orthogonal rational functions and quadrature on the unit circle. *Numerical Algorithms.* To appear.

[10] **V.I. Krylov.** Approximate calculations of integrals. Translated by A. H. Stroud. Macmillan, New York (1962)

[11] **M.M. Djrbashian.** A survey on the theory of orthogonal systems and some open problems. In P. Nevai, editor, *Orthogonal polynomials: Theory and practise,* vol. 294 of series C: *Mathematical and Physical Sciences,* 135-146, NATO-ASI, Kluwer Academic Publishers (1990)

[12] **N. I. Achieser.** Theory of approximation. Frederick Ungar Publ. Co.,New York (1956)

[13] **P.J. Davis and P. Rabinowitz.** Methods of numerical integration. Second

Edition. Academic Press (1984)

[14] **A. Bultheel, P. González-Vera, E. Hendriksen and O. Njastad.** Orthogonal rational functions similar to Szegö polynomials. *IMACS annals on computing and applied mathematics.* Vol. 9, 195-204 (1991)