Recurrence relations for orthogonal functions

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Abstract

Let a sequence $\{\alpha_n\}$ of numbers in the extended complex plane of one of the following four kinds be given: 1) All α_n are interior to or exterior to the unit circle. 2) All α_n are on the unit circle. 3) All α_n are above or below the real line. 4) All α_n are on the real line. In each case the sequence $\{\alpha_n\}$ gives rise to a space of rational functions and an orthogonal base in this space with respect to a given functional. Various kinds of recurrence relations satisfied by the elements of this orthogonal base are discussed.

1 Introduction

In this paper we shall use the following notations: $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$, $\mathbf{E} = \{z \in \mathbf{C} : |z| > 1\}$, $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$, $\mathbf{H}_+ = \{z \in \mathbf{C} : Im\, z > 0\}$, $\mathbf{H}_- = \{z \in \mathbf{C} : Im\, z < 0\}$, $\mathbf{R} = \{z \in \mathbf{C} : Im\, z = 0\}$.

The Cayley transform τ shall here be defined by the formulas

$$Z = \tau(z) = i\frac{1+z}{1-z}, z = \tau^{-1}(Z) = \frac{Z-i}{Z+i}, \;\; z \in \mathbf{D}, \;\; Z \in \mathbf{H}_+ \tag{1.1}$$

The transform $z \to Z$ maps $\mathbf{D}$ onto $\mathbf{H}_+$, $\mathbf{E}$ onto $\mathbf{H}_-$ and $\mathbf{T}$ onto $\mathbf{R}$.

The substar conjugate f_* of a function f is defined as

$$f_*(z) = \overline{f(1/\overline{z})}. \tag{1.2}$$

When f is a rational function, this may be written as

$$f_*(z) = \overline{f}(1/z), \tag{1.3}$$

where the bar denotes complex conjugates of the coefficients. Similarly the subtilde conjugate $f_\sim$ is defined by

$$f_\sim(z) = \overline{f(\overline{z})}. \tag{1.4}$$

When f is a rational function, this may be written as

$$f_\sim(z) = \overline{f}(z). \tag{1.5}$$

We set $\mathcal{L}_{n*} = \{\mathcal{L}_* : L \in \mathcal{L}_n\}$, $\mathcal{L}_{n\sim} = \{L_\sim : L \in \mathcal{L}_n\}$.

The setting of this paper will be certain spaces of rational functions with a functional M defined on them. The functional M gives rise to an inner product $<>$ (not necessarily definite) though the formula

$$< f, g >= M(f(z) \cdot g_*(z)) \tag{1.6}$$

or

$$< f, g >= M(f(z) \cdot g_\sim(z)). \tag{1.7}$$

When $< f, f > \neq 0$ for all f in the space under consideration, we shall call M and $<,>$ quasi-definite, and when $< f, f >> 0$ for these f we call M and $<,>$ positive definite.

When $<,>$ is defined by (1.6) we shall talk abaout the circle-disk situation, and when $<,>$ is defined by (1.7) we shall talk about the line-plane situation.

When μ is a finite (positive) Borel measure with infinite support on T or R, positive definite functionals M (an thereby positive definite inner products) are defined through

$$M(f) = \int_{-\pi}^{\pi} f(e^{i\theta})d\mu(\theta) \tag{1.8}$$

in the circle-disk situation,

$$M(f) = \int_{-\infty}^{\infty} f(t)d\mu(t) \tag{1.9}$$

in the line-plane situation.

The spaces of rational functions that will be treated arise in connection with Nevanlinna-Pick interpolation theory. The aim of this paper is to study certain properties of orthogonal bases for these spaces, more specifically recurrence relations connecting elements of these bases.

The theory of orthogonal sequences in these spaces (for the case that all interpolation points are in D) was initiated by Djrbashian in 1969 (see [14]), and independently by Bultheel, Dewilde and Dym (see [3,5,13]). For a general introduction to recent work on this theory, see [6]. Other papers discussing these and related spaces are [9,10,25,29]. For earlier work on recurrence relations in these spaces, see [6,7,8,10,16,17,27,28].

When all interpolation points coalesce at one point, the situation is essentially a polynomial situation, and when the interpolation points consist of two points cyclically repeated, the situation is to a large extent a Laurent polynomial situation. In particular when the one point is the origin in the circle-disk situation, orthogonal polynomials on T are obtained, see [15,21,22,34], while when the one point is the point at infinity in the line-plane situation, orthogonal polynomials on R are obtained (see [1,2,11]). When the two points are the origin and the point at infinity in the line-plane situation, orthogonal Laurent polynomials on R are obtained (see [18,19,20,24,32]).

2   Interpolation in the unit disk

Let $\{\alpha_n : n = 1, 2, ...\}$ be an arbitrary sequence of (not necessarily distinct) points (interpolation points) in $D \cup E$. We shall always assume that $\alpha_j \neq 1/\overline{\alpha_k}$ for $j, k = 1, 2, ...$ Note

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that $1/\overline{\alpha_j} \in E$ if and only if $\alpha_j \in D$, hence $\alpha_j \neq 1/\overline{\alpha_k}$ is always satisfied if $\alpha_n \in D$ for all n or $\alpha_n \in E$ for all n.

We define the Blaschke factor ζ_n as the function

$$\zeta_n(z) = \frac{\overline{\alpha_n}(\alpha_n - z)}{|\alpha_n|(1 - \overline{\alpha_n}z)}, \quad n = 1, 2, ... \tag{2.1}$$

(By convention we set $\frac{\overline{\alpha_n}}{|\alpha_n|} = -1$ if $\alpha_n = 0$.)

The Blaschke products B_n are defined by

$$B_0(z) = 1, \quad B_n(z) = \prod_{k=1}^{n} \zeta_k(z), \quad n = 1, 2, ... \tag{2.2}$$

We define the spaces $\mathcal{L}_n$, $\mathcal{L}$ by

$$\mathcal{L}_n = Span\{B_k : k = 0, 1, ..., n\}, \quad \mathcal{L} = \cup_{n=0}^{\infty} \mathcal{L}_n. \tag{2.3}$$

The functions in $\mathcal{L}_n$ are exactly the functions that may be written in the form

$$L(z) = \frac{p_n(z)}{\pi_n(z)}, \tag{2.4}$$

where

$$\pi_n(z) = \prod_{k=1}^{n}(1 - \overline{\alpha_k}z), \quad n = 1, 2, ... \tag{2.5}$$

and $p_n \in \Pi_n$ (the space of polynomials of degree at most n). This follows by partial fraction decomposition. In particular the situation reduces to the polynomial case $\mathcal{L}_n = \Pi_n$ when $\alpha_n = 0$ for all n.

For $f \in \mathcal{L}_n$ we define its superstar conjugate f^* by

$$F^*(z) = B_n(z)f_*(z). \tag{2.6}$$

Note that this transformation depends on n. It must be clear from the context what n is. Also note that $f^* \in \mathcal{L}_n$ when $f \in \mathcal{L}_n$.

Let M be a quasi-definite functional on $\mathcal{L}_n + \mathcal{L}_{n*}$, and let $<,>$ be defined by (1.6). Let the sequence $\{\Phi_n : n = 0, 1, 2, ...\}$ be obtained by orthogonalization of the sequence $\{B_n\}$ with respect to $<,>$. We shall assume that all Φ_n are monic, which means that the leading coefficient $\kappa_n = \kappa_n^{(n)}$ in the decomposition

$$\Phi_n(z) = \sum_{k=0}^{n} \kappa_k^{(n)} B_k(z) \tag{2.7}$$

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equals one, i.e. $\kappa_n^{(n)} = 1$.

The following orthogonalitty properties are valid:

$$< f, \Phi_n >= 0 \quad \text{for} \quad f \in \mathcal{L}_{n-1} \tag{2.8}$$

$$< g, \Phi_n^* >= 0 \quad \text{for} \quad g \in \zeta_n \mathcal{L}_{n-1}. \tag{2.9}$$

Recall that we may write

$$\Phi_n(z) = \frac{P_n(z)}{\pi_n(z)}, \quad P_n \in \Pi_n. \tag{2.10}$$

By substituting α_n for z in the expression for $\Phi_n^*(z)$ obtained from (2.4), we easily verify that

$$\Phi_n^*(\alpha_n) = 1. \tag{2.11}$$

We shall call the index n and the function Φ_n **degenerate** if $\Phi_n^*(\alpha_{n-1}) = 0$, **non-degenerate** otherwise. We shall call the index n and the function Φ_n **exceptional** if $\Phi_n(\alpha_{n-1}) = 0$, **non-exceptional** otherwise. Note that in the polynomial case ($\alpha_n = 0$ for all n), $\Phi_n^*(\alpha_{n-1}) = \Phi_n^*(\alpha_n) = \Phi_n^*(0) \neq 0$, so that the degenerate case can not occur.

The following Christoffel-Darboux formula can be shown to be valid (see [6], where only the case $\alpha_n \in D$ for all n is explicitely treated) when the inner product $<,>$ is **positive definite**:

$$[\Phi_n^*(\alpha_n)]^2[\Phi_n^*(z)\overline{\Phi_n^*(w)} - \Phi_n(z)\overline{\Phi_n(w)}] = [1 - \zeta_n(z)\overline{\zeta_n(w)}] \sum_{k=0}^{n-1} [\Phi_k^*(\alpha_k)]^2 \Phi_k(z)\overline{\Phi_k(w)}. \tag{2.12}$$

Proposition 2.1 *Assume that the functional M is positive definite, and that $\alpha_n \in D$ for all n or $\alpha_n \in E$ for all n. Then all Φ_n are non-degenerate.*

Proof:

By setting $w = z$ in (2.12) we obtain

$$[1 - |\zeta_n(z)|^2]^{-1}[|\Phi_n^*(z)|^2 - |\Phi_n(z)|^2] > 0. \tag{2.13}$$

From this it easily follows that $\Phi_n^*(z) \neq 0$ for $z \in D$ if $\alpha_n \in D$, $\Phi_n(z) \neq 0$ for $z \in E$ if $\alpha_n \in E$. Thus $\Phi_n^*(\alpha_{n-1}) \neq 0$ for all n if $\alpha_n \in D$ for all n or $\alpha_n \in E$ for all n. $\square$

Cf. the discussion above of the special case $\alpha_n = 0$ for all n, i.e. the polynomial situation.

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3   Recurrence in the disk situation

For the sake of completeness we state a version of the fundamental result on Szegö type recurrence for the orthogonal rational functions.

Theorem 3.1 *The sequence $\{\Phi_n\}$ satisfies the following recurrence relation:*

$$\Phi_n(z) = \varepsilon_n \frac{z - \alpha_{n-1}}{1 - \overline{\alpha_n} z} \Phi_{n-1}(z) + \delta_n \frac{1 - \overline{\alpha_{n-1}} z}{1 - \overline{\alpha_n} z} \Phi_{n-1}^*(z), \quad n = 1, 2, \ldots \tag{3.1}$$

$$\Phi_n^*(z) = \overline{\delta_n} \frac{z - \alpha_{n-1}}{1 - \overline{\alpha_n} z} \Phi_{n-1}(z) + \overline{\varepsilon_n} \frac{1 - \overline{\alpha_{n-1}} z}{1 - \overline{\alpha_n} z} \Phi_{n-1}^*(z), \quad n = 1, 2, \ldots \tag{3.2}$$

$$\alpha_0 = 0, \quad \Phi_0 = 1, \quad \Phi_0^* = 1. \tag{3.3}$$

The coefficients δ_n, ε_n are given by

$$\delta_n = \frac{(1 - \alpha_{n-1} \overline{\alpha_n}) \Phi_n(\alpha_{n-1})}{(1 - |\alpha_{n-1}|^2) \overline{\Phi_n^*(\alpha_{n-1})}} \tag{3.4}$$

$$\varepsilon_n = -\frac{\overline{\alpha_n}(1 - \overline{\alpha_{n-1}} \alpha_n) \overline{\Phi_n^*(\alpha_{n-1})}}{|\alpha_n|(1 - |\alpha_{n-1}|^2)}. \tag{3.5}$$

Proof:

The results follows from [7, Theorem 4.1]. $\qquad\square$

Corrollary 3.2 *A function Φ_n (or an index n) can not be both degenerate and exceptional at the same time.*

Proof:

Follows from (3.1)-(3.4) since $\Phi_n(z) \not\equiv 0$. $\qquad\square$

We note that in the polynomial case we have $\Phi_n^*(0) = 1$ for all n, hence $\delta_n = \Phi_n(0)$, $\varepsilon_n = 1$ (recall that $\frac{\overline{\alpha_n}}{(\alpha_n)} = -1$ in this case), and (3.1)-(3.2) reduces to the classical Szegö formulas

$$\Phi_n(z) = z\Phi_{n-1}(z) + \Phi_n(0)\Phi_{n-1}^*(z) \tag{3.6}$$

$$\Phi_n^*(z) = \overline{\Phi_n(0)} z \Phi_{n-1}(z) + \Phi_{n-1}^*(z). \tag{3.7}$$

Since in this case $\varepsilon_n = 1$, Φ_n can never be degenerate.

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Proposition 3.3 *If the functional M is positive definite, and all $\alpha_n \in D$ or all $\alpha_n \in E$, then $|\delta_n| < |\varepsilon_n|$ for all n.*

Proof:

We recall that in this situation $\varepsilon_n \neq 0$ (see Proposition 2). From (3.4)-(3.5) we get

$$\frac{\delta_n}{\varepsilon_n} = -(\frac{\overline{\alpha_n}}{|\alpha_n|})^{-1} \frac{\Phi_n(\alpha_{n-1})}{\Phi_n^*(\alpha_{n-1})}. \tag{3.8}$$

The proof of Proposition 2.1 shows that $|\Phi_n^*(\alpha_m)| > |\Phi_n(\alpha_m)|$ for all m, n when all $\alpha_m \in D$ and when all $\alpha_m \in E$. It follows from (3.8) that $|\frac{\delta_n}{\varepsilon_n}| < 1$, hence $|\delta_n| < |\varepsilon_n|$. $\square$

Theorem 3.4 *Assume that Φ_n is non-degenerate. Then the following recurrence relation is satisfied:*

$$\Phi_n(z) = \frac{\delta_n}{\overline{\varepsilon}_n} \Phi_n^*(z) + \frac{1}{\overline{\varepsilon}_n}(|\varepsilon_n|^2 - |\delta_n|^2)\frac{z - \alpha_{n-1}}{1 - \overline{\alpha}_n z}\Phi_{n-1}(z), \quad n = 1, 2, ... \tag{3.9}$$

In particular, this is the case when the functional M is positive definite and all $\alpha_n \in D$ or all $\alpha_n \in E$.

Proof:

This formula is obtained by substitution for $\Phi_{n-1}^*(z)$ from (3.2) into (3.1), ε_n being different from zero. $\square$

We note that in the polynomial case (3.9) reduces to the well-known formula (see e.g. [22])

$$\Phi_n(z) = \delta_n \Phi_n^*(z) + (1 - |\delta_n|^2)z\Phi_{n-1}(z). \tag{3.10}$$

We shall call the sequence $\{\Phi_n\}$ **non-degenerate** if all Φ_n are non-degenerate. We observe that if $\{\Phi_n\}$ is non-degenerate then (3.2) and (3.9) together define a three-term recurrence relation in the sequence $\{\Phi_0, \Phi_1^*, \Phi_1, \Phi_2^*, \Phi_2, ..., \Phi_n^*, \Phi_n, ...\}$, and thus the elements of this sequence are denominators of a continued fraction. (For basic information on continued fractions, see [23].) We state this property formally in a theorem.

Theorem 3.5 *Let $\{\Phi_n\}$ be a non-degenerate sequence, and define*

$$Q_{2m}(z) = \Phi_m^*(z), \quad m = 1, 2, ... \tag{3.11}$$

$$Q_{2m+1}(z) = \Phi_m(z), \quad m = 0, 1, 2, ... \tag{3.12}$$

Then the sequence $\{Q_n\}$ form the denominators of a continued fraction $K_{n=1}^{\infty}\dfrac{a_n(z)}{b_n(z)}$, where the elements $a_n(z)$, $b_n(z)$ are given by

$$a_{2m}(z) = \bar{\varepsilon}_m \frac{1 - \overline{\alpha_{m-1}}z}{1 - \overline{\alpha_m}z}, \quad a_{2m+1}(z) = \frac{1}{\bar{\varepsilon}_n}(|\varepsilon_m|^2 - |\delta_m|^2)\frac{z - \alpha_{m-1}}{1 - \overline{\alpha_m}z}, \quad m = 1, 2, \ldots \quad (3.13)$$

$$b_{2m}(z) = \overline{\delta_m}\frac{z - \alpha_{m-1}}{1 - \overline{\alpha_m}z}, \quad b_{2m+1}(z) = \frac{\delta_m}{\bar{\varepsilon}_m}, \quad m = 1, 2, \ldots. \quad (3.14)$$

The recurrence relation satisfied by the denominators $\{Q_n\}$ are thus

$$Q_{2m}(z) = \overline{\delta_m}\frac{z - \alpha_{m-1}}{1 - \overline{\alpha_m}z}Q_{2m-1}(z) + \bar{\varepsilon}_m \frac{1 - \overline{\alpha_{m-1}}z}{1 - \overline{\alpha_m}z}Q_{2m-2}(z),, \quad m = 1, 2, \ldots \quad (3.15)$$

$$Q_{2m+1}(z) = \frac{\delta_m}{\bar{\varepsilon}_m}Q_{2m}(z) + \frac{1}{\bar{\varepsilon}_m}(|\varepsilon_m|^2 - |\delta_m|^2)\frac{z - \alpha_{m-1}}{1 - \overline{\alpha_m}z}Q_{2m-1}(z), \quad m = 1, 2, \ldots \quad (3.16)$$

$$\alpha_0 = 0, \quad Q_0 = 1, \quad Q_1 = 1. \quad (3.17)$$

In particular the above results hold when the functional M is positive definite and all $\alpha_n \in D$ or all $\alpha_n \in E$.

In the polynomial case (all $\alpha_n = 0$) the continued fractions above reduce to the well-known PC-fractions (Perron-Carathéodory fractions), see [21,22]. The PC-fractions are closely related to the Carathéodory Coefficient Problem. The continued fractions described in Theorem 3.5 shall be called NP-fractions (Nevanlinna-Pick fractions). These continued fractions are related to the Nevanlinna-Pick Interpolation Problem (cf. [1,26,33]) in a way that directly generalizes the relationship between PC-fractions and the Carathéodory Coefficient Problem. For connections between the Nevanlinna-Pick Problem and general system theory, see e.g. [12].

We shall call the sequence $\{\Phi_n\}$ **non-exceptional** if all Φ_n are non-exceptional.

Theorem 3.6 *Assume that Φ_{n-1} is non-exceptional. Then the following recurrence relation is valid:*

$$\Phi_n(z) = \frac{1}{1 - \overline{\alpha_n}z}[\varepsilon_n(z - \alpha_{n-1}) + \bar{\varepsilon}_n\frac{\delta_n}{\delta_{n-1}}(1 - \overline{\alpha_{n-1}}z)]\Phi_{n-1}(z) \quad (3.18)$$

$$-\frac{\delta_n}{\delta_{n-1}}\frac{(z - \alpha_{n-2})}{(1 - \overline{\alpha_n}z)}[|\varepsilon_{n-1}|^2 - |\delta_{n-1}|^2]\Phi_{n-2}(z) \quad n = 2, 3, \ldots$$

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$$\Phi_n^*(z) = \frac{1}{1 - \overline{\alpha_n}z}[\overline{\varepsilon_n}(1 - \overline{\alpha_{n-1}}z) + \frac{\varepsilon_{n-1}\overline{\delta_n}}{\overline{\delta_{n-1}}}(z - \alpha_{n-1})]\Phi_{n-1}^*(z) \qquad (3.19)$$

$$-\frac{\overline{\delta_n}}{\overline{\delta_{n-1}}}[|\varepsilon_{n-1}|^2 - |\delta_{n-1}|^2]\frac{(z - \alpha_{n-1})(1 - \overline{\alpha_{n-2}}z)}{(1 - \overline{\alpha_n}z)(1 - \overline{\alpha_{n-1}}z)}\Phi_{n-2}^*(z) \quad n = 2, 3, \dots.$$

$$\Phi_0 = 1, \quad \Phi_0^* = 1, \quad \Phi_1(z) = \frac{\delta_1 + \varepsilon_1 z}{1 - \overline{\alpha_1}z_1}\Phi_1^*(z) = \frac{\overline{\varepsilon_1} + \overline{\delta_1}z}{1 - \overline{\alpha_1}z}. \qquad (3.20)$$

Proof:

Immediately from (3.1)-(3.2) with n replaced by $n - 1$ we obtain

$$\Phi_{n-2}^*(z) = \frac{1}{\delta_{n-1}} \cdot \frac{1 - \overline{\alpha_{n-1}}z}{1 - \overline{\alpha_{n-2}}z}\Phi_{n-1}(z) - \frac{\varepsilon_{n-1}}{\delta_{n-1}}\frac{z - \alpha_{n-2}}{1 - \overline{\alpha_{n-2}}z}\Phi_{n-2}(z) \qquad (3.21)$$

$$\Phi_{n-2}(z) = \frac{1}{\overline{\delta_{n-1}}} \cdot \frac{1 - \overline{\alpha_{n-1}}z}{z - \alpha_{n-2}}\Phi_{n-1}^*(z) - \frac{\overline{\varepsilon_{n-1}}}{\overline{\delta_{n-1}}} \cdot \frac{1 - \overline{\alpha_{n-2}}z}{z - \alpha_{n-2}}\Phi_{n-2}^*(z). \qquad (3.22)$$

By substituting from (3.21) for Φ_{n-1}^* and Φ_{n-2}^* in (3.2) with n replaced by $n - 1$, we obtain (3.18). Similarly by substituting from (3.22) for Φ_{n-1} and Φ_{n-2} in (3.1) with n replaced by $n - 1$, we obtain (3.19). $\qquad\square$

Theorem 3.7 *Let $\{\Phi_n\}$ be a non-exceptional sequence. Then $\{\Phi_n\}$ form the denominators of a continued fraction $K_{n=1}^\infty \frac{a_n(z)}{b_n(z)}$, where the elements $a_n(z)$, $b_n(z)$ are given by*

$$a_n(z) = \frac{\delta_n}{\delta_{n-1}}\frac{z - \alpha_{n-2}}{1 - \overline{\alpha_n}z}(|\delta_{n-1}|^2 - |\varepsilon_{n-1}|^2) \qquad (3.23)$$

$$b_n(z) = \frac{1}{1 - \overline{\alpha_n}z}[\varepsilon_n(z - \alpha_{n-1}) + \overline{\varepsilon_n}\frac{\delta_n}{\delta_{n-1}}(1 - \overline{\alpha_{n-1}}z)], \quad n = 2, 3, \dots, \qquad (3.24)$$

and $\{\Phi_n^*\}$ form the denominators of a continued fraction $K_{n=1}^\infty \frac{c_n(z)}{d_n(z)}$, where the elements $c_n(z)$, $d_n(z)$ are given by

$$c_n(z) = \frac{\overline{\delta_n}}{\overline{\delta_{n-1}}}[|\delta_{n-1}|^2 - |\varepsilon_{n-1}|^2]\frac{(z - \alpha_{n-1})(1 - \overline{\alpha_{n-2}}z)}{(1 - \overline{\alpha_n}z)(1 - \overline{\alpha_{n-1}}z)}, \qquad (3.25)$$

$$d_n(z) = \frac{1}{1 - \overline{\alpha_n}z}[\overline{\varepsilon_n}(1 - \overline{\alpha_{n-1}}z) + \frac{\varepsilon_{n-1}\overline{\delta_n}}{\overline{\delta_{n-1}}}(z - \alpha_{n-1})], \quad n = 2, 3, \dots. \qquad (3.26)$$

The recurrence relations have the form (3.18)-(3.20).

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Proof:

Immediate from Theorem 3.6.   □

In the polynomial case ($\alpha_n = 0$ for all n), the formulas (3.18)-(3.19) reduce to the formulas

$$\Phi_n(z) = (z + \frac{\delta_n}{\delta_{n-1}})\Phi_{n-1}(z) - (1 - |\delta_{n-1}|^2)\frac{\delta_n}{\delta_{n-1}}z\Phi_{n-2}(z) \tag{3.27}$$

$$\Phi_n^*(z) = (1 + \frac{\overline{\delta_n}}{\delta_{n-1}}z)\Phi_{n-1}^*(z) - \frac{\overline{\delta_n}}{\delta_{n-1}}(1 - |\delta_{n-1}|^2)z\Phi_{n-2}^*(z). \tag{3.28}$$

These are M-fractions and general T-fractions, respectively. (See [23].) Continued fractions of the form (3.18) or (3.19) are instances of MP-fractions (Multipoint Padé continued fractions), see Section 5 (cf. [17,30]). They bear the same relationship to multipoint Padé approximation as general T-fractions and M-fractions to two-point Padé approximation.

For a short treatment of continued fractions obtained from the basic recurrence formulas (3.1)-(3.2), see also [6], with reference to [4].

4   Interpolation on the unit circle

Let $\{\alpha_n : n = 1, 2, ...\}$ be an arbitrary sequence of (not necessarily distinct) points on T. Note that $1/\overline{\alpha_n} = \alpha_n$. We introduce functions ω_n by

$$\omega_0 = 1, \quad \omega_n(z) = \prod_{k=1}^{n}(z - \alpha_k), \quad n = 1, 2, ..., \tag{4.1}$$

and define the spaces $\mathcal{L}_n$ and $\mathcal{L}$ by

$$\mathcal{L}_n = Span\{\frac{1}{\omega_0}, \frac{1}{\omega_1}, ..., \frac{1}{\omega_n}\}, \quad \mathcal{L} = \cup_{n=0}^{\infty}\mathcal{L}_n. \tag{4.2}$$

We may also write

$$\mathcal{L}_n = Span\{\frac{z^{m_0}}{\omega_0}, \frac{z^{m_1}}{\omega_1(z)}, ..., \frac{z^{m_n}}{\omega_n(z)}\}, \quad 0 \le m_k \le k \tag{4.3}$$

and

$$\mathcal{L}_n = Span\{\sigma_0, \sigma_1, ..., \sigma_n\}, \tag{4.4}$$

where

$$\sigma_0 = 1, \quad \sigma_n = \frac{i^n(1 + z)^n \prod_{k=1}^{n}(1 + \alpha_k)}{\omega_n(z)}, \quad n = 1, 2, ... \tag{4.5}$$

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(We have for convenience assumed that $z = -1$ is not among the points α_n. If $\alpha_n = -1$ for some n, a slightly different definition can be used.)

We observe that $\sigma_{n_*} = \sigma_n$, and $\mathcal{L}_{n_*} = \{L_* : L \in \mathcal{L}_n\} = \mathcal{L}_n$.

The functions in $\mathcal{L}_n$ are exactly the functions that may be written in the form

$$L(z) = \frac{p_n(z)}{\omega_n(z)}, \quad p_n \in \Pi_n. \tag{4.6}$$

(This follows by partial fraction decomposition).

We shall have occasion to work also with the spaces $\mathcal{L}_n \cdot \mathcal{L}_n$ and $\mathcal{L} \cdot \mathcal{L}$. Note that $\mathcal{L}_n \cdot \mathcal{L}_n \subset \mathcal{L}_n$, $\mathcal{L} \cdot \mathcal{L} \subset \mathcal{L}$, since $1 \in \mathcal{L}_n$ for all n.

Let M be a linear functional on $\mathcal{L} \cdot \mathcal{L}$, and assume that M has real values for all functions in $\mathcal{L}$ that are real on T. As before we define the inner product $<,>$ by (1.6), and assume that this inner product is quasidefinite.

Let $\{\Phi_n\}$ be the monic orthogonal system obtained by the Gram-Schmidt process from the sequence $\{\sigma_n\}$ (or equivalently from the sequence $\{\frac{1}{\omega_n}\}$). That Φ_n is monic means that the coefficient of σ_n in the expansion of Φ_n is one. Thus

$$\Phi_n(z) = \sum_{m=0}^{n} b_m^{(n)} \sigma_m(z), \quad b_n^{(n)} = 1, \quad b_m^{(n)} \in R. \tag{4.7}$$

We may write

$$\Phi_n(z) = \frac{P_n(z)}{\omega_n(z)}, \quad P_n \in \Pi_n. \tag{4.8}$$

We easily verify that

$$\Phi_{n_*}(z) = \Phi_n(z), \quad n = 0, 1, 2, .. \tag{4.9}$$

Note that $P_n(\alpha_n) \neq 0$ for all n. We shall call the function Φ_n and the index n **singular** if $P_n(\alpha_{n-1}) = 0$. (Note that the properties of being degenerate and of being exceptional coincide in the circle situation, because of (4.9), and we use the word singular for these coinciding phenomena in this case.) We shall call Φ_n and n **regular** if $P_n(\alpha_{n-1}) \neq 0$.

5 Recurrence in the circle situation

We shall in this section state without proof a general result on three-term recursion in the circle situation, and briefly discuss two special cases with one interpolation point repeated and with two interpolation points cyclically repeated.

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Theorem 5.1 *Assume that the system $\{\Phi_n\}$ is regular. Then a recurrence relation of the following form holds:*

$$\Phi_n(z) = \left(\frac{A_n}{z - \alpha_n} + B_n\frac{z - \alpha_{n-2}}{z - \alpha_n}\right)\Phi_{n-1}(z) + C_n\frac{z - \alpha_{n-2}}{z - \alpha_n}\Phi_{n-2}(z), \quad n = 2, 3, \ldots \quad (5.1)$$

with the initial condition

$$\alpha_0 = 0. \quad (5.2)$$

The constants A_n, B_n, C_n satisfy the inequalities

$$A_n + B_n(\alpha_{n-1} - \alpha_{n-2}) \neq 0, \quad n = 2, 3, \ldots \quad (5.3)$$

$$C_n \neq 0, \quad n = 2, 3, \ldots . \quad (5.4)$$

Proof:

See [10]. Cf. also [16], which treats the cyclic situation in the line case (discussed in Section 6). □

It follows that the orthogonal functions $\{\Phi_n\}$ are denominators of a continued fraction $K_{n=1}^{\infty}\frac{a_n(z)}{b_n(z)}$, with elements

$$a_n(z) = \frac{C_n(z - \alpha_{n-2})}{z - \alpha_n} \quad (5.5)$$

$$b_n(z) = \left(\frac{A_n}{z - \alpha_n} + B_n\frac{z - \alpha_{n-2}}{z - \alpha_n}\right). \quad (5.6)$$

Continued fractions of this form have been called MP-fractions (Multipoint Padé continued fractions). See [17] for further discussion in the cyclic case. Note that if $\alpha_n = 1$ for all n, then Φ_n is of the form

$$\Phi_n(z) = \sum_{m=1}^{n} \frac{\beta_m}{(z - 1)^m}. \quad (5.7)$$

Corrollary 5.2 *Let $\alpha_n = 1$ for $n = 1, 2, \ldots$. Then the sequence $\{\Phi_n\}$ satisfies a recurrence relation of the following form:*

$$\Phi_n(z) = \left(\frac{A_n}{z - 1} + B_n\right)\Phi_{n-1}(z) + C_n\Phi_{n-2}(z), \quad n = 2, 3, \ldots \quad (5.8)$$

$$A_n \neq 0 \quad for \quad n = 2, 3, \ldots \quad (5.9)$$

$$C_n \neq 0 \quad for \quad n = 2, 3, \ldots \quad (5.10)$$

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Proof:

Follows directly from Theorem 5.1, when we recall that Φ_n is always regular in this case. (The case $n = 2$ must be handled separately.) $\square$

For technical reasons we introduced the assumption $\alpha_n \neq -1$ for all n. It can be shown that a recurrence relation of the form (5.1) is valid also in the case that $\alpha_n = -1$ for some n.

Corrollary 5.3 *Let* $\alpha_{2m} = 1$, $m = 1, 2, ...$, $\alpha_{2m+1} = -1$, $m = 0, 1, 2, ...$ *Assume that the sequence* $\{\Phi_n\}$ *is regular. Then* $\{\Phi_n\}$ *satisfies a recurrence relation of the following form:*

$$\Phi_{2m}(z) = (\frac{A_{2m}}{z-1} + B_{2m})\Phi_{2m-1}(z) + C_{2m}\Phi_{2m-2}(z) \ m = 1, 2, ... \tag{5.11}$$

$$\Phi_{2m+1}(z) = (\frac{A_{2m+1}}{z+1} + B_{2m+1})\Phi_{2m}(z) + C_{2m+1}\Phi_{2m-1}(z) \ m = 1, 2.. \tag{5.12}$$

Proof:

Follows directly from Theorem 5.1. $\square$

6 Orthogonality and recurrence in the plane situation

Let $\{A_n\}$ be an arbitrary sequence of (not necessarily distinct) points in $\boldsymbol{H}_+ \cup \boldsymbol{H}_-$, and assume that $A_j \neq \overline{A_k}$ for all j, k. Set

$$\alpha_n = \frac{A_n - i}{A_n + i}. \tag{6.1}$$

cf. (1.1).

Then $\alpha_n \in \boldsymbol{D} \cup \boldsymbol{E}$, and $\alpha_j \neq 1/\overline{\alpha_k}$ for $j \neq k$. To the functions $\zeta_n(z)$ correspond the functions $\Gamma_n(Z)$ defined by

$$\Gamma_n(Z) = \frac{z - A_n}{z - \overline{A_n}}. \tag{6.2}$$

Define Δ_n by

$$\Delta_0 = 1, \ \Delta_n(Z) = \prod_{k=1}^{n} \Gamma_k(Z), \ n = 1, 2, ..., \tag{6.3}$$

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and set

$$\mathcal{M}_n = Span\{\Delta_0, \Delta_1, ..., \Delta_n\}, \quad \mathcal{M} = \cup_{n=0}^{\infty} \mathcal{M}_n. \tag{6.4}$$

We set $\mathcal{M}_{n\sim} = \{M_\sim : M \in \mathcal{M}_n\}, \mathcal{M}_\sim = \cup_{n=0}^{\infty} \mathcal{M}_{m\sim}$. Let M be a linear functional on $\mathcal{M} + \mathcal{M}_\sim$, and let $<,>$ be defined by (1.7). The functions F_n defined by

$$F_n(Z) = \Phi_n(z) = \Phi_n(\frac{Z-i}{Z+i}) \tag{6.5}$$

are orthogonal functions corresponding to the bases $\{\mathcal{M}_n\}$. (Here $\{\Phi_n\}$ are orthogonal with respect to the inner product (1.6) corresponding to the bases $\{\mathcal{L}_n\}$.)

We define

$$f_n^\sim(Z) = \Delta_n(Z)f_{n\sim}(Z) \tag{6.6}$$

when $f_n \in \mathcal{M}_n$, and note that $f_n^\sim \in \mathcal{M}_n$.

We further observe that

$$F_n^\sim(Z) = \Phi_n^*(z), \quad z = \frac{Z-i}{Z+i} \tag{6.7}$$

We call F_n **degenerate** if $F_n^\sim(A_{n-1}) = 0$, **exceptional** if $F_n(A_{n-1}) = 0$.

Note that if $Z = \tau(z), \quad A = \tau(\alpha)$ (cf.(1.1)), then

$$z - \alpha = \frac{2i(Z-A)}{(Z+i)(A+i)} \tag{6.8}$$

$$1 - \overline{\alpha}z = \frac{2i(Z-\overline{A})}{(i-\overline{A})(Z+i)}. \tag{6.9}$$

Theorem 6.1 *The sequence $\{F_n\}$ satisfies a recurrence relation of the following form:*

$$F_n(Z) = E_n \frac{Z - A_{n-1}}{Z - \overline{A}_n} F_{n-1}(Z) + D_n \frac{Z - \overline{A_{n-1}}}{Z - \overline{A}_n} F_{n-1}^\sim(Z) \tag{6.10}$$

$$F_n^\sim(Z) = \overline{D_n} \frac{Z - A_{n-1}}{Z - \overline{A}_n} F_{n-1}(Z) + \overline{E_n} \frac{Z - \overline{A_{n-1}}}{Z - \overline{A}_n} F_{n-1}^\sim(Z), \quad n = 1, 2, ... \tag{6.11}$$

$$A_0 = i, \quad F_0 = 1, \quad F_0^\sim = 1. \tag{6.12}$$

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Proof:

Follows from Theorem 3.1 by the substitutions (6.7), (6.8), (6.9). □

Theorem 6.2 *Assume that F_n is non-degenerate. Then F_n satisfies a recurrence relation of the following form:*

$$F_n(Z) = \frac{\overline{D_n}}{\overline{E_n}} F_n^{\sim}(Z) + \frac{1}{\overline{E_n}}(|E_n|^2 - |D_n|^2)\frac{Z - A_n}{Z - \overline{A_n}}F_{n-1}(Z) \tag{6.13}$$

In particular this is the case if the functional M is positive definite and all $A_n \in \boldsymbol{H}_+$ or all $A_n \in \boldsymbol{H}_-$.

Proof:

Follows as above from Theorem 3.4. □

It follows by combining (6.11) and (6.13) that if the sequence $\{F_n\}$ is non-degenerate, then the elements of the sequence $\{F_0, F_1^{\sim}, F_1, F_2^{\sim}, F_2, ..., F_n^{\sim}, F_n, ...\}$ are the denominators of a continued fraction, just as in the circle case. We shall call also these continued fractions NP-fractions.

Theorem 6.3 *Assume that $\{F_n\}$ is non-exceptional. Then $\{F_n\}$ and $\{F_n^{\sim}\}$ satisfy recurrence relations of the following form:*

$$F_n(Z) = \left[\frac{E_n(Z - A_{n-1})}{Z - \overline{A_n}} + \frac{\overline{E_n}D_n(Z - \overline{A_{n-1}})}{D_{n-1}(Z - \overline{A_n})}\right]F_{n-1}(Z) \tag{6.14}$$
$$-\frac{D_n(Z - A_{n-2})}{D_{n-1}(Z - \overline{A_n})}[|E_{n-1}|^2 - |D_{n-1}|^2]F_{n-2}(Z), \quad n = 2, 3, ...$$

$$F_n^{\sim}(Z) = \left[\frac{\overline{E_n}(Z - \overline{A_{n-1}})}{Z - \overline{A_n}} + \frac{E_{n-1}(Z - A_{n-1})}{D_{n-1}(Z - \overline{A_n})}\right]F_{n-1}^{\sim}(Z) \tag{6.15}$$
$$-\frac{\overline{D_n}}{\overline{D_{n-1}}}[|E_{n-1}|^2 - |D_{n-1}|^2]\frac{(Z - A_{n-1})(Z - \overline{A_{n-2}})}{(Z - \overline{A_n})(Z - \overline{A_{n-1}})}F_{n-2}^{\sim}(Z), \quad n = 2, 3, ...$$

$$F_0 = 1, \quad F_0^{\sim} = 1, \quad F_1(Z) = \frac{D_1 + E_1 Z}{Z - \overline{A_1}}, F_1^{\sim} = \frac{\overline{E_1} + \overline{D_1}Z}{Z - \overline{A_1}}. \tag{6.16}$$

Proof:

Follows as above from Theorem 3.6. □

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It follows that if the sequence $\{F_n\}$ is non-exceptional, then the elements of the sequences $\{F_n\}$ and $\{F_n^\sim\}$ are the denominators of two continued fractions. These are again instances of MP-fractions. (See Section 4,5.)

In the special case $A_n = i$ for all n (corresponding to the polynomial case in the circle situation) these formulas reduce to

$$F_n(Z) = [E_n\frac{Z-i}{Z+i} + \frac{\overline{E_n}D_n}{D_{n-1}}]F_{n-1}(Z) - \frac{D_n}{D_{n-1}}[|E_{n-1}|^2 - |D_{n-1}|^2]\frac{Z-i}{Z+i}F_{n-2}(Z) \quad (6.17)$$

$$F_n^\sim(Z) = [\overline{E_n} + \frac{\overline{E_{n-1}}}{\overline{D_{n-1}}}\frac{(Z-i)}{(Z+i)}]F_{n-1}^\sim(Z) - \frac{\overline{E_n}D_n}{\overline{D_{n-1}}}[|E_{n-1}|^2 - |D_{n-1}|^2]\frac{Z-i}{Z+i}F_{n-2}^\sim(Z). \quad (6.18)$$

7 Orthogonality and recurrence in the line situation

Let $\{A_n\}$ be an arbitrary sequence of (not necessarily distinct) points on $\mathbf{R}$. Define the points α_n as in Section 6:

$$\alpha_n = \frac{A_n - i}{A_n + i}. \quad (7.1)$$

Then $\alpha_n \in \mathbf{T}$. To the functions $\omega_n(z)$ defined in (4.1) correspond the functions $\Omega_n(Z)$ defined by

$$\Omega_0 = 1, \quad \Omega_n = \prod_{k=1}^n (Z - A_k), \quad n = 1, 2, \dots. \quad (7.2)$$

We set

$$\mathcal{M}_n = Span\{\frac{1}{\Omega_0}, \ \frac{1}{\Omega_n}, \dots, \frac{1}{\Omega_n}\}, \ \mathcal{M} = \cup_{n=0}^\infty \mathcal{M}_n. \quad (7.3)$$

Let $\mathcal{L}_n$, $\mathcal{L}$ be the spaces defined by (4.2).

Let M be a functional on $\mathcal{M} \cdot \mathcal{M}$ with $<, >$ defined by (1.7). The functions F_n defined by

$$F_n(Z) = \Phi_n(z) = \Phi_n(\frac{Z-i}{Z+i}) \quad (7.4)$$

are orthogonal functions corresponding to the bases $\{\mathcal{M}_n\}$. (Here $\{\Phi_n\}$ are orthogonal with respect to the inner product (1.6), corresponding to the bases $\{\mathcal{L}_n\}$.)

We may write F_n in the form

$$F_n(Z) = \frac{P_n(Z)}{\Omega_n(Z)}, \quad P_n \in \Pi_n. \quad (7.5)$$

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We shall call F_n **singular** if $P_n(A_{n-1}) = 0$. (Here as in the circle situation, the properties of being degenerate and of being exceptional coincide, and we use the word singular for these coinciding phenomena in this case.) We call F_n **regular** if $P_n(A_{n-1}) \neq 0$.

Theorem 7.1 *Assume that the system $\{F_n\}$ is regular. Then a recurrence relation of the following form holds:*

$$F_n(Z) = [\frac{a_n}{Z - A_n} + \frac{b_n(Z - A_{n-2})}{Z - A_n}]F_{n-1}(Z) + C_n\frac{Z - A_{n-2}}{Z - A_n}F_{n-2}(Z), \quad n = 2, 3, \dots, \quad (7.6)$$

with initial condition

$$A_0 = i. \tag{7.7}$$

Proof:

Follows from Theorem 5.1 by the substitutions (1.1) and (7.4). $\qquad\square$

Let $A_n = \infty$ for all n. By (1.1) this corresponds to $\alpha_n = 1$ for all n. In this case $\Phi_n(z)$ are of the form

$$\Phi_n(z) = \sum_{k=0}^{n} \frac{\beta_k}{(z - 1)^k}, \tag{7.8}$$

see (5.7). By substituting for z from (1.1) we get

$$F_n(Z) = \sum_{k=0}^{n} \gamma_k Z^k, \tag{7.9}$$

which is a polynomial. Thus $A_n = \infty$ for all n represents the polynomial situation.

Corrollary 7.2 *Let $A_n = \infty$ for all n. Then the sequence $\{F_n\}$ satisfies a recurrence relation of the form*

$$F_n(Z) = (a_nZ + b_n)F_{n-1}(Z) + C_nF_{n-2}(Z), \quad n = 2, 3, 4\dots \tag{7.10}$$

Proof:

Follows from Corollary 5.2 by the substitutions (1.1) and (7.4). $\qquad\square$

Let $A_{2m} = \infty$, $m = 1, 2, \dots$, $A_{2m+1} = 0$, $m = 0, 1, 2, \dots$. Then the orthogonal functions $\Phi_n(z)$ are of the form

$$\Phi_{2m}(z) = \sum_{k=0}^{m} \frac{\beta_k}{(z - 1)^k} + \sum_{k=0}^{m} \frac{\gamma_k}{(z + 1)^k}, \tag{7.11}$$

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$$\Phi_{2m+1}(z) = \sum_{k=0}^{m} \frac{\beta_k}{(z-1)^k} + \sum_{k=0}^{m+1} \frac{\gamma_k}{(z+1)^k}. \tag{7.12}$$

By substituting in (7.11)-(7.12) from (1.1) and (7.4) we get

$$F_{2m}(Z) = \sum_{k=-m}^{m} \kappa_k Z^k \tag{7.13}$$

$$F_{2m+1}(Z) = \sum_{k=-(m+1)}^{m} \kappa_k Z^k. \tag{7.14}$$

These functions are orthogonal Laurent polynomials.

Corrollary 7.3 *Let $A_{2m} = \infty$ for $m = 1, 2, ...$, $A_{2m+1} = 0$ for $m = 0, 1, 2, ...$ Assume that the sequence $\{F_n\}$ is regular. Then $\{F_n\}$ satisfies a recurrence relation of the following form:*

$$F_{2m}(Z) = (a_{2m}Z + b_{2m})(F_{2m-1}(Z) + c_{2m}F_{2m-2}(Z), \quad m = 1, 2, 3, ... \tag{7.15}$$

$$F_{2m+1}(Z) = (\frac{a_{2m+1}}{Z} + b_{2m+1})F_{2m}(Z) + c_{2m+1}F_{2m-1}(Z), \quad m = 0, 1, 2, ... \tag{7.16}$$

Proof:

Follows from Corrollary 5.3 by the substitutions (1.1) and (7.4). □

The above recurrence relations are essentially the well-known recurrence relations for regular orthogonal Laurent polynomials. (See [20,32]. Cf. also [31].)

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