Subdivide and conquer resolution

for signals, images, and surfaces

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13th July 2006

Survey

- 1D signals signal
  - wavelets
  - multiresolution
  - lifting
  - subdivision
- 2D signal (images)
  - wavelets
  - case study: normal offsets
- 3D signals (surfaces)
  - Powell-Sabin splines
  - subdivision
  - hierarchical basis, wavelet basis

1D signals

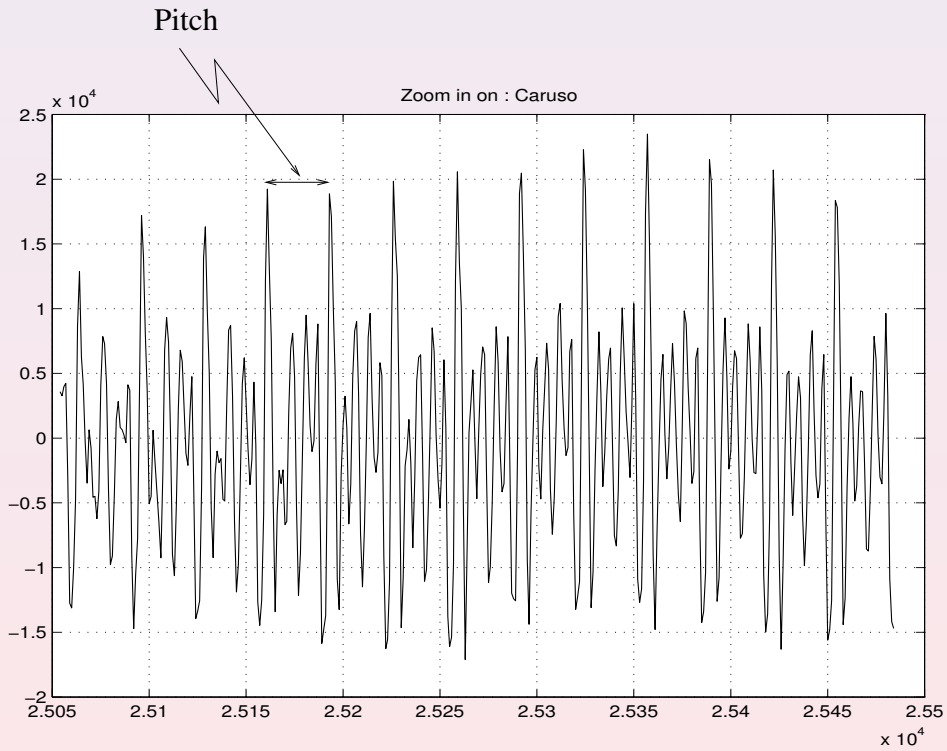


Which signals?

- Speech signal
- Biomedical signal
- Economic signal
- Seismic signal
- ...

1D signals

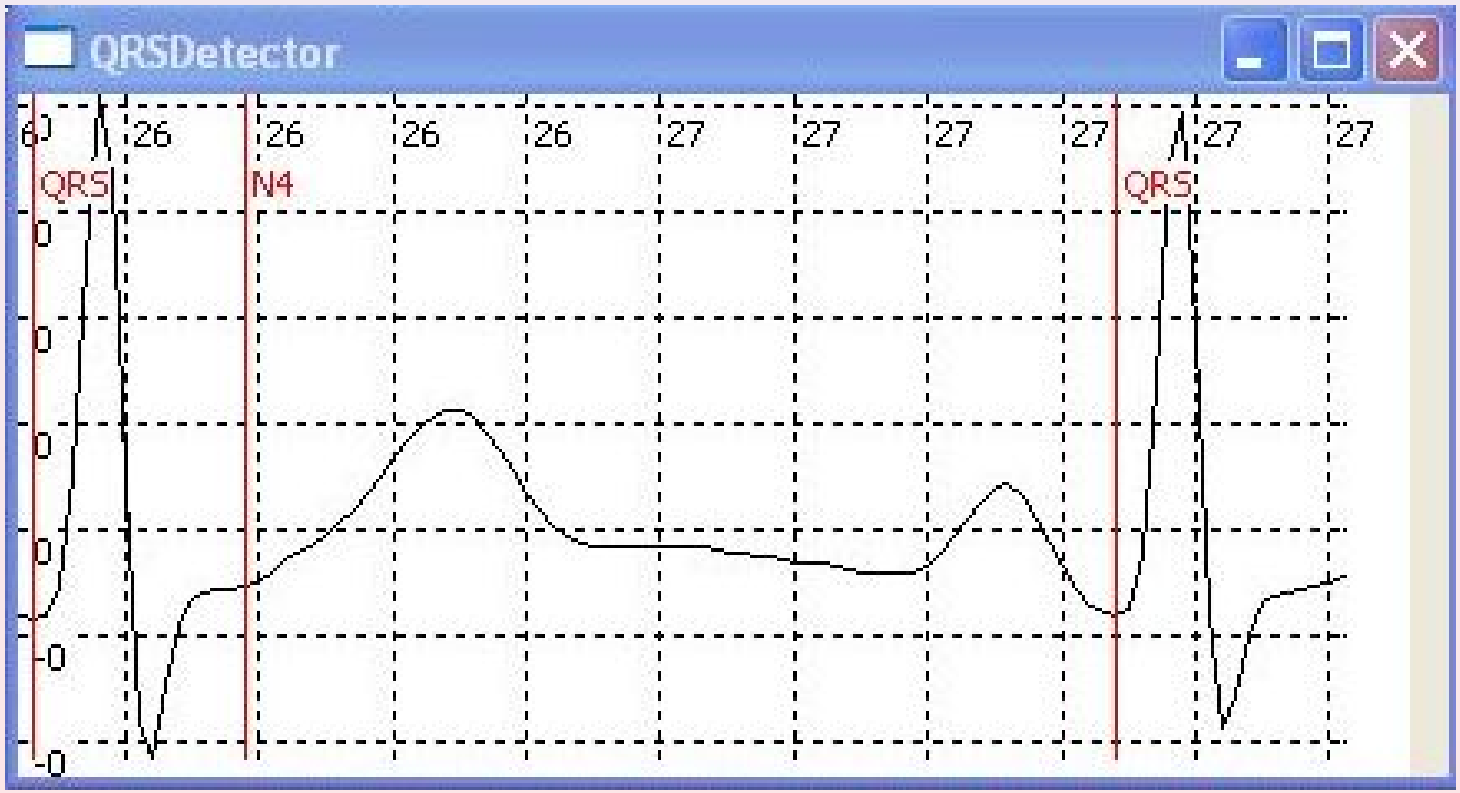
- Speech signal



- denoising
- compression

1D signals

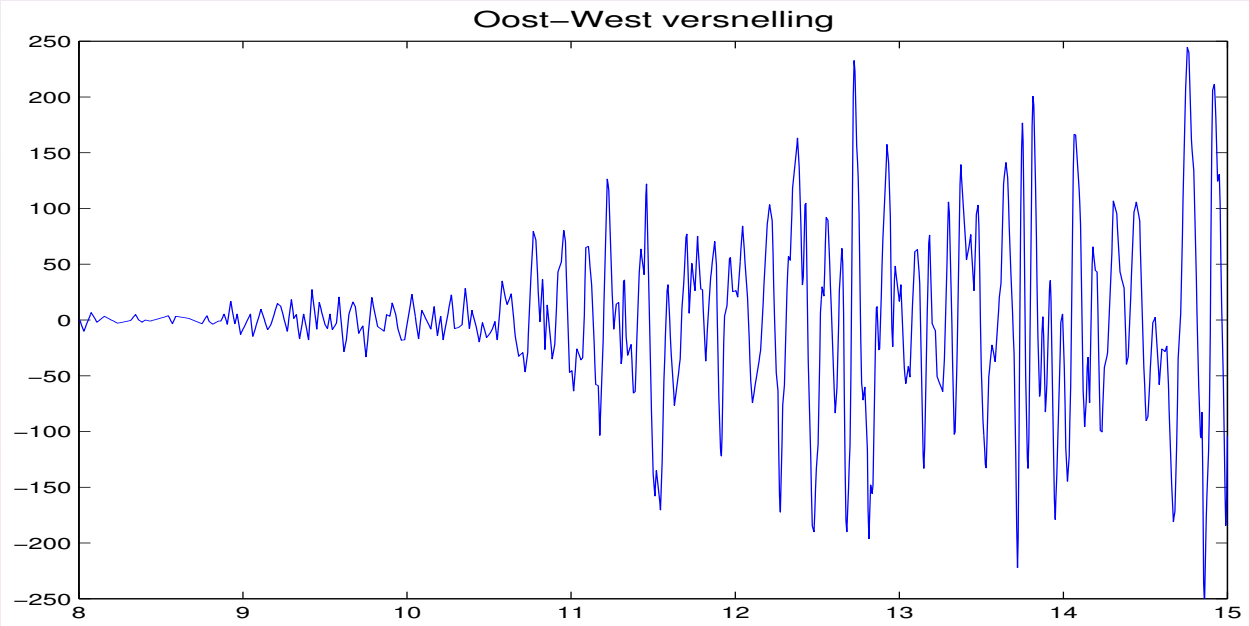
- Biomedical signal



- ECG QRS detection
- remove noise

1D signals

- Seismic signal



1989 California earthquake

- predict eruption
- oil detection

1D signals

- Economic signal



German DAX index

- seasonal movement
- trend detection
- crash alarm

Fourier

$$
\begin{aligned}
x(t) &= \int_{-\infty}^{\infty} X(\omega) e^{i\omega t} \mathrm{d}\omega \\
&\approx \sum_{k=-\infty}^{\infty} X(\omega_k) e^{i\omega_k t} \\
&= \sum_{k=0}^{\infty} [a_k \cos(\omega_k t) + b_k \sin(\omega_k t)] \\
&= \sum_{k} \hat{x}_k \psi_k(t)
\end{aligned}
$$

The ψ_k are basis functions.

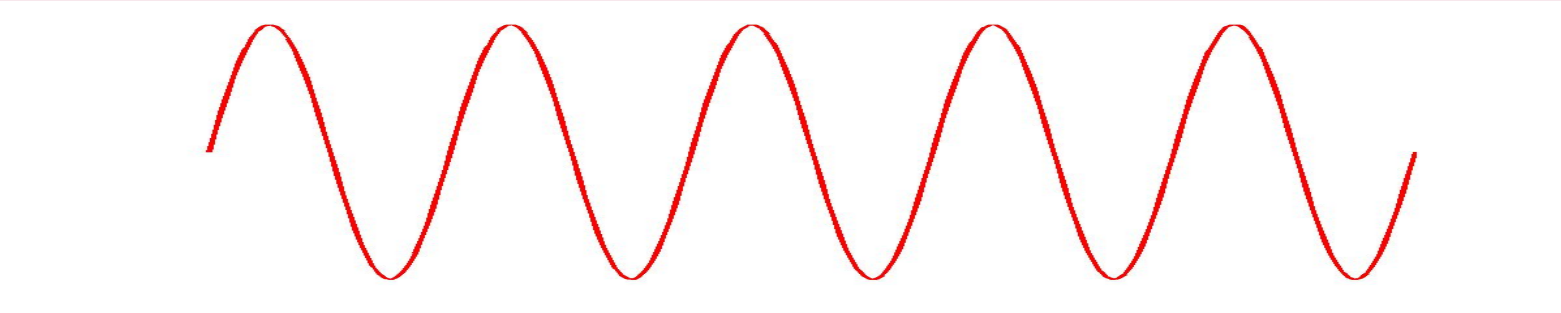
Fourier

Advantages

- every ψ_k corresponds to exactly 1 frequency
- orthogonal system
- generated by 1 "mother function"

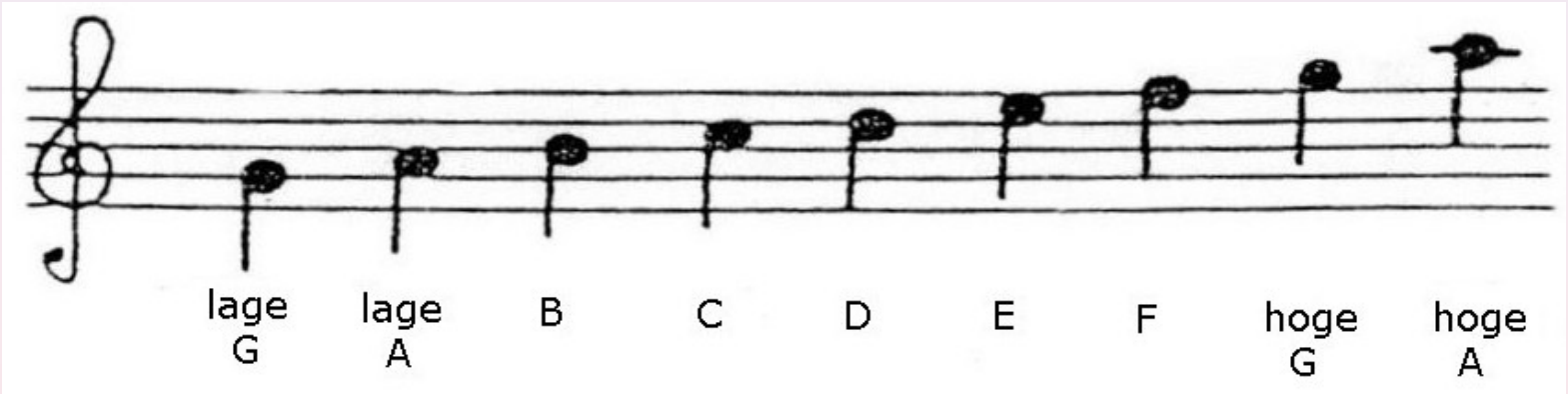
Disadvantage

- infinite support in time domain $\sum_k c_k \psi_k(t)$



Wavelets

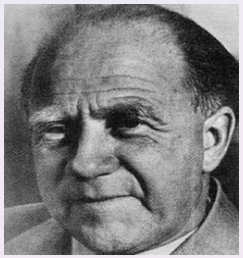
Look for basis functions that are local in time and frequency



- one specific frequency
- at one specific moment

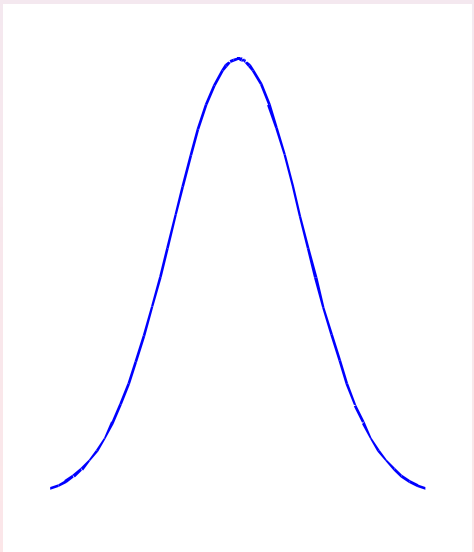
Wavelets

Heisenberg uncertainty principle

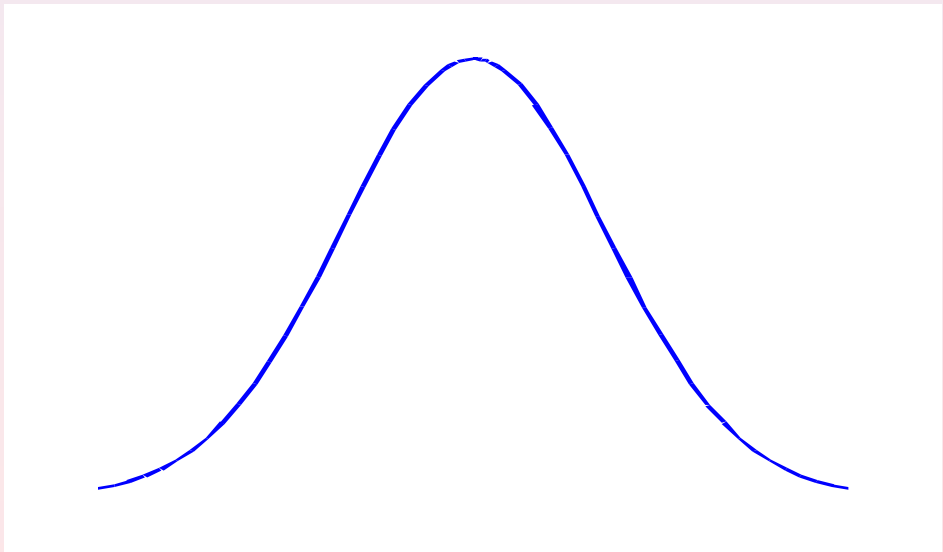


1901-1976

time domain



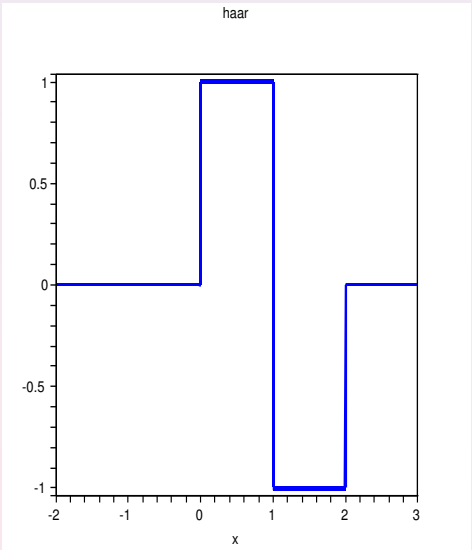
frequency domain

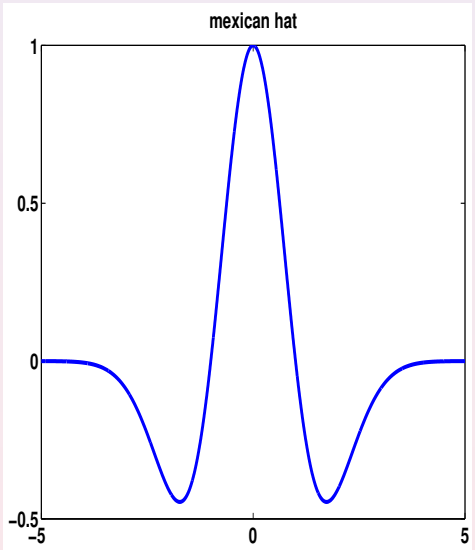


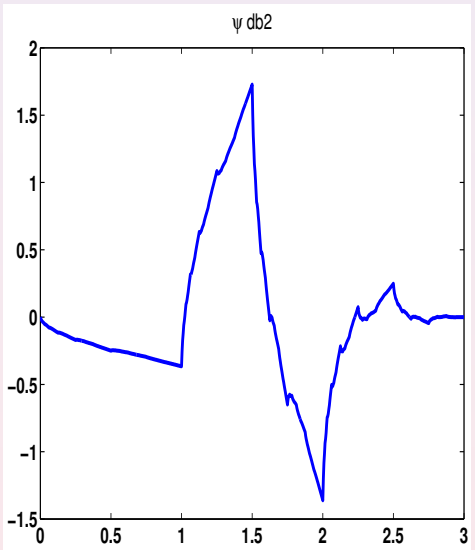
$$\sigma_t \sigma_\omega \geq \frac{1}{4}$$

Wavelets

- limited frequency content
- limited support in time









A. Haar
1885-1933

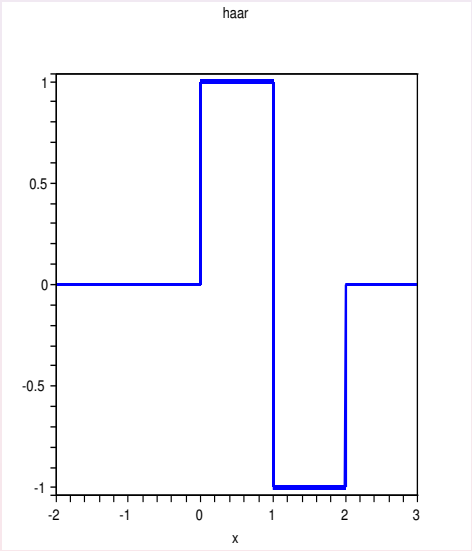


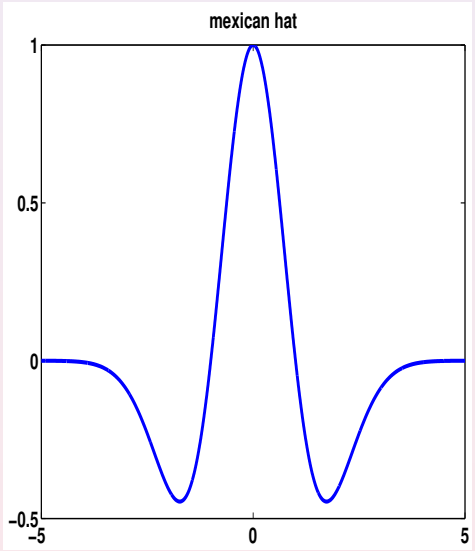
I. Daubechies
1954

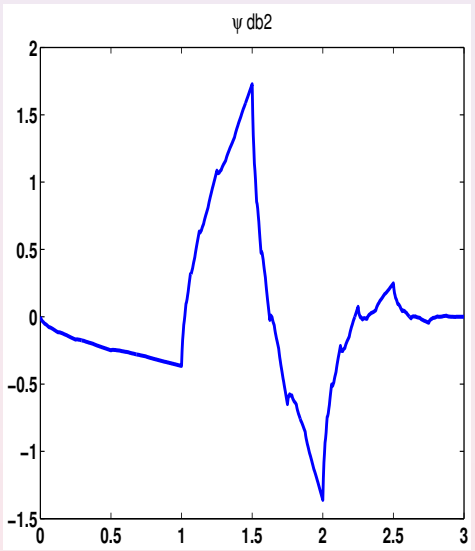
The integral is zero. Hence *wavelet*

Wavelets

- limited frequency content
- limited support in time









A. Haar
1885-1933

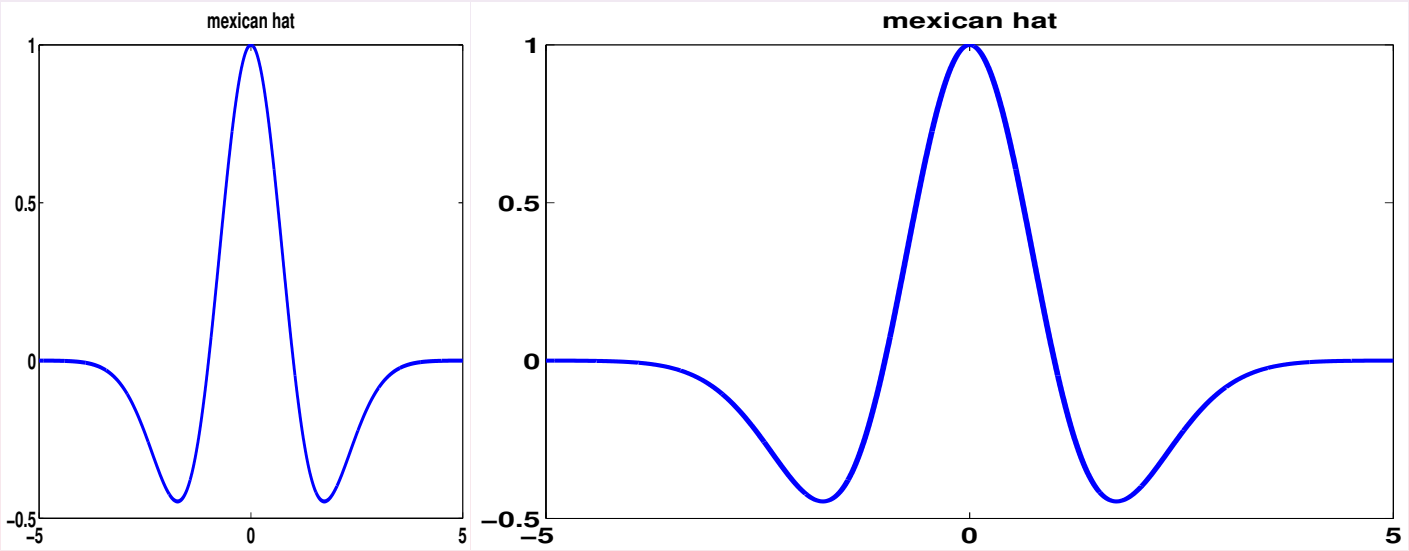


I. Daubechies
1954



Wavelets

- compact support $\Rightarrow$ needs translates + dilates



- two parameters, 1 "mother function"

$$x(t) = \sum_{k,l} c_{k,l}\psi_{k,l}(t), \quad \psi_{k,l}(t) = 2^{k/2}\psi(2^k t - l)$$

Wavelets

Advantages

- local effect in both domains
- fast algorithm
- (bi)orthogonal basis
- additional properties (symmetry, smoothness,...)
- multiresolution/hierarchy

Disadvantage

- no explicit formula

Multiresolution

$$W_k = \operatorname{span}\{\psi_{k,l}(t)\}_{l=-\infty}^{\infty} = \{\psi_{k,0} \text{ and its translates}\}$$

$$x(t) = \underbrace{\underbrace{x_0(t)}_{\in V_0} + \underbrace{w_0(t)}_{\in W_0}}_{x_1(t) \in V_1} + \underbrace{w_1(t)}_{\in W_1} + \cdots$$

- Each W_k adds a layer of detail
- WT splits signal in layers of resolution
- $V_n = V_0 + W_0 + W_1 + \cdots + W_{n-1}$ = approximation space
- $V_n = \operatorname{span}\{\varphi_{n,l}\}_l = \{\varphi_{n,0} \text{ and translates }\}$

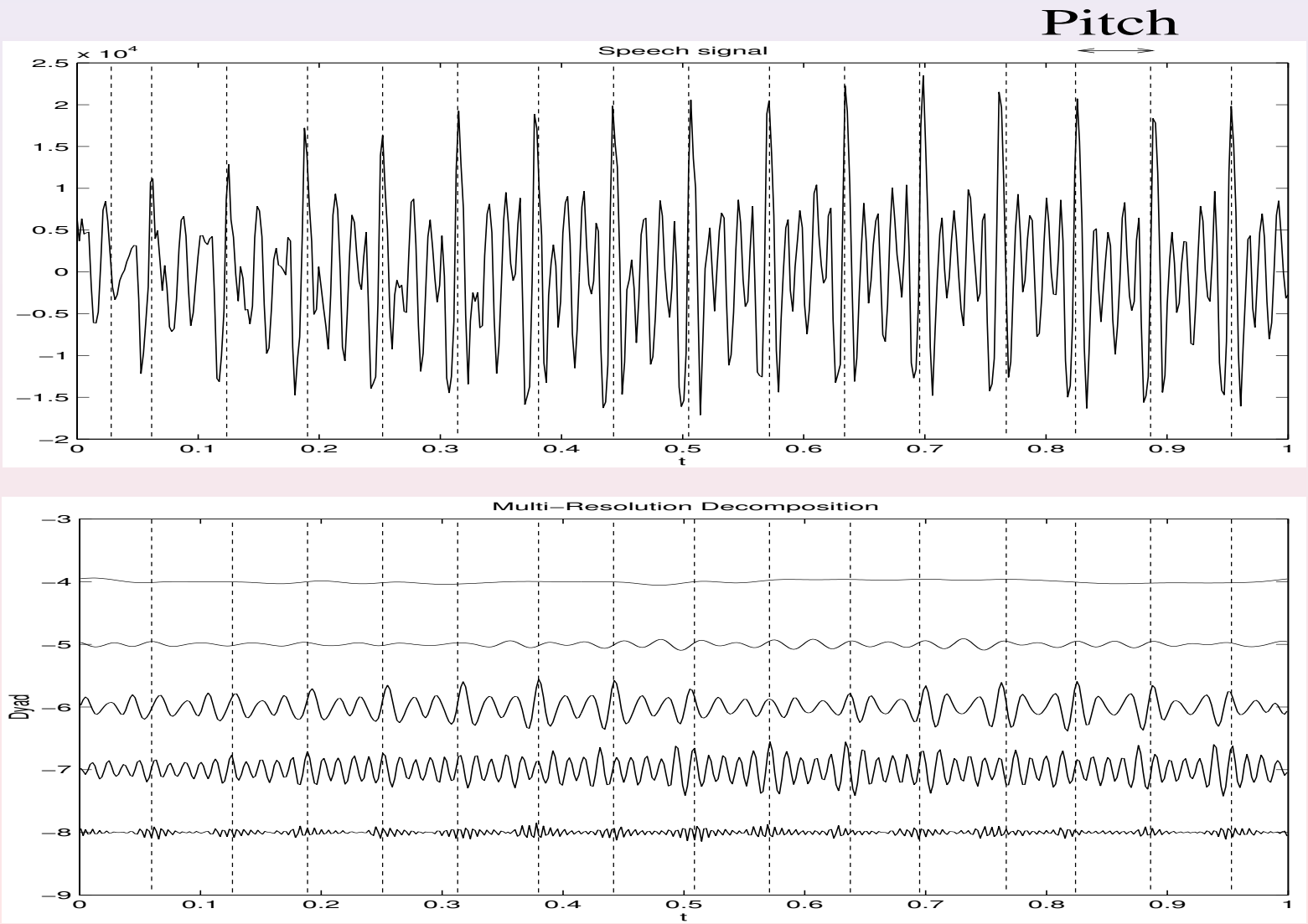
Multiresolution

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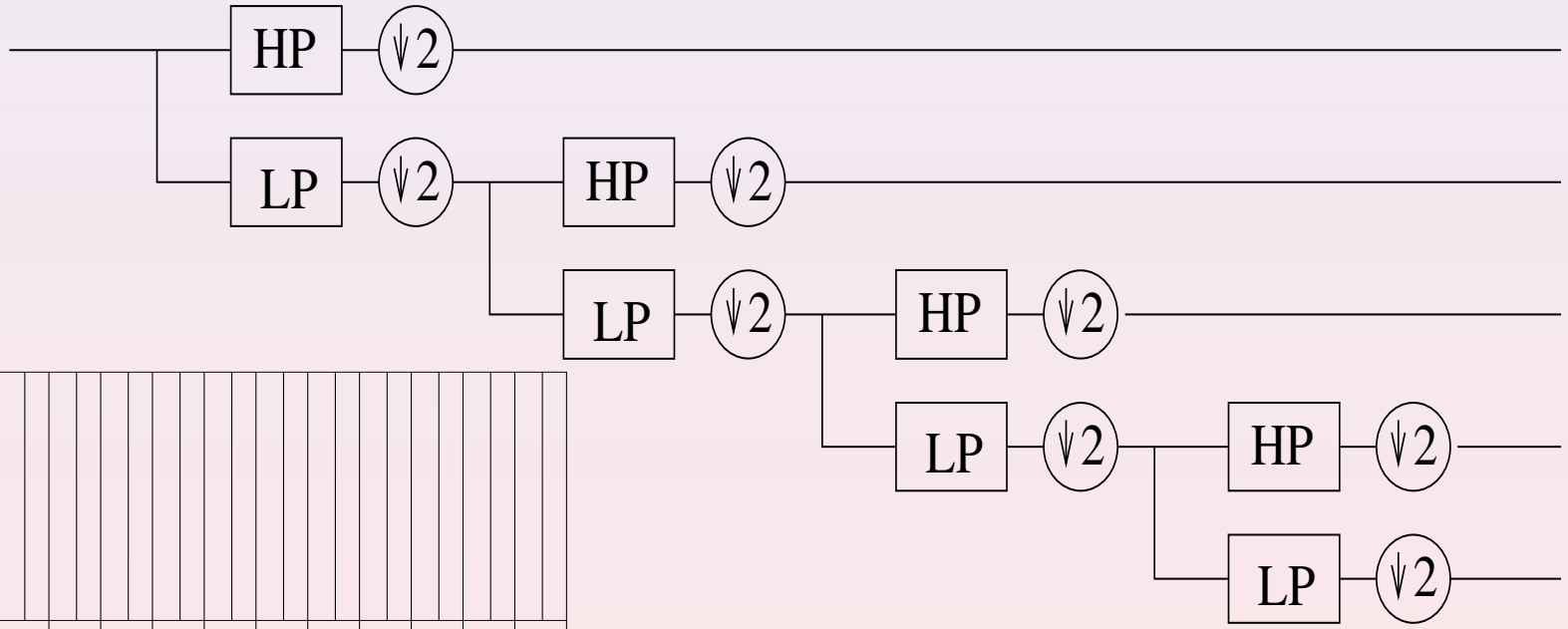
$$\begin{aligned} V_n &= V_{n-1} + W_{n-1} \quad \Rightarrow \quad x_n(t) = x_{n-1}(t) + w_{n-1}(t) \\ V_n &= \text{span}\{\varphi_{n,l}(t)\}_l \\ &= \text{span}\{\varphi_{n-1,l}(t)\}_l \cup \{\psi_{n-1,l}(t)\}_l \end{aligned}$$

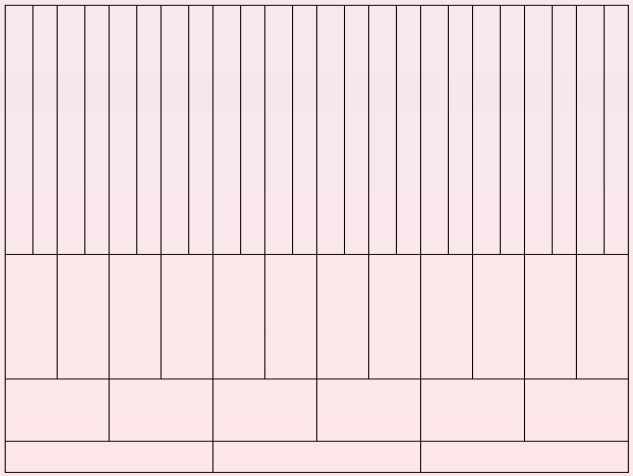
Multiresolution



Multiresolution

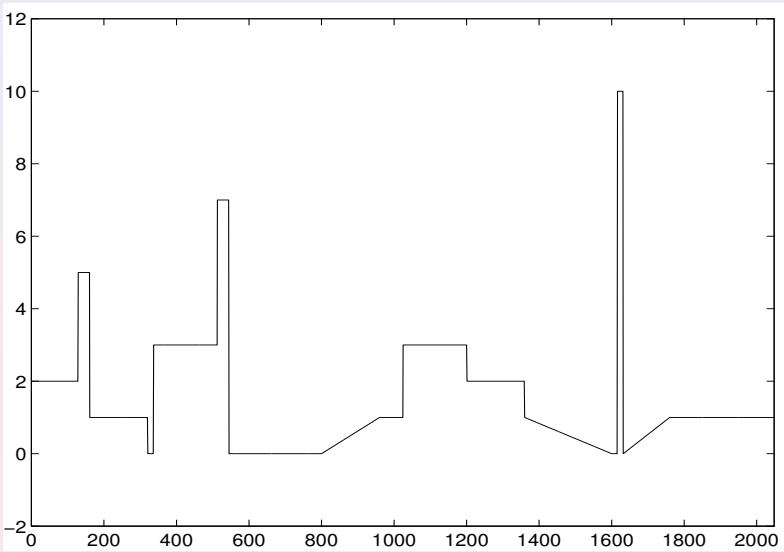
Recursive filterbank

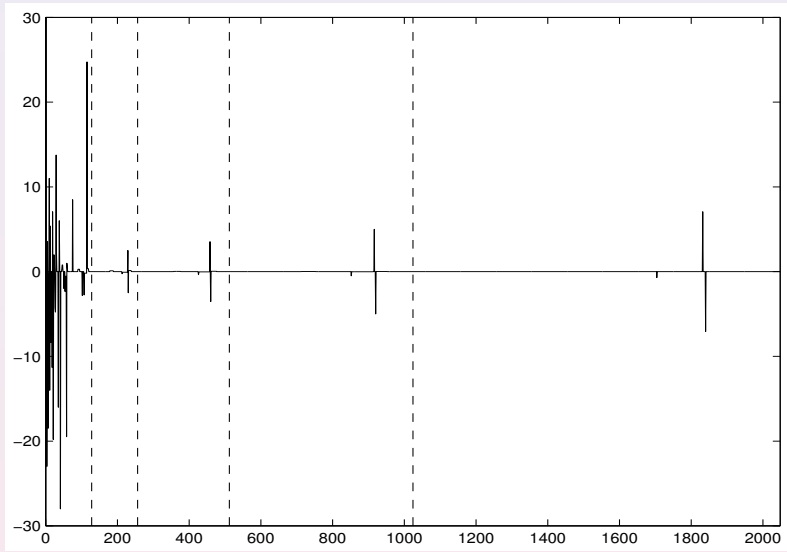


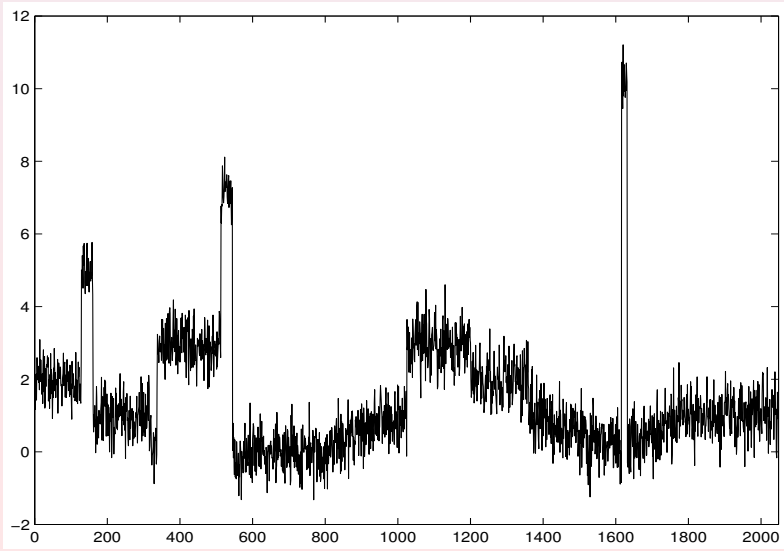


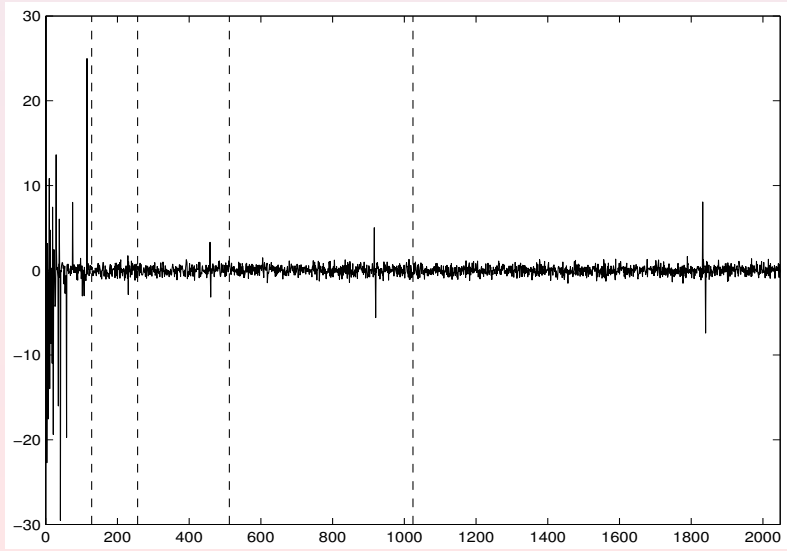
dyadic division of time-frequency plane

Noise removal

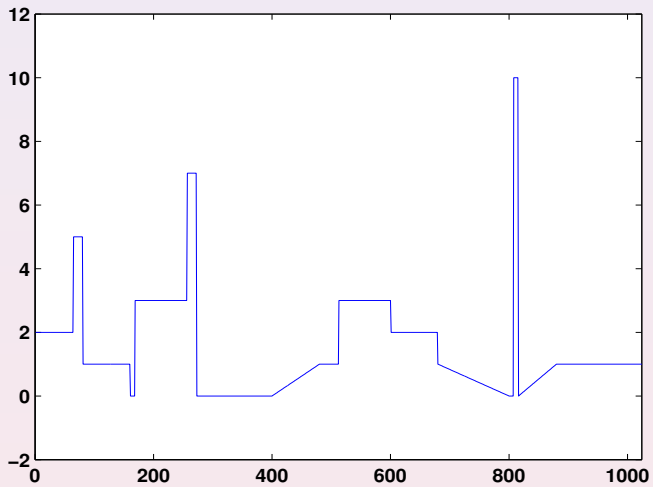


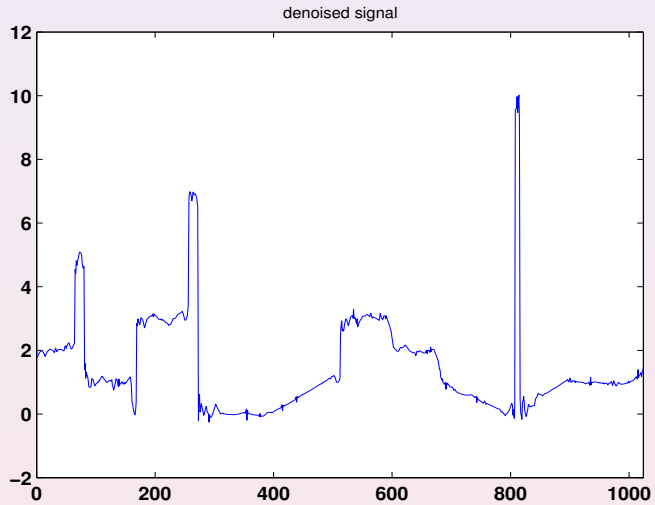






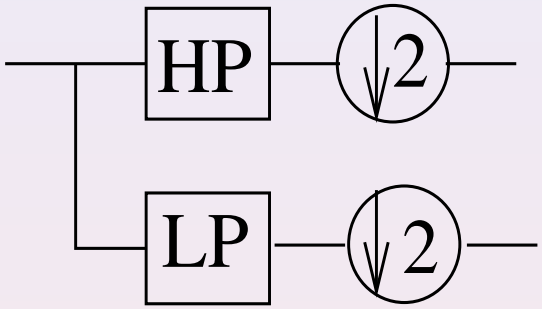
Noise removal

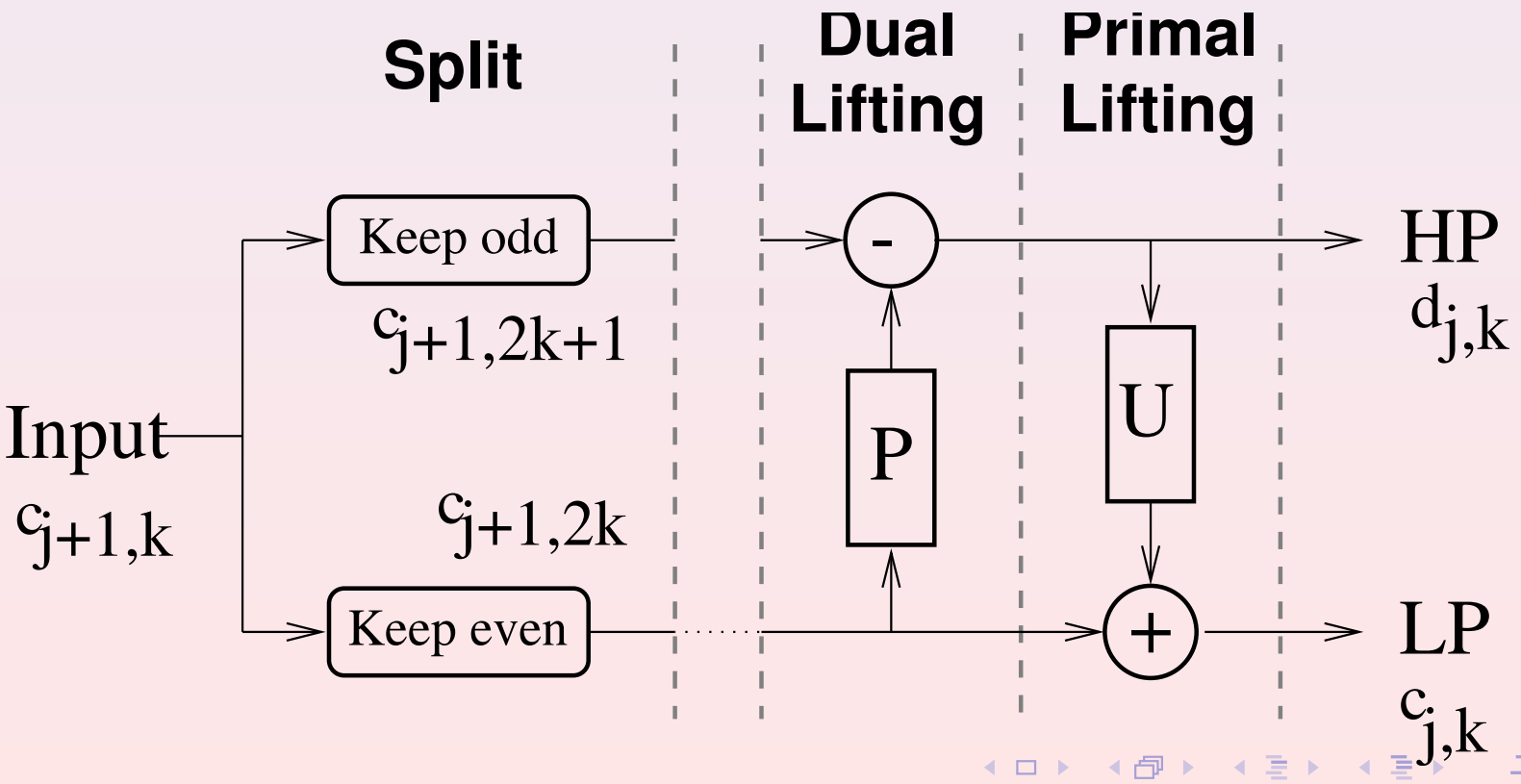




- estimate threshold
- non-additive, non-white noise
- non-equidistant samples
- ...

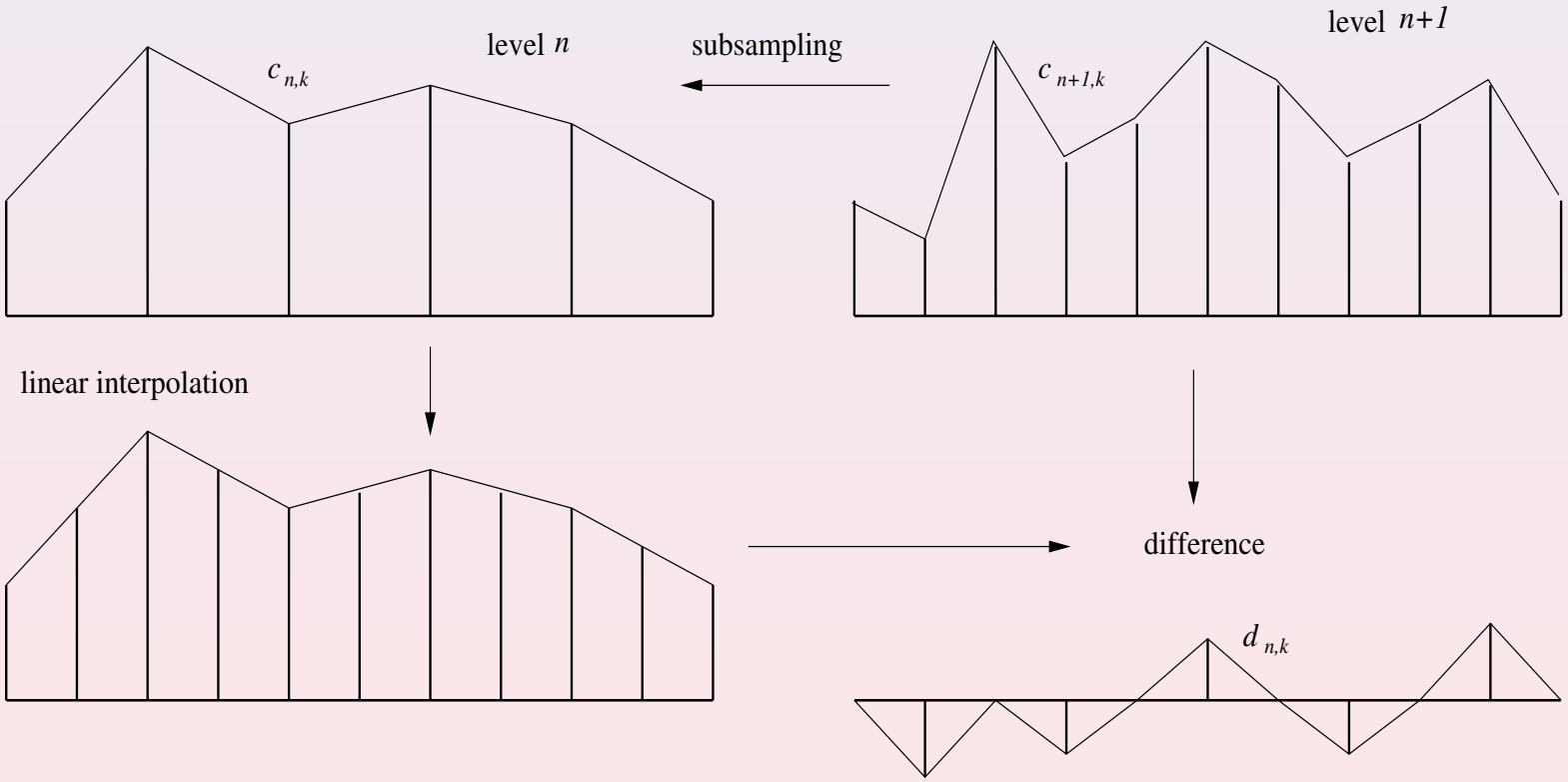
Lifting scheme





Subdivison

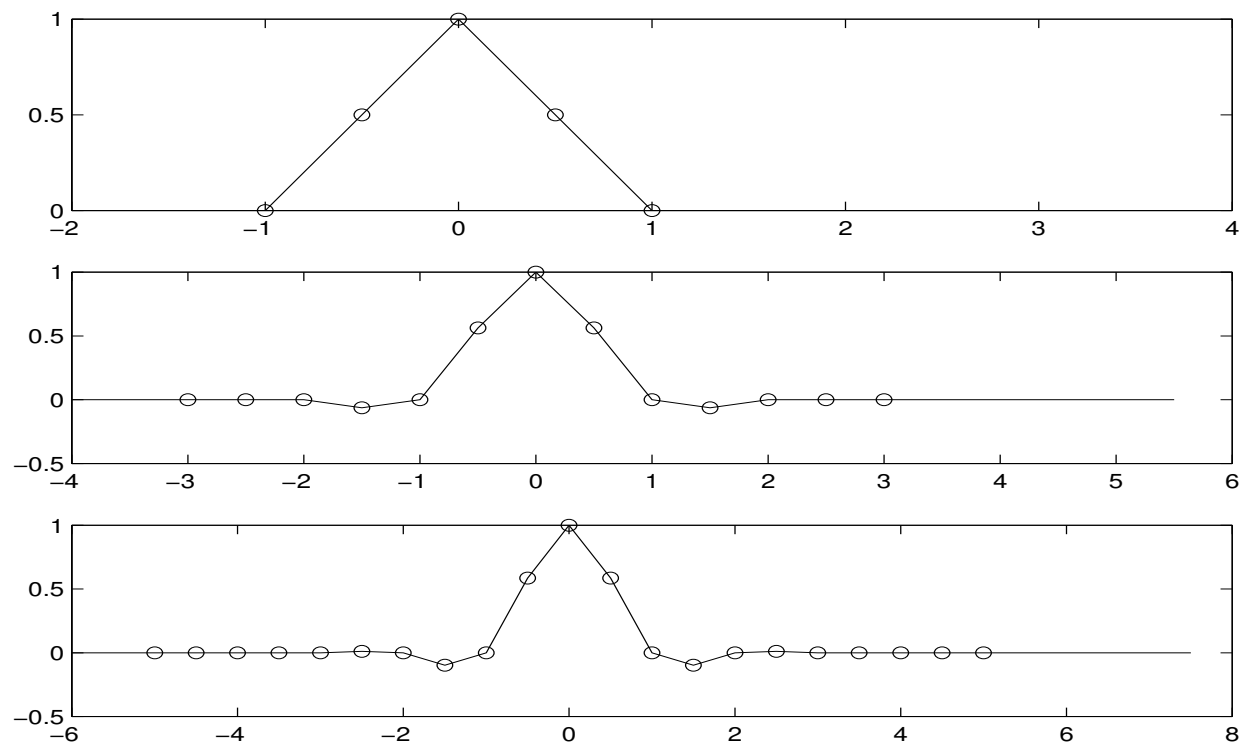
linear interpolation



Deslauriers-Dubuc subdivision

Subdivison

Scaling function φ by subdivision of $(\ldots, 0, 0, 1, 0, 0, \ldots)$, e.g. $n = 1, 3, 5$



All other φ_{kl} are dilates and translates of φ.

Summary 1D

- Wavelets
- Multiresolution
- Lifting scheme
- Subdivision

Images = 2D signals



- great variety of images
- many types of applications
- some examples ...

Image processing tasks

JPEG 2000 compression



original



compression 400



compression 600



compression 900

Image processing tasks

Image denoising



original



lowpass



wavelet

Image processing tasks

Database consulting










Image processing tasks

Edge detection



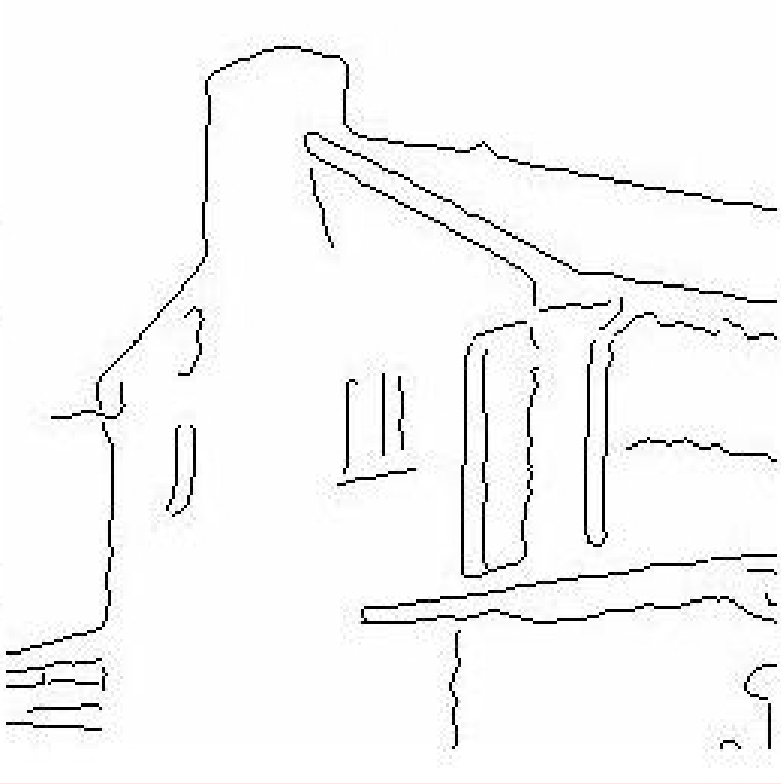
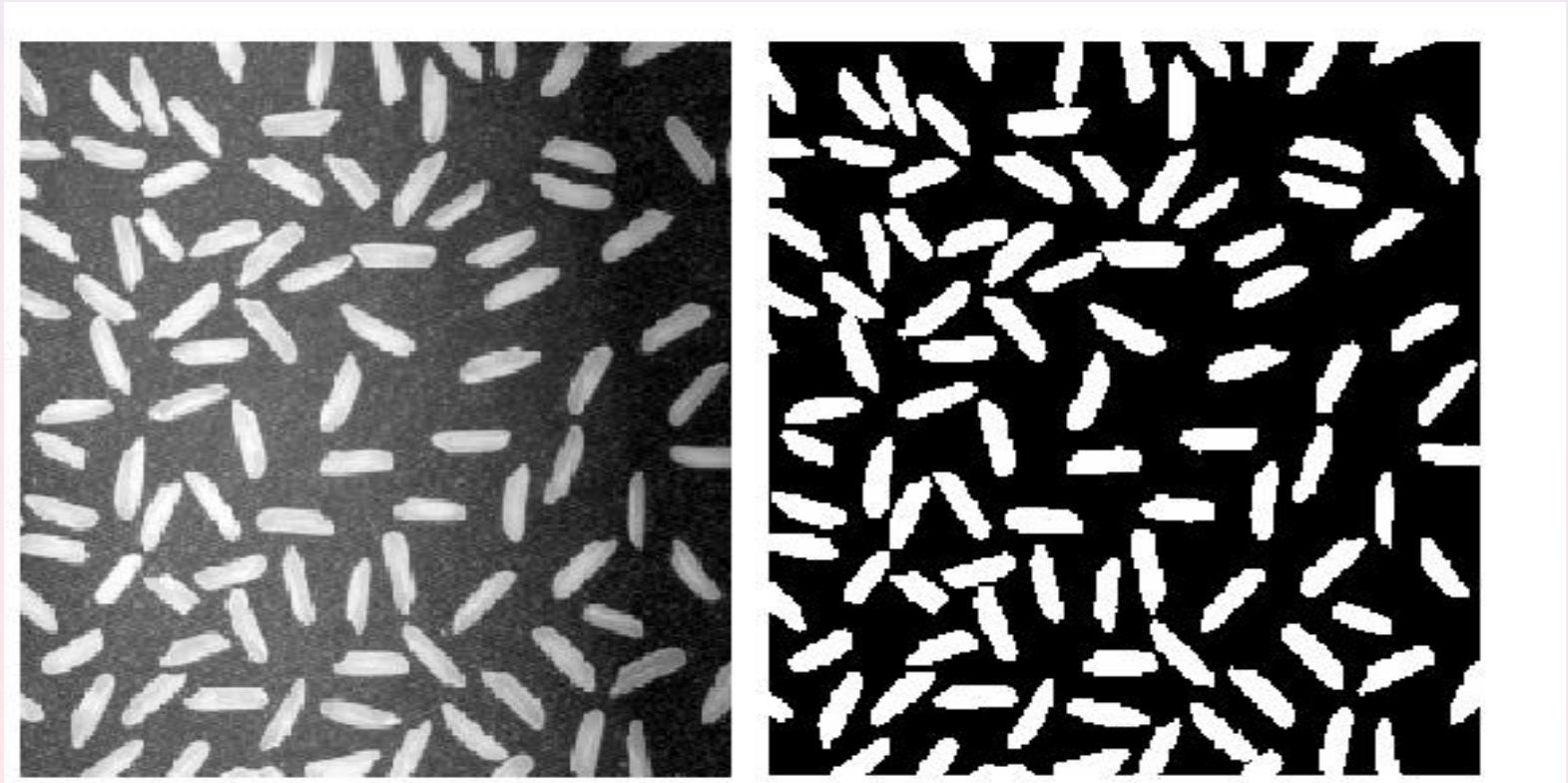


Image processing tasks

Segmentation (needs edges)



texture recognition

Image processing tasks

Segmentation

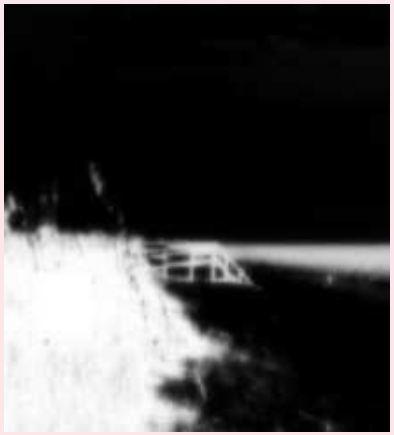
given image



sand ownership



grass ownership



sky ownership



Image processing tasks

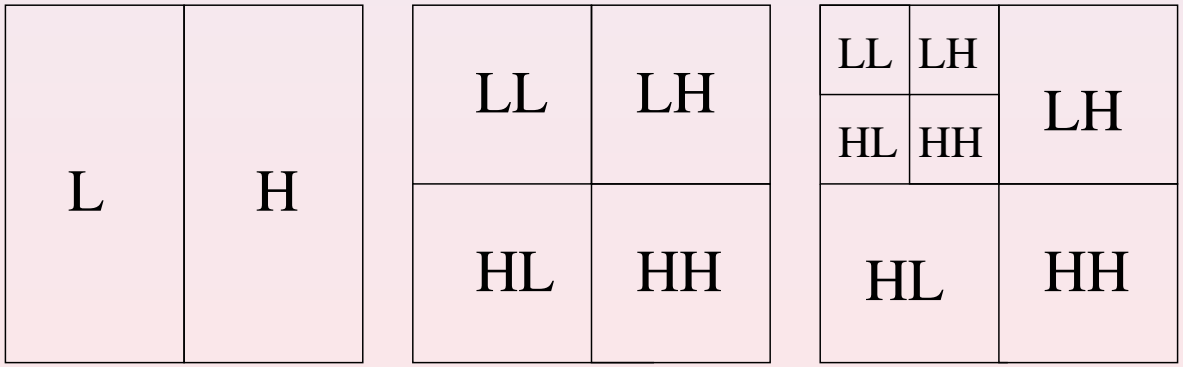
Contrast enhancement (needs edges)



2D wavelets

Tensor product wavelets

- first do wavelet transform of the rows
- then wavelet transform of columns
- repeat on LL part



2D wavelets

Tensor product wavelets



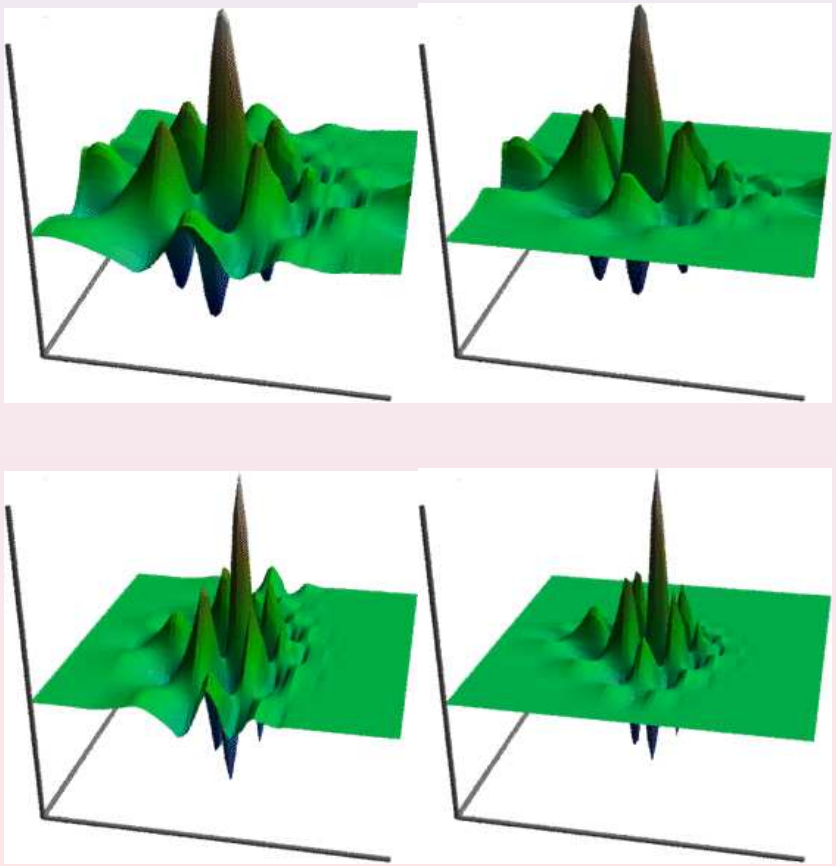


Note that

- LH = vertical edges
- HL = horizontal edges
- HH = diagonal edges

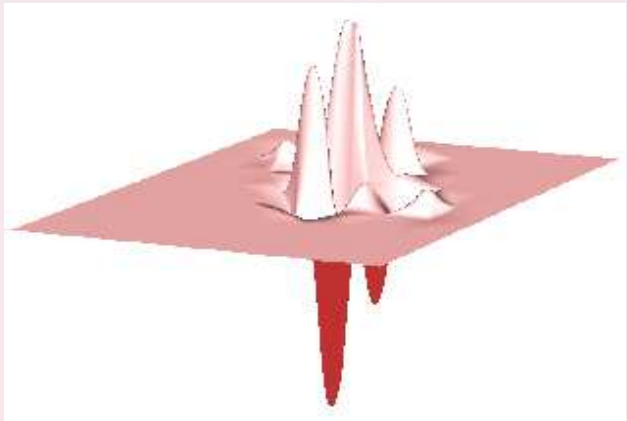
2D wavelets

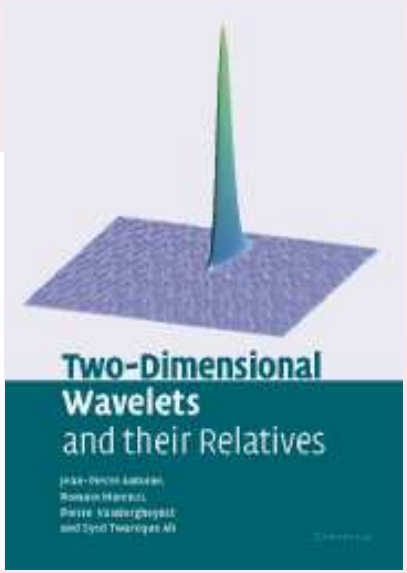
Tensor product wavelets



2D wavelets

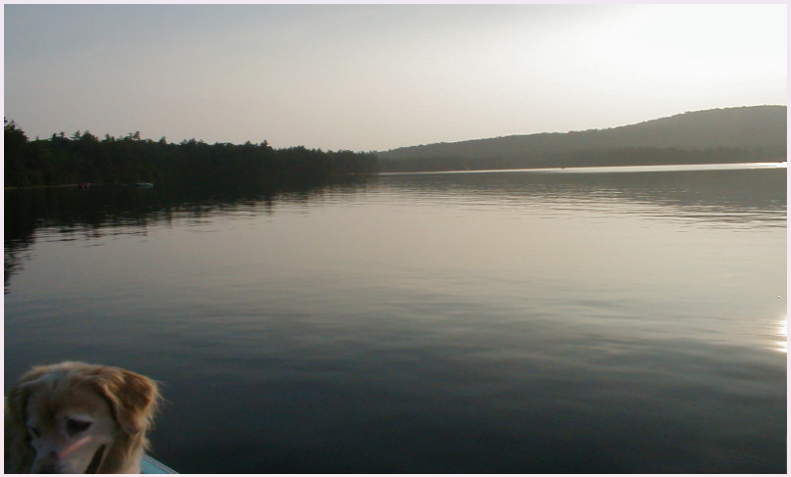
Nonseparable wavelets, curvelets, ridgelets,...





Horizon images

smooth areas with smooth edges

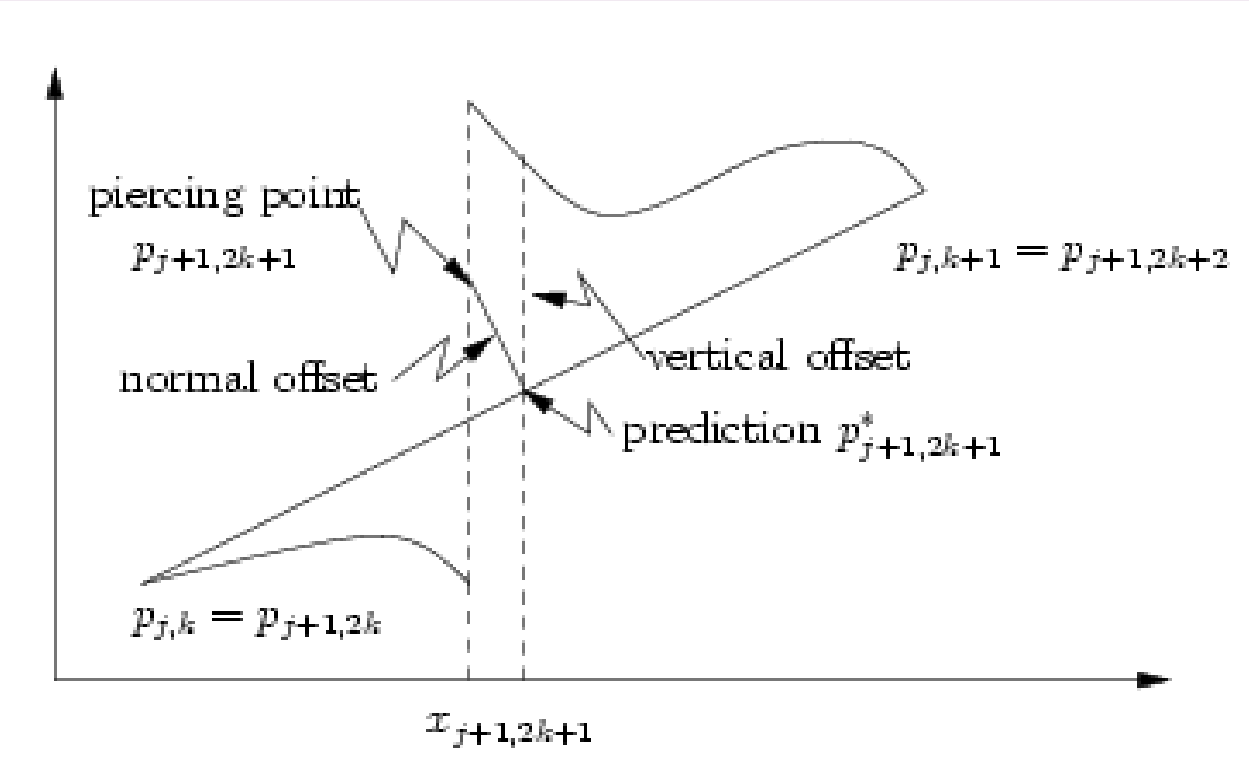






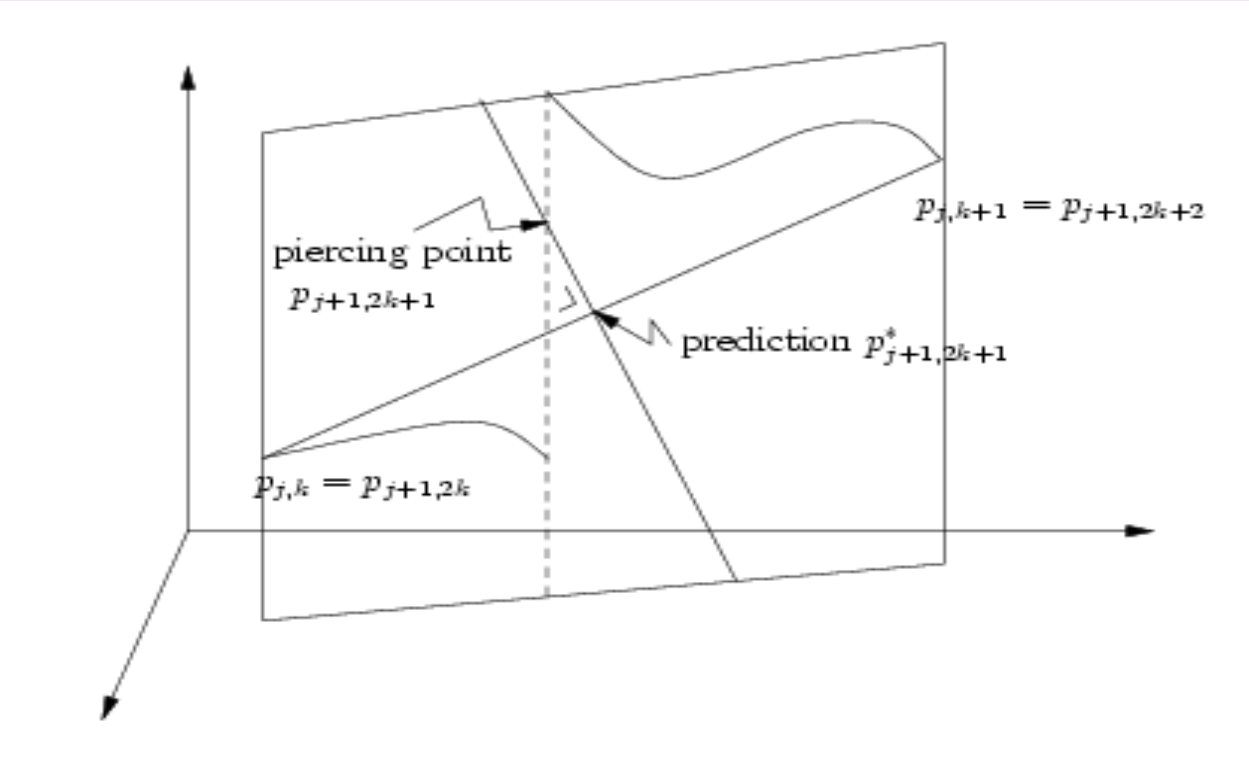
Normal offsets (1D)

Difference subdivision (lifting) step



Normal offsets (2D)

subdivision (lifting) step



length and sign of offset suffice

Normal offsets

- Loose regular grid (triangulation)
- Subdivision points attracted by singularities
- Requires extra book-keeping
- Subdivision points are on edges of coarse triangulation
- Smooth contours approximated by polylines
- Discourage edges crossing contours
- This is non-uniform splitting = extra book-keeping

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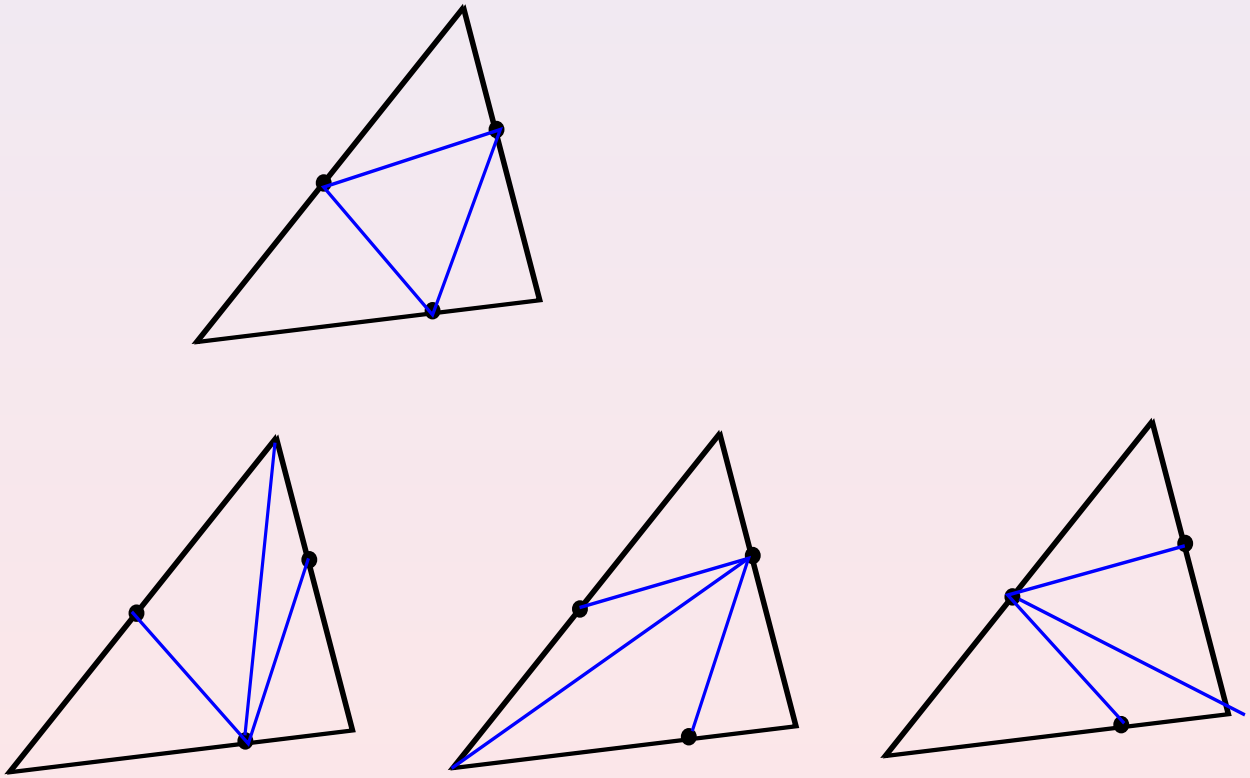
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Normal offsets (2D)

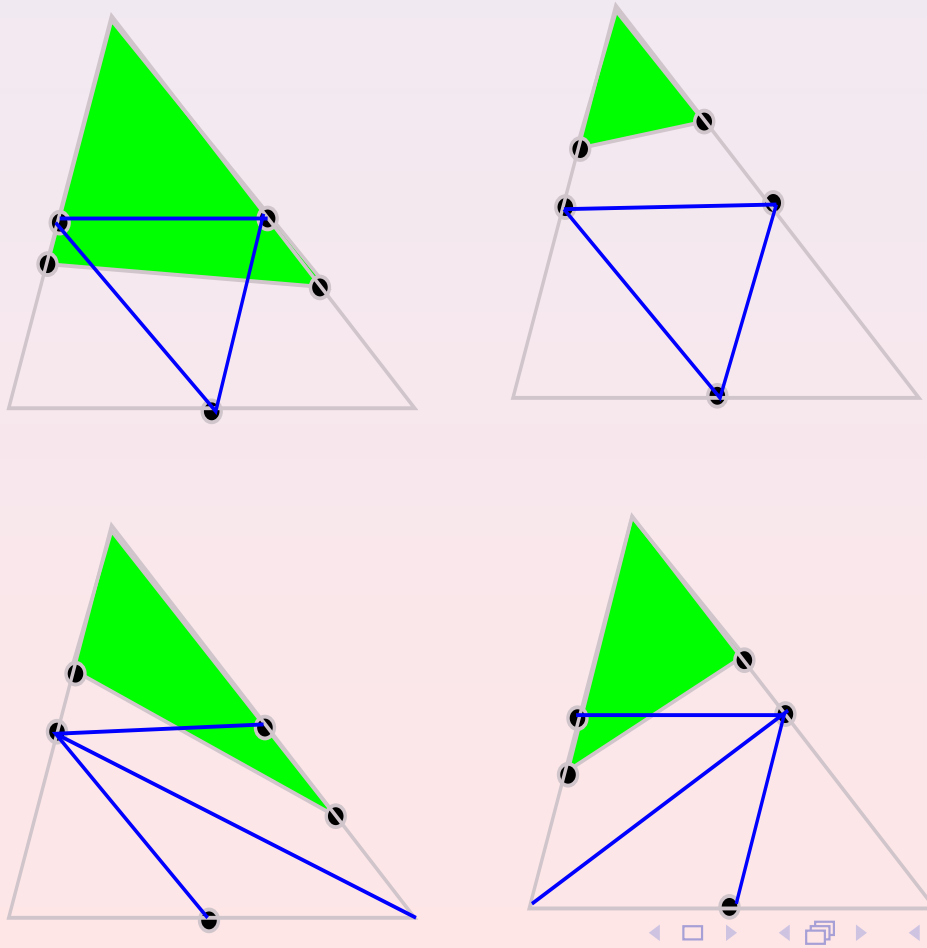
Adaptive topology



4 different ways of refining the triangulation.

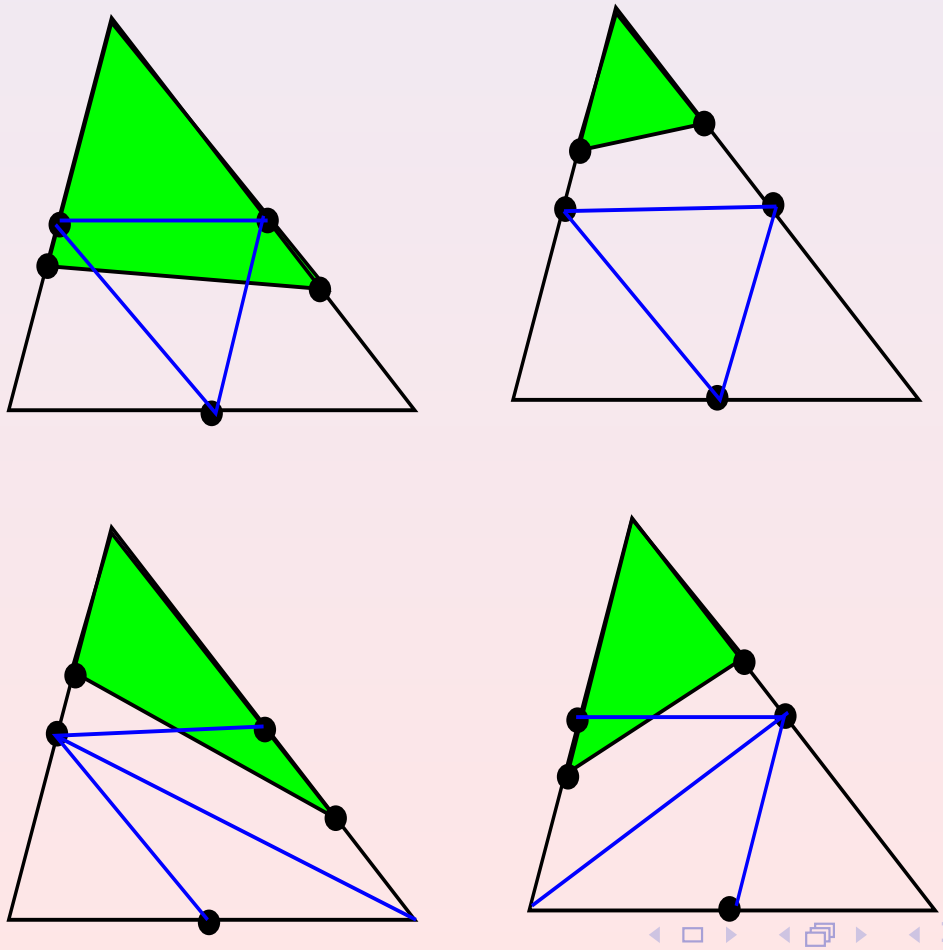
Normal offsets (2D)

- Decide on approximation error (e.g. L_2)
- For horizon images, this is much simpler:



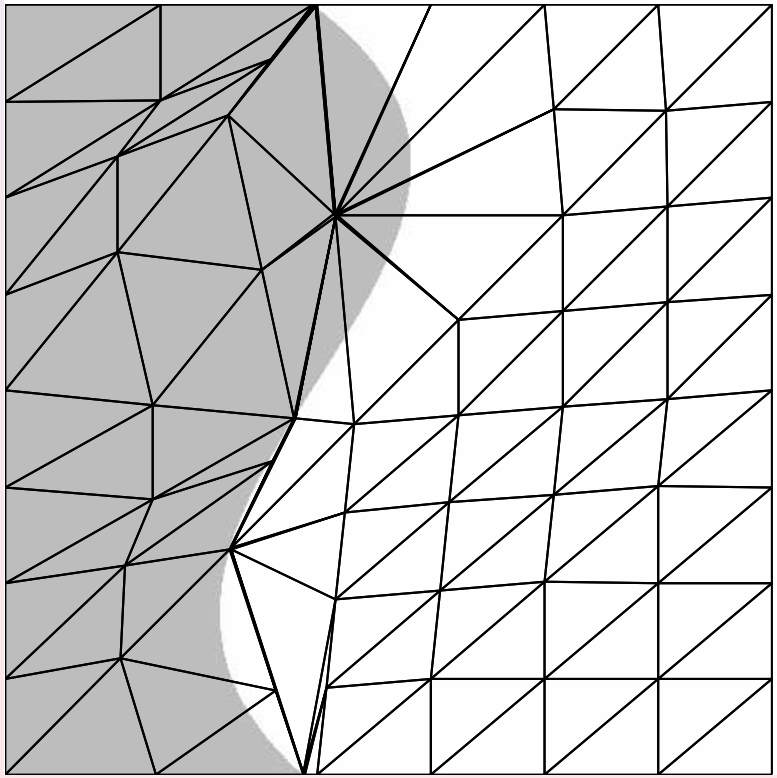
Normal offsets (2D)

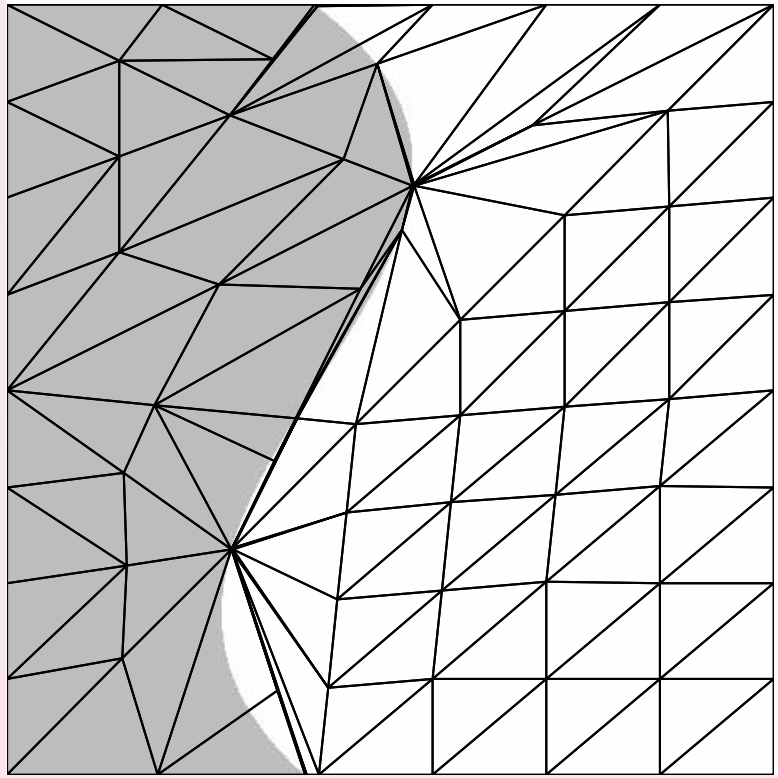
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Normal offsets (2D)

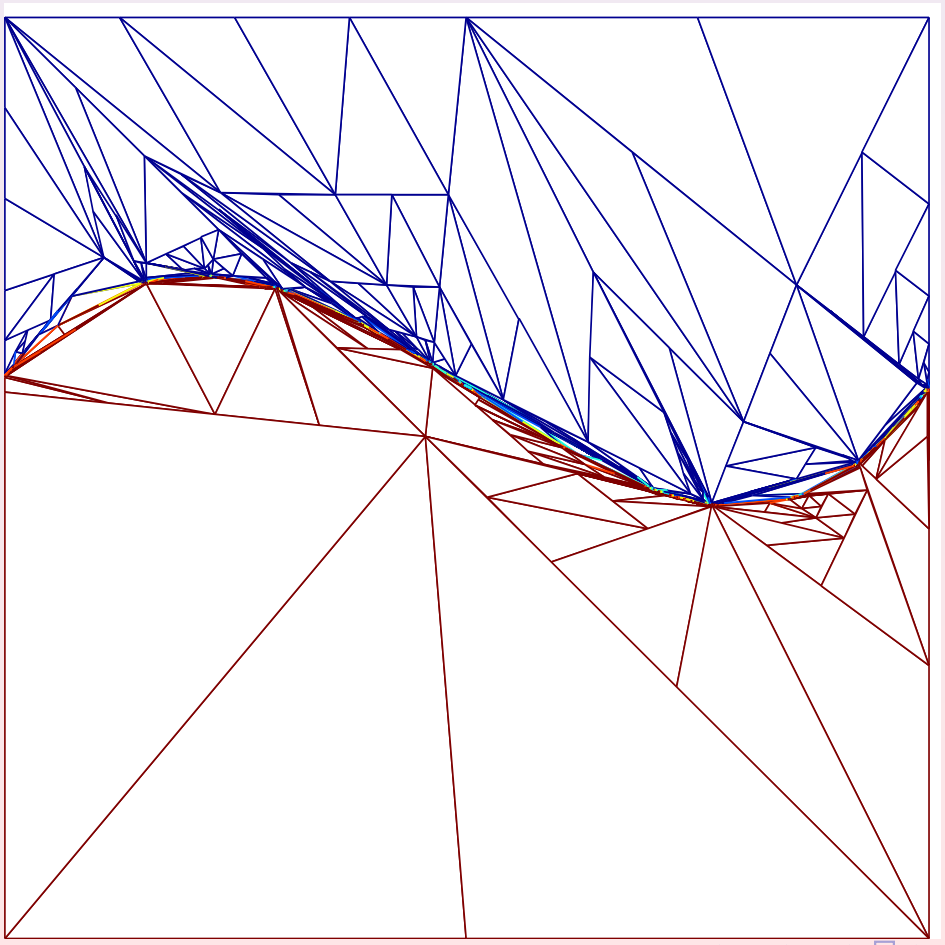
Uniform vs adaptive splitting (horizon image level 3)





Normal offsets (2D)

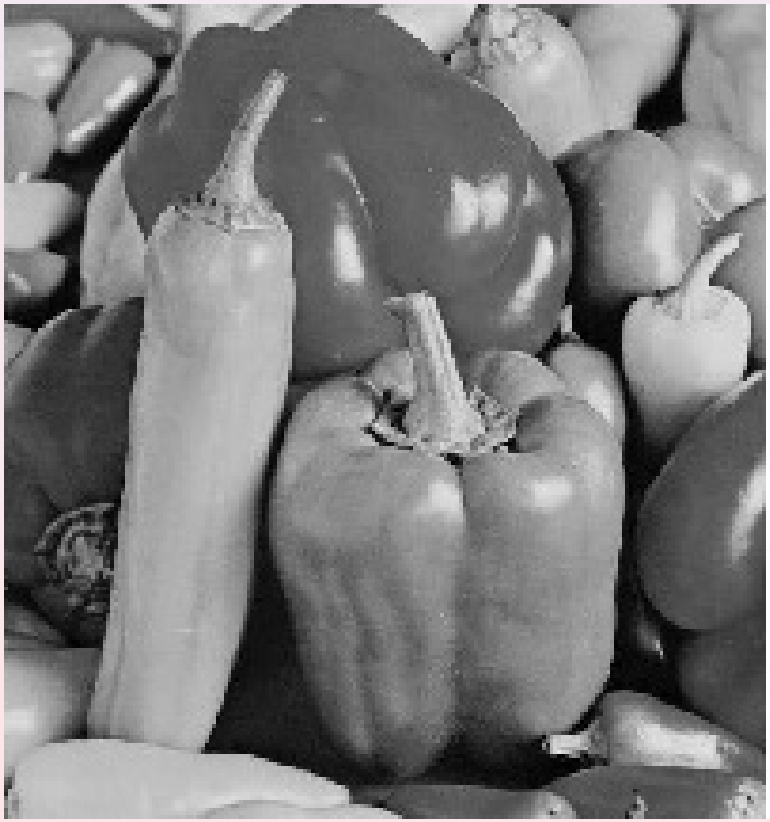
Compression: only subdivide triangles that have edge-crossings.
e.g. horizon image level 7.



Normal offsets (2D)

Compression after 7 steps

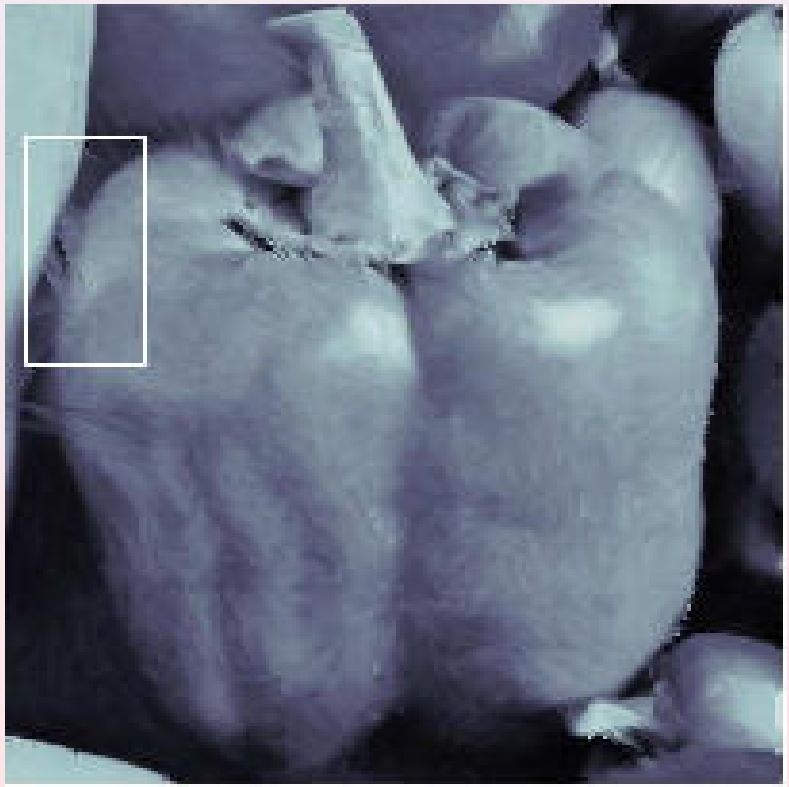
$256^2 \rightarrow \#\text{edges} = 65536 \rightarrow 5570 = 12 \rightarrow 1$ (no proning)

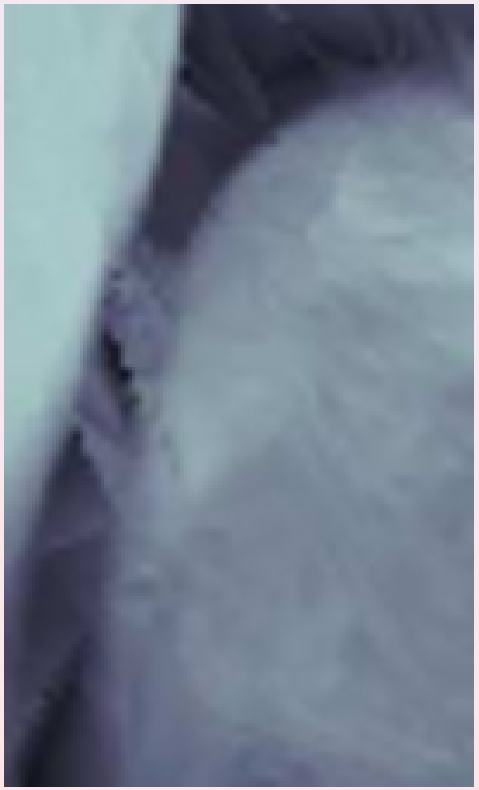




Normal offsets (2D)

Normal offsets may give artifacts for edges that are close.
Detail of previous compressed image.





Normal offsets triangulation

Triangles contain a lot of information (edges)

use normal offsets



use wavelets here

triangulation



3d view



Summary 2D

- Many different problems
- Edges are important
- Subdivision: one example = normal offsets

$$\mathbf{p}^{j+1} = \mathbf{S}\mathbf{p}^{j}$$

S = nonlinear, p^j = points in 3D

images are like surfaces

Surface representations



Different representations

- Mathematical expression
- Polygon model
- Procedural definition

Mathematical expressions

- Most used in current software
- Splines, NURBS
- Coordinates given by continuous, bivariate, single valued polynomials

$$\mathbf{P}(u,v) = (x,y,z) \quad \text{where} \quad \begin{aligned} x &= X(u,v) \\ y &= Y(u,v) \\ z &= Z(u,v) \end{aligned}$$

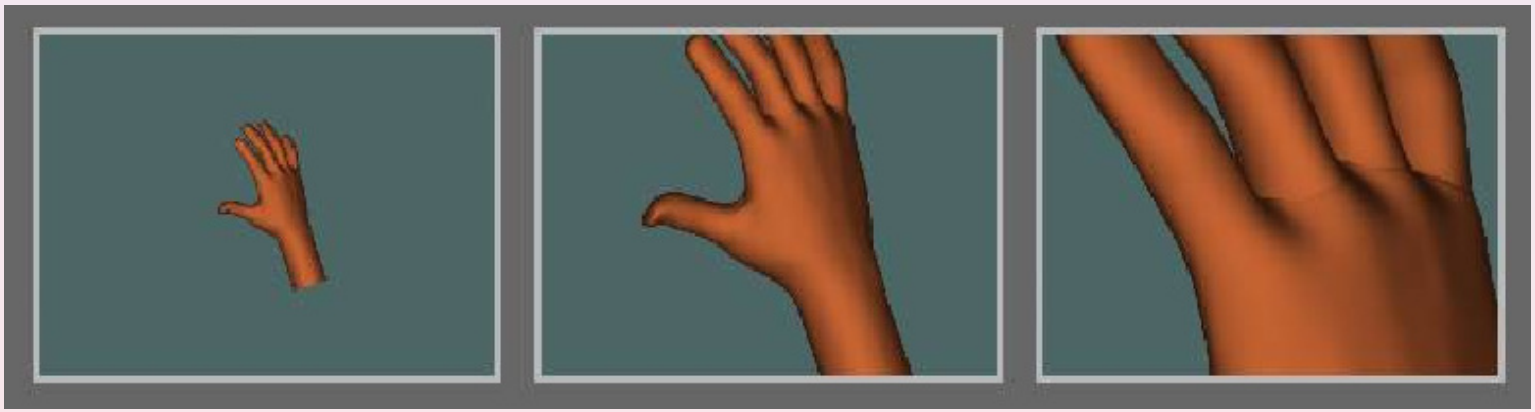
- Defined over planar domain (u,v)

Drawbacks

- Surfaces homomorphic to pieces of the plane
- Divide surfaces into patches
- Stich patches together for complex surfaces
- Preserve smoothness across boundaries
- Difficult to avoid seams
- Difficult to represent fine detail

Drawbacks

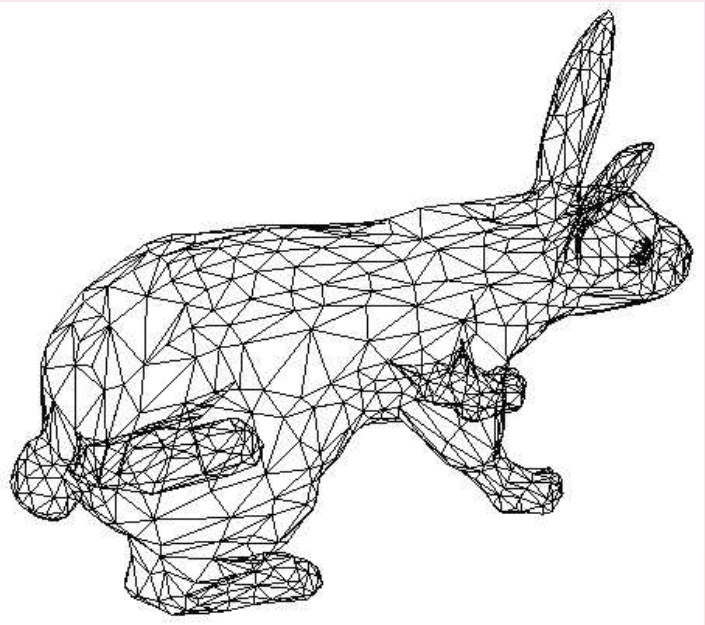
Stitch patches together..



..difficult to avoid seams

Polygon models

- Scanner output, triangulate point clouds
- Hardware rendering: triangle scan conversion
- Can represent arbitrary topology
- No stitching patches together
- Describe fine detail

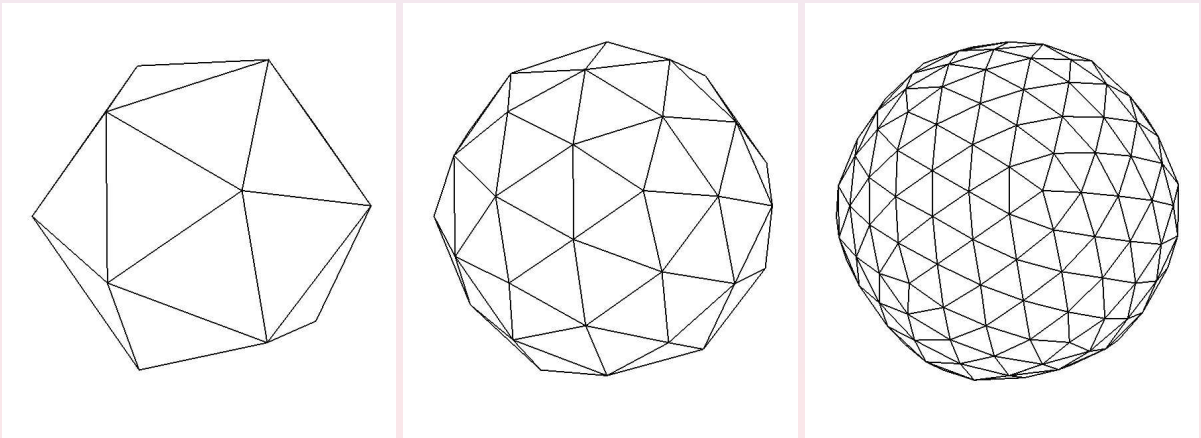


Drawbacks

- Great number of polygons needed for complex surfaces
- Difficult to manipulate
- Not well suited for smooth changes over large regions
- No underlying mathematical expression

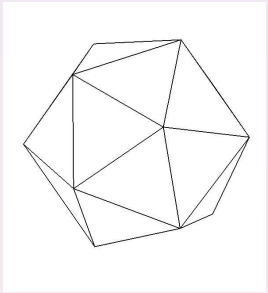
Procedural formulation

Subdivision defines a surface as the limit of a sequence of successive *refinements* of an *initial control polyhedron*



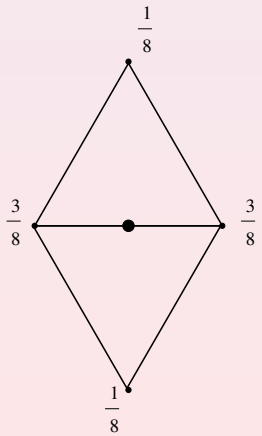
Idea of subdivision

Coarse base polyhedron:



controls overall shape of the surface

Refinement rule:



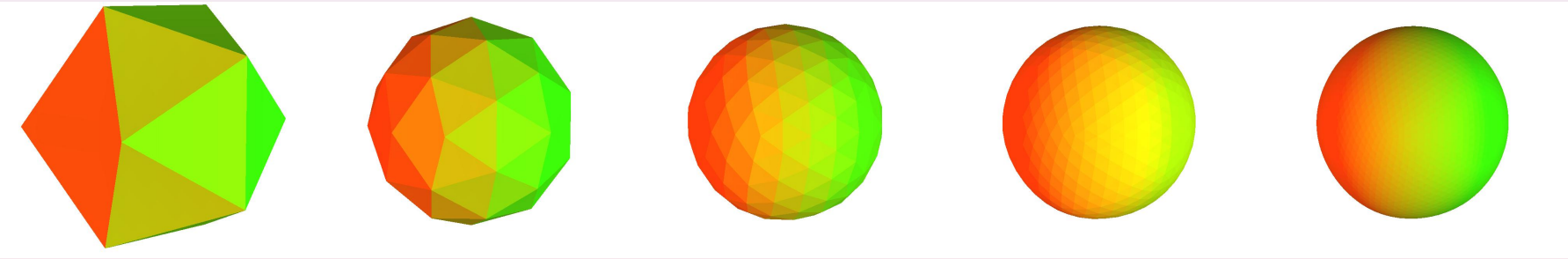
compute inbetween control points on finer scale

Advantages

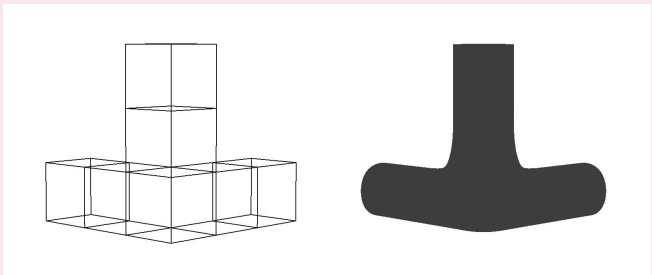
- Not restricted to images of bivariate functions
- Arbitrary topology
- No constraint management between patches
- Multiresolution applications
- Level of detail rendering
- Code simplicity
- Rendering is polygonal

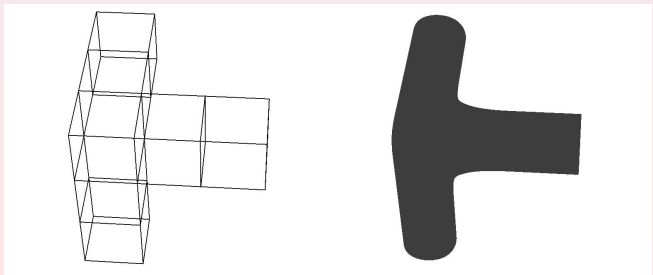
What makes a good scheme?

- Recursive application leads to smooth surface



- Affine invariance





What makes a good scheme?

- Floating point efficient
- Compact support
- Local definition
- Simplicity
- Continuity

Classification

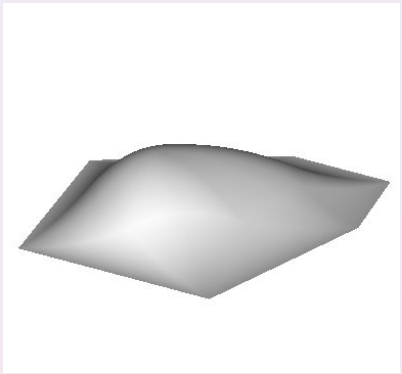
Classification criteria

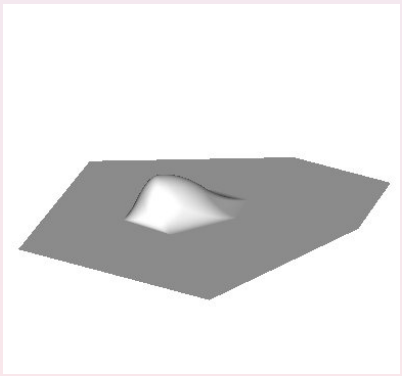
- Stationary, nonstationary
- Type of mesh, triangular, quad,..
- Approximating or interpolating

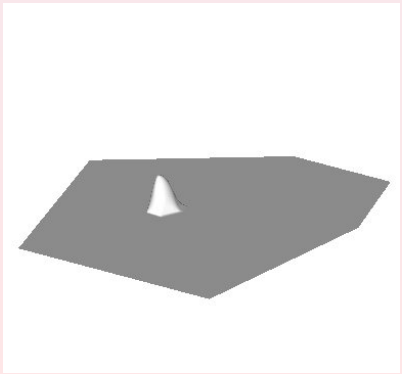
Classification

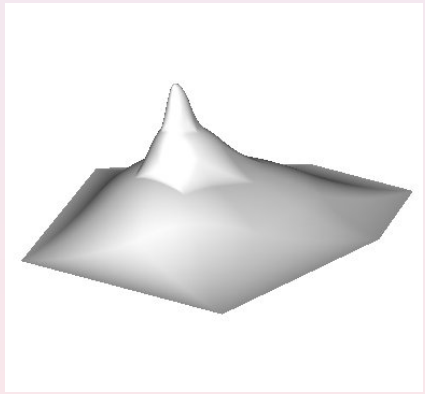
- Interpolating
  - Related to onedimensional interpolation
  - Initial control points on limit surface
  - In place implementation
  - Control surface in intuitive manner
- Approximating
  - Based on uniform spline subdivision
  - Control points change each step
  - Often better surface quality

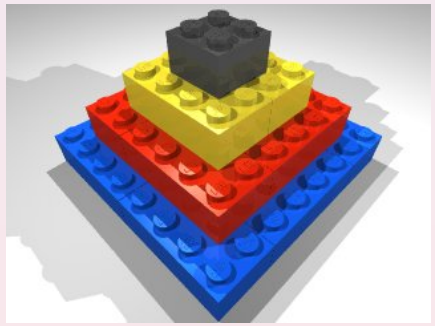
Splines as mathematical building blocks



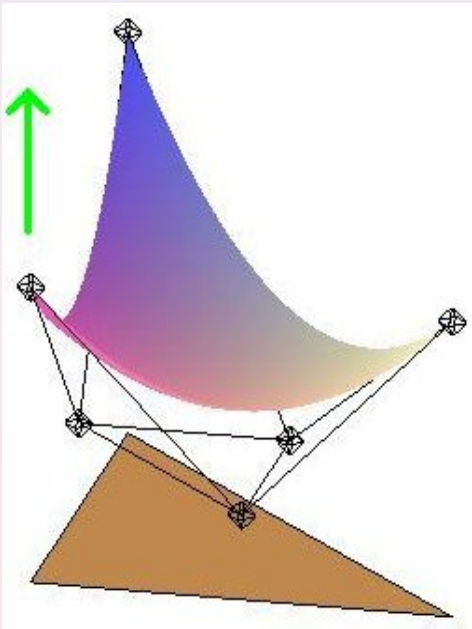




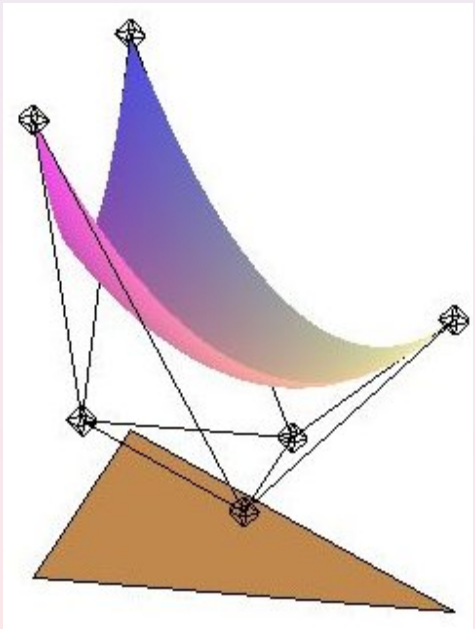




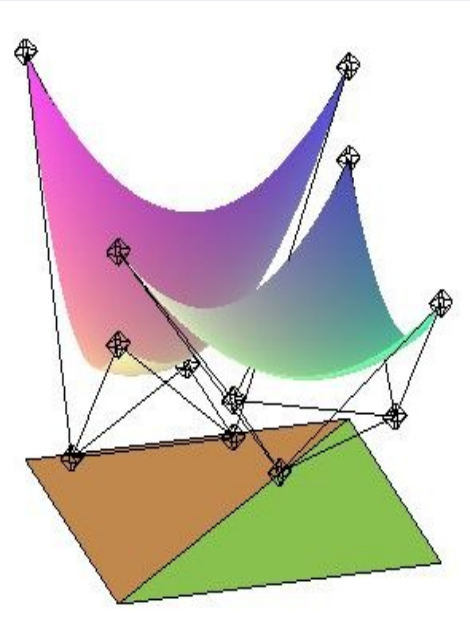
Bernstein–Bézier representation



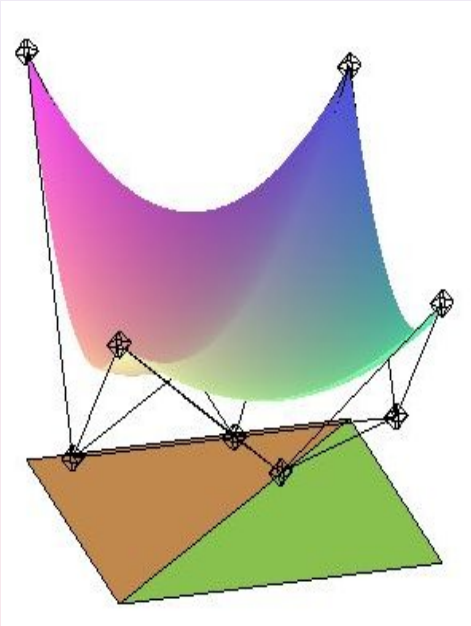
$\Longrightarrow$

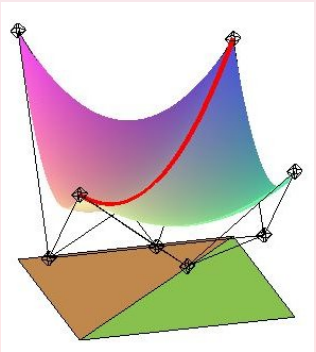


Stitching together Bézier triangles



$\Longrightarrow$

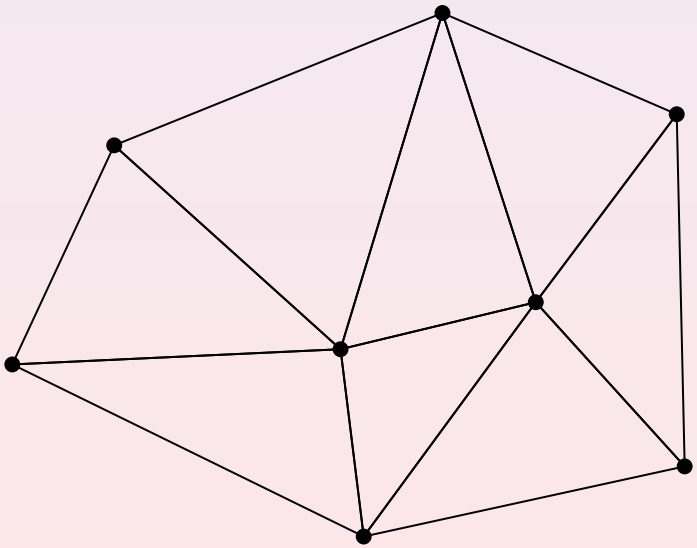




No C^1 continuity at the red curve

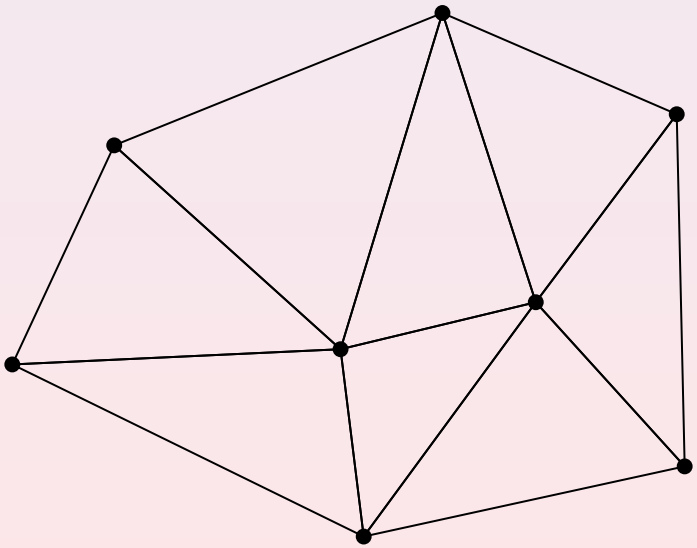
Powell-Sabin refinement

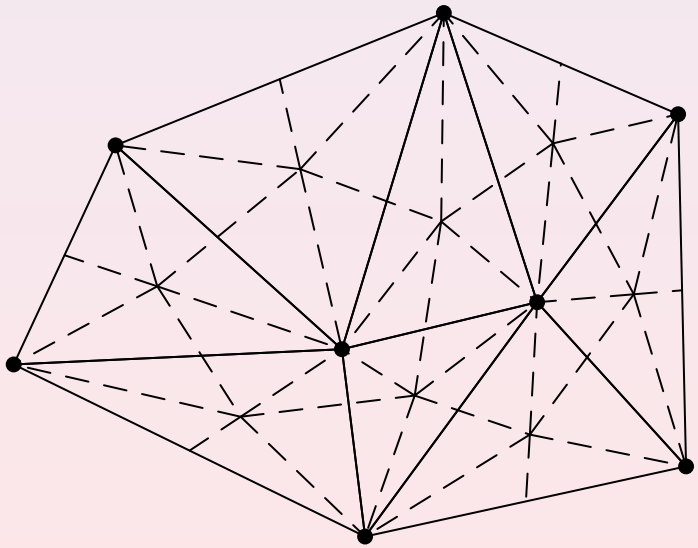
- Conformal triangulation Δ
- PS 6-split Δ^*.



Powell-Sabin refinement

- Conformal triangulation Δ
- PS 6-split Δ^*.

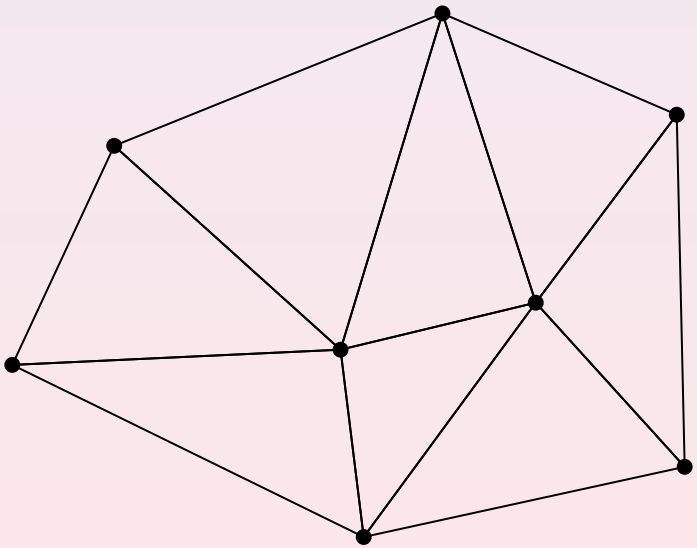


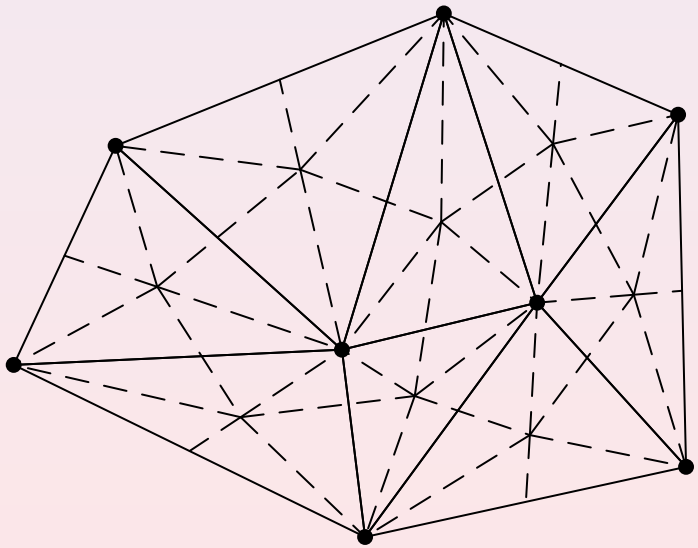


Powell-Sabin spline

$S_2^1(\Delta^*)$ = space of PS-splines

- Quadratic polynomial on triangles $t^* \in \Delta^*$
- C^1 continuity (function values and first derivative)

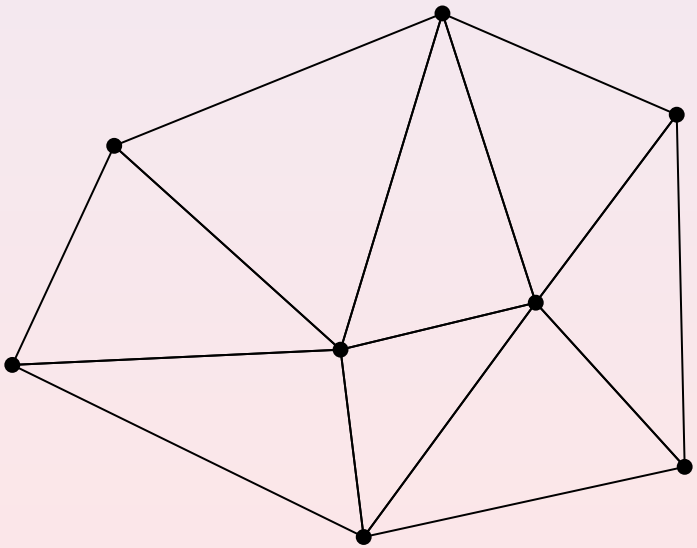


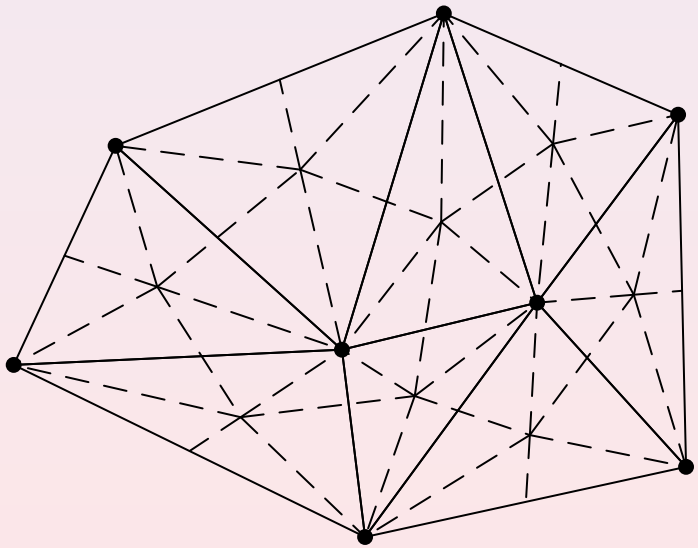


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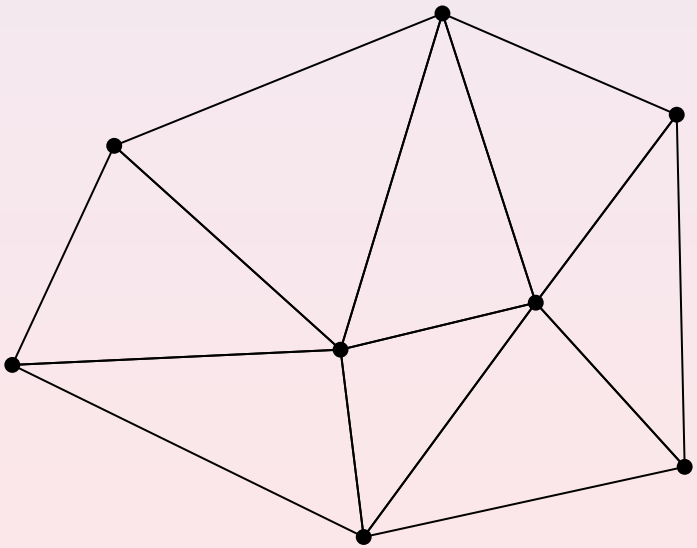


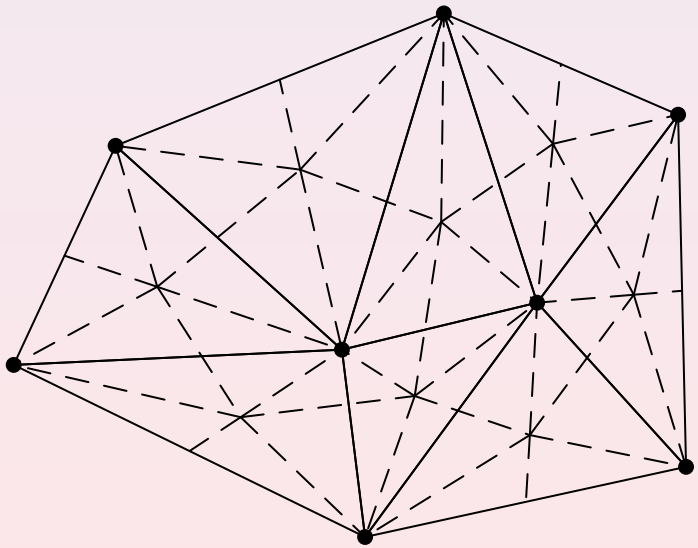


Powell-Sabin spline

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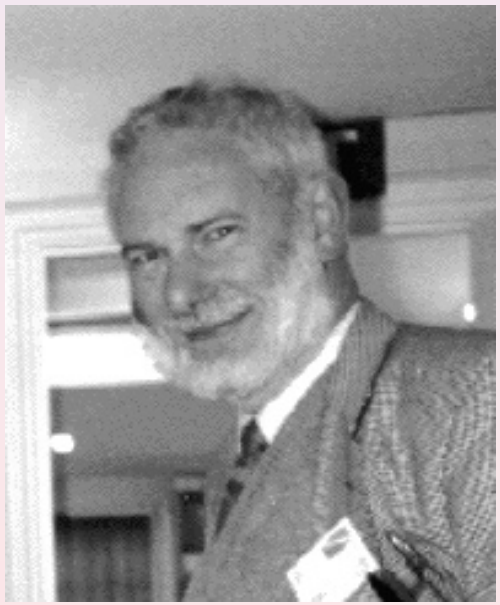


Powell-Sabin spline

$S_2^1(\Delta^*)$ = space of PS-splines

- Quadratic polynomial on triangles $t^* \in \Delta^*$
- C^1 continuity (function values and first derivative)





M.J.D. Powell & M.A. Sabin

Powell-Sabin spline Hermite interpolation

Dimension of $S_2^1(\Delta^*)$ is $3N$, $N = \#V$.
There is exactly one solution to the Hermite interpolation problem

- $s \in S_2^1(\Delta^*)$
- $s(V_i) = f_i$ *(same functionvalue)*
- $\nabla s(V_i) = \nabla f_i$ ($\nabla = [D_x, D_y]$) *(same normal)*

Thus $s(x, y)$ completely defined by the data

$$(s(V_i), s'_x(V_i), s'_y(V_i)) = (\alpha_i, \beta_i, \gamma_i), \quad i = 1, \ldots, N$$

Powell-Sabin spline, B-spline basis

$\forall V_i$: 3 basis functions B_{ij} for 3 independent $(\alpha_{ij}, \beta_{ij}, \gamma_{ij})$ values.

B_{ij}, $i = 1, \ldots, N$, $j = 1, 2, 3$

- $[B_{ij}(V_k), D_x B_{ij}(V_k), D_y B_{ij}(V_k)] = [0, 0, 0]$ for all $k \neq i$
- $[B_{ij}(V_i), D_x B_{ij}(V_i), D_y B_{ij}(V_i)] = [\alpha_{ij}, \beta_{ij}, \gamma_{ij}]$ for $j = 1, 2, 3$

Partition of unity:

- $\sum_{i=1}^{N} \sum_{j=1}^{3} B_{ij}(x, y) = 1,$
- $B_{ij}(x, y) \geq 0$

$$s(x, y) = \sum_{i=1}^{N} \sum_{j=1}^{3} c_{ij} B_{ij}(x, y)$$

Powell-Sabin spline, B-spline basis

$\forall V_i$: 3 basis functions B_{ij} for 3 independent $(\alpha_{ij}, \beta_{ij}, \gamma_{ij})$ values.
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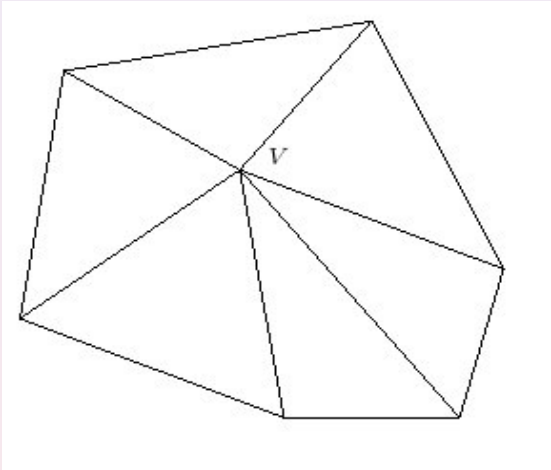
Partition of unity:

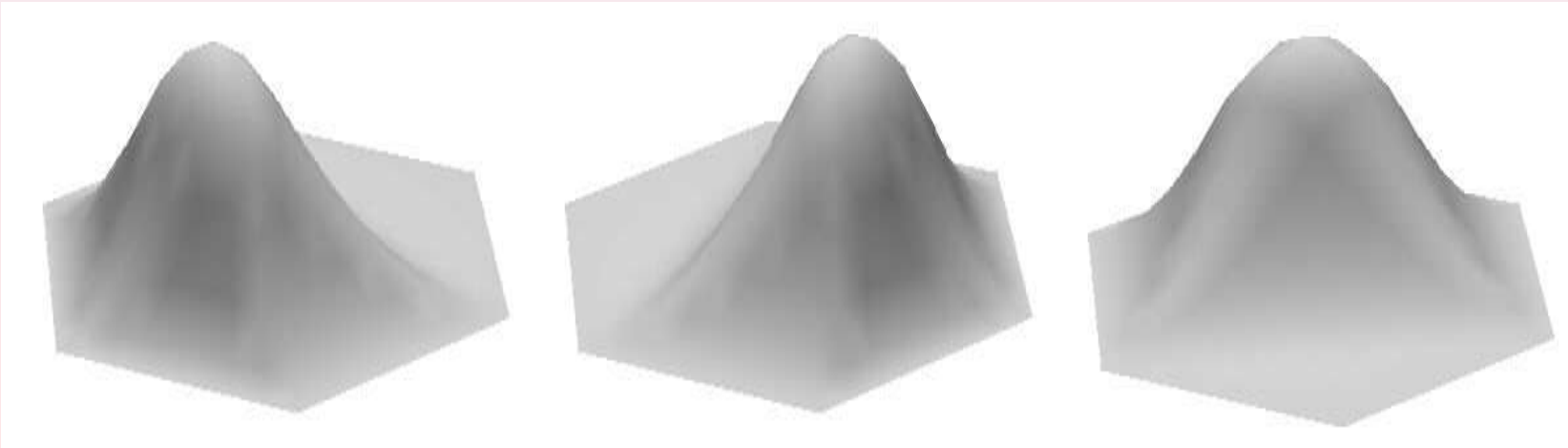
- $\sum_{i=1}^{N} \sum_{j=1}^{3} B_{ij}(x, y) = 1,$
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Powell-Sabin spline, B-spline basis

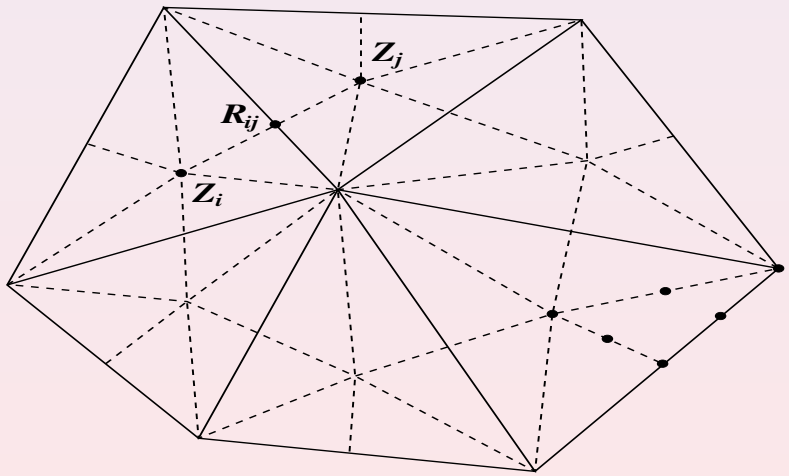
three B-splines





Powell-Sabin spline, B-spline basis

B-splines are local basis functions.
They differ from zero on a molecule M_i
I.e. all triangles of Δ having V_i as a vertex.



Powell-Sabin spline, control triangle

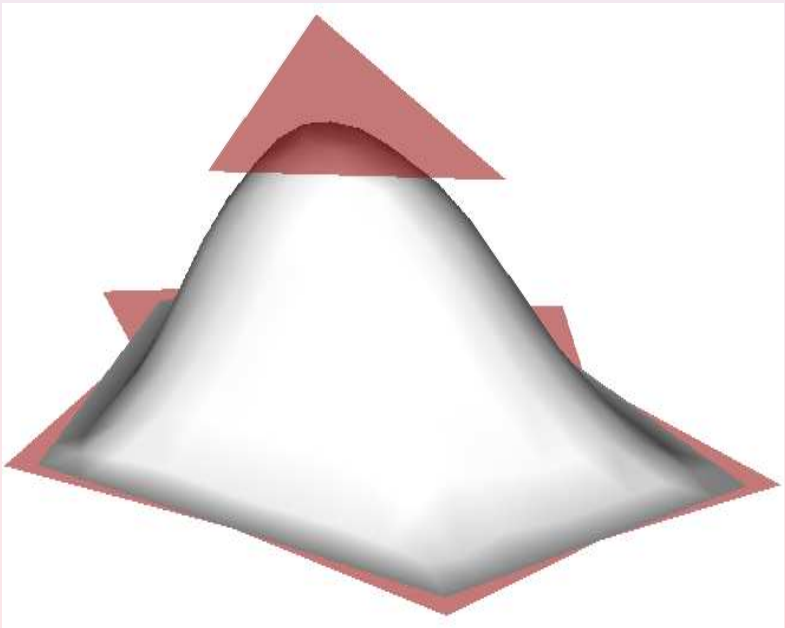
With B_{ij}, $i = 1, \ldots, N$, $j = 1, 2, 3$ one may represent any linear polynomial exactly

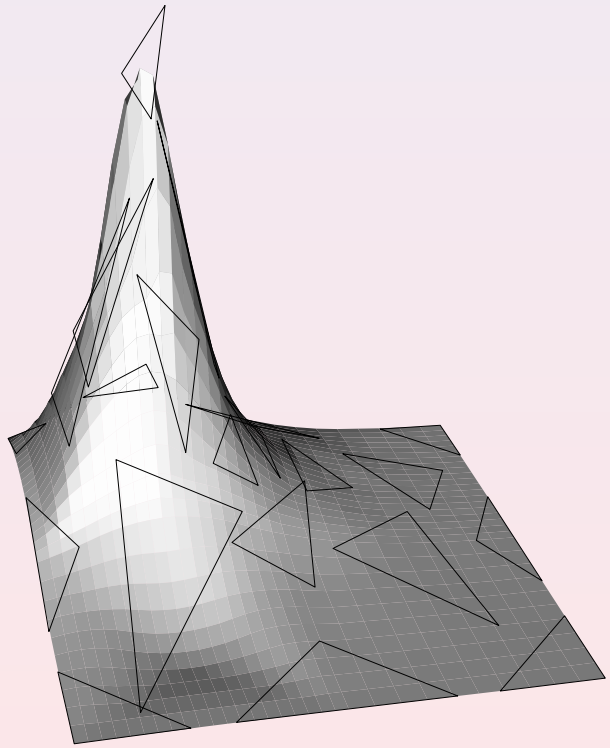
$$\begin{aligned} x &= \sum_{i=1}^{N} \sum_{j=1}^{3} X_{ij} B_{ij}(x, y) \\ y &= \sum_{i=1}^{N} \sum_{j=1}^{3} Y_{ij} B_{ij}(x, y) \\ s(x, y) &= \sum_{i=1}^{N} \sum_{j=1}^{3} c_{ij} B_{ij}(x, y) \end{aligned}$$

The (X_{ij}, Y_{ij}, c_{ij}), $j = 1, 2, 3$ are the vertices of a control triangle that is tangent to the surface in V_i for $i = 1, \ldots, N$.
It 'controls' the local shape of the spline surface.

Powell-Sabin spline, control triangle

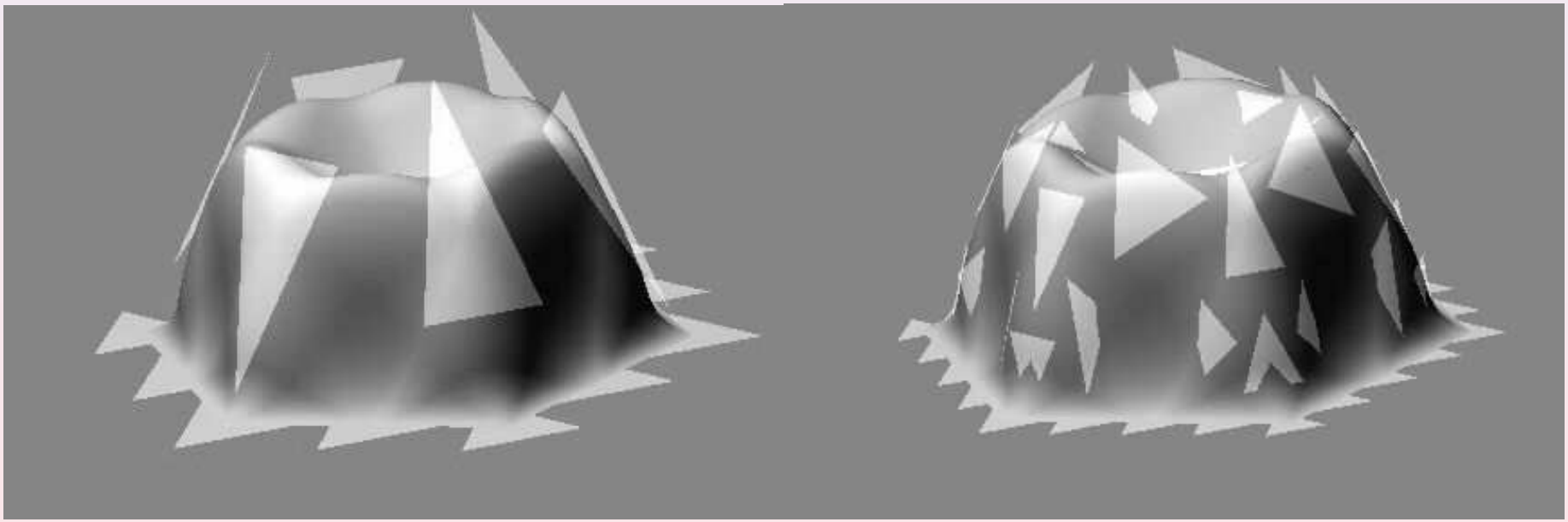
PS surfaces and control triangles





Powell-Sabin spline, control triangle

PS surfaces and control triangles



Powell-Sabin spline, evaluation

- Easy transformation of control triangles to Bézier control net
- Evaluation of Bézier polynomials and derivatives is fast de Casteljau algorithm





Pierre Étienne Bézier (1910-1999)

Paul de Faget de Casteljau

(1910-1999)

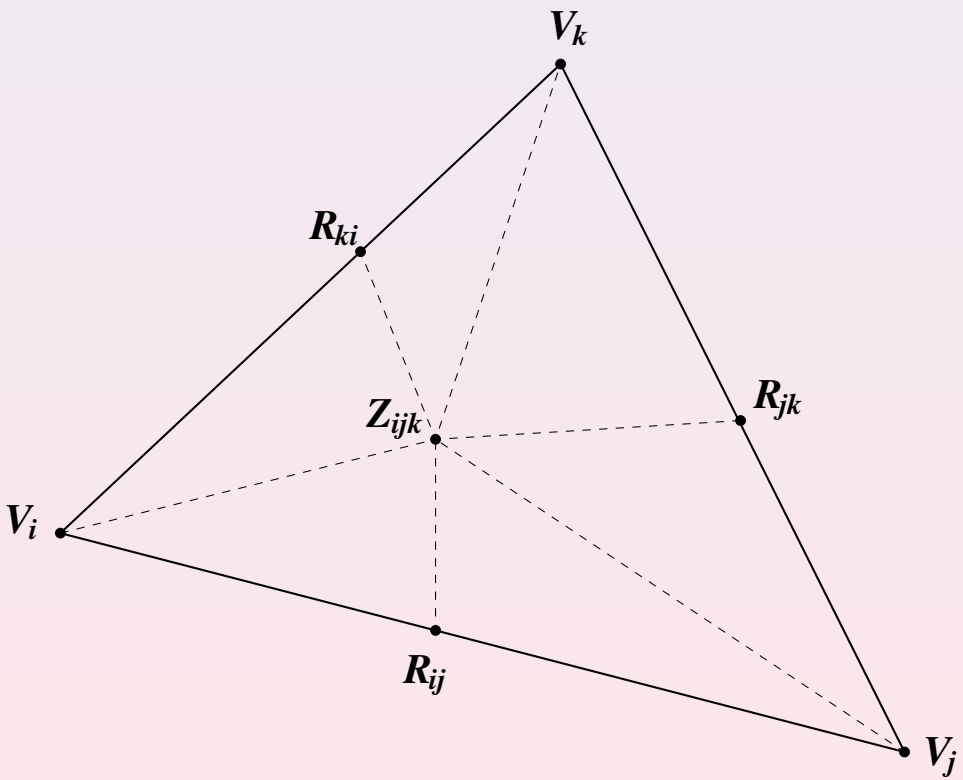
Subdivision

Possible refinements: dyadic or triadic

dyadic subdivision: take PS 6-split points on the edges.
central point has to be in the central triangle!!

Subdivision

Possible refinements: dyadic

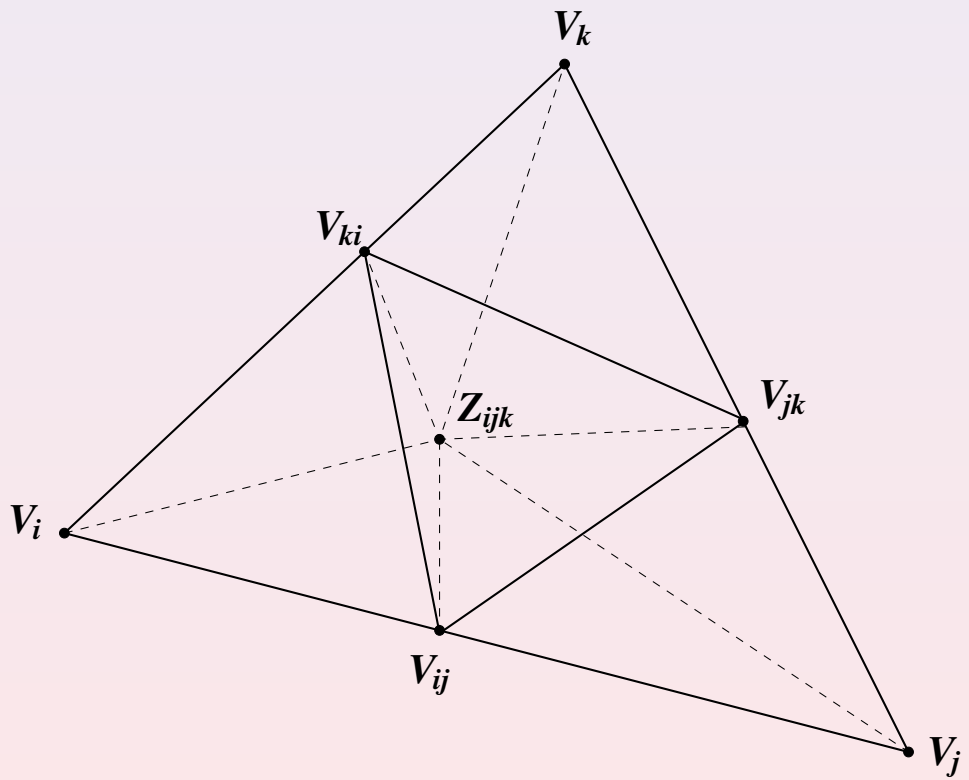


dyadic subdivision: take PS 6-split points on the edges.

central point has to be in the central triangle!!

Subdivision

Possible refinements: dyadic

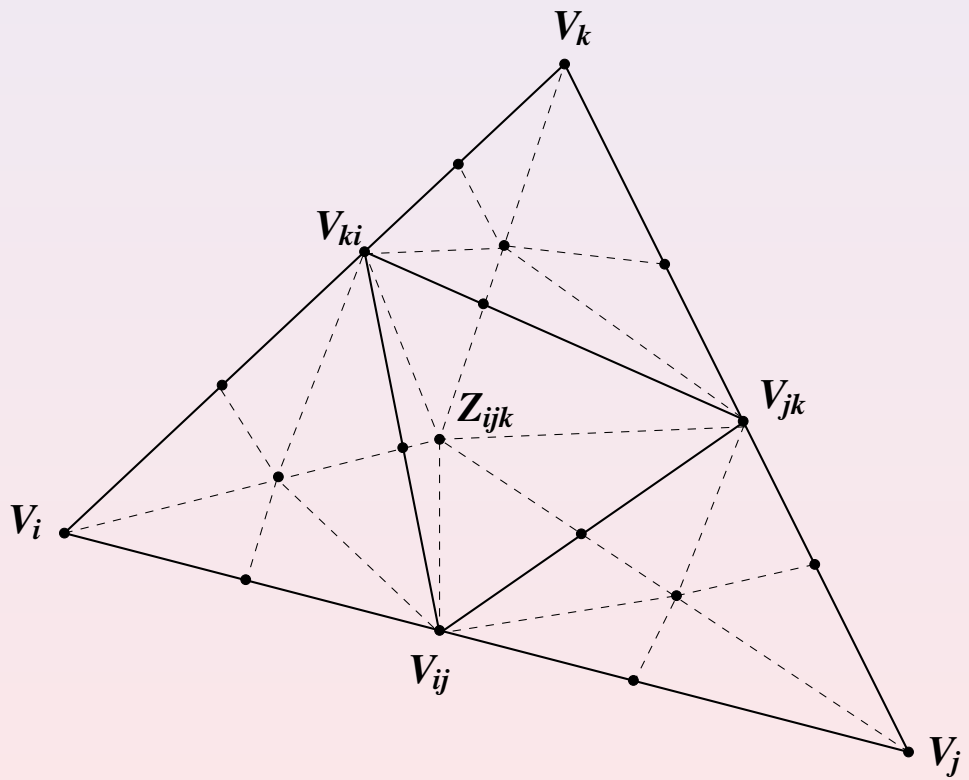


dyadic subdivision: take PS 6-split points on the edges.

central point has to be in the central triangle!!

Subdivision

Possible refinements: dyadic

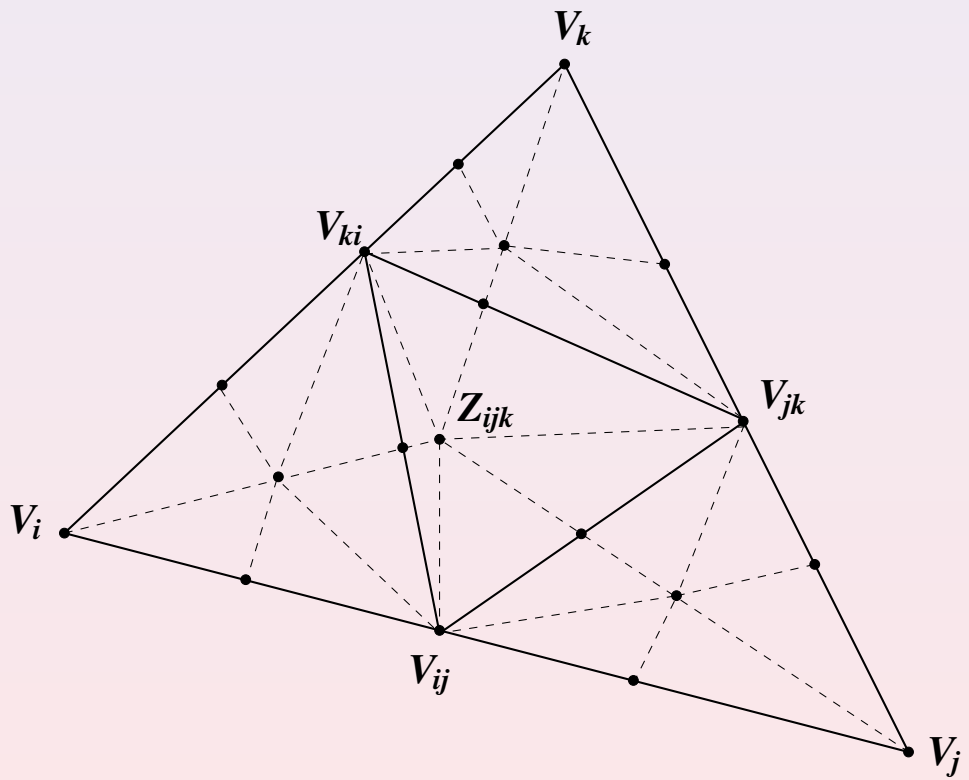


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Subdivision

Possible refinements: dyadic



dyadic subdivision: take PS 6-split points on the edges.
central point has to be in the central triangle!!

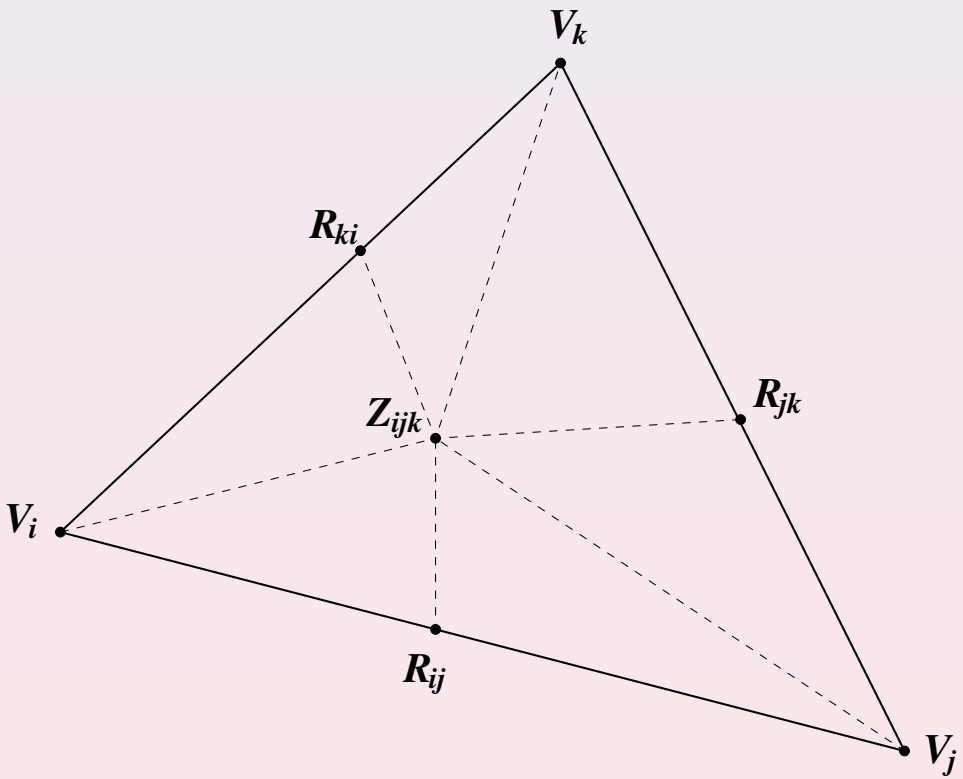
Subdivision

Possible refinements: dyadic or triadic

triadic subdivision: take 2 points both sides of PS 6-split points on the edges. This is always possible

Subdivision

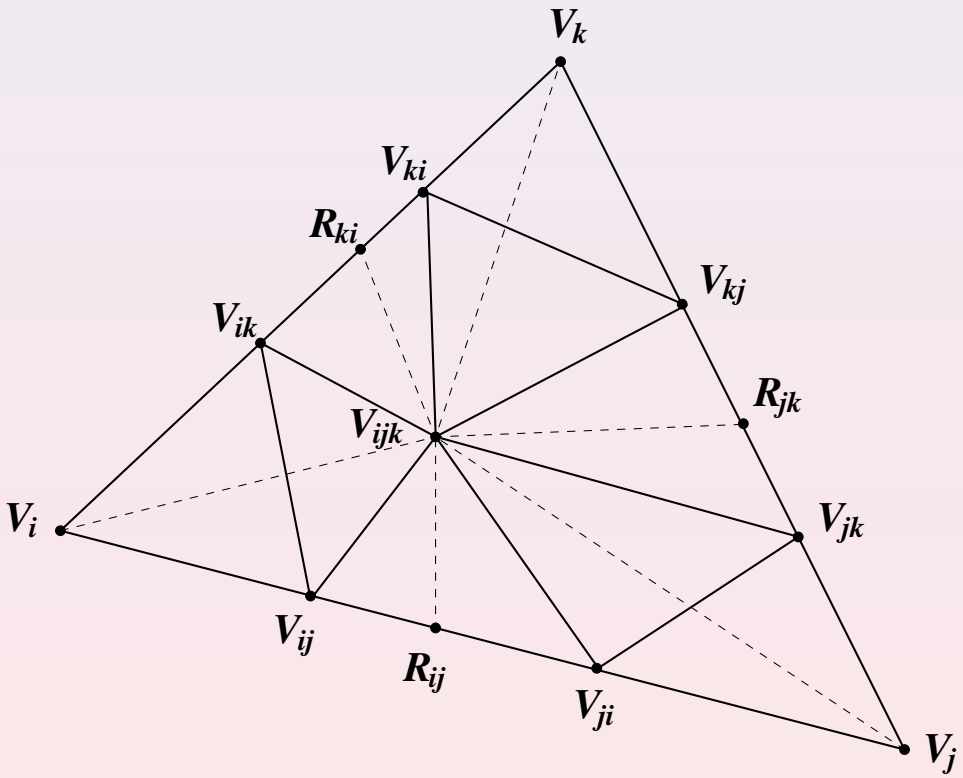
Possible refinements: triadic



triadic subdivision: take 2 points both sides of PS 6-split points on the edges. This is always possible

Subdivision

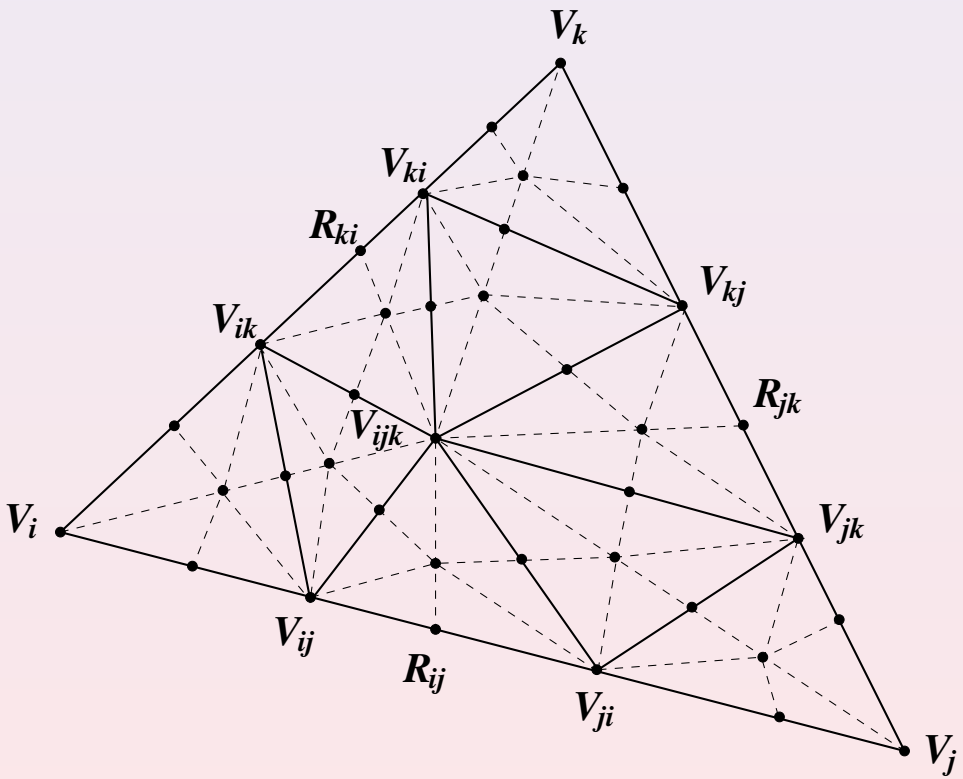
Possible refinements: triadic



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Subdivision

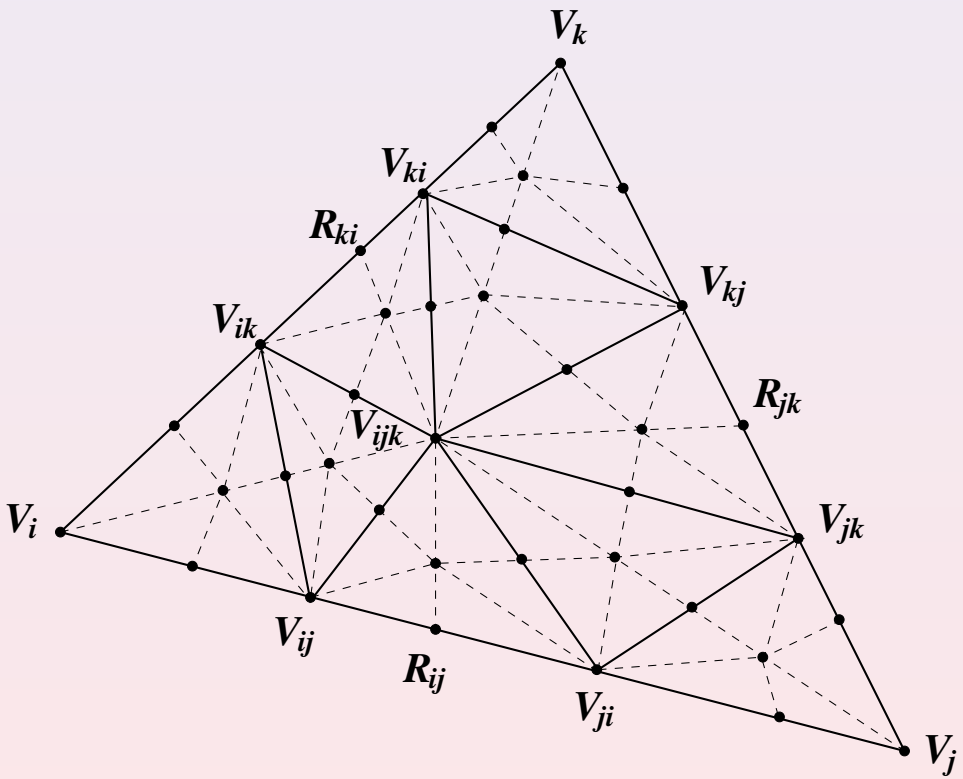
Possible refinements: triadic



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Subdivision

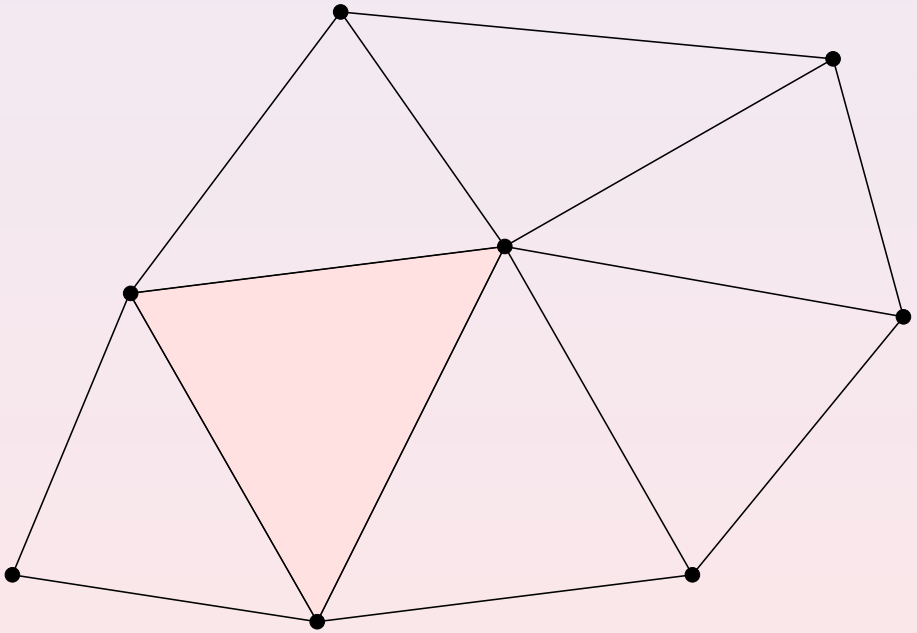
Possible refinements: triadic



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Subdivision

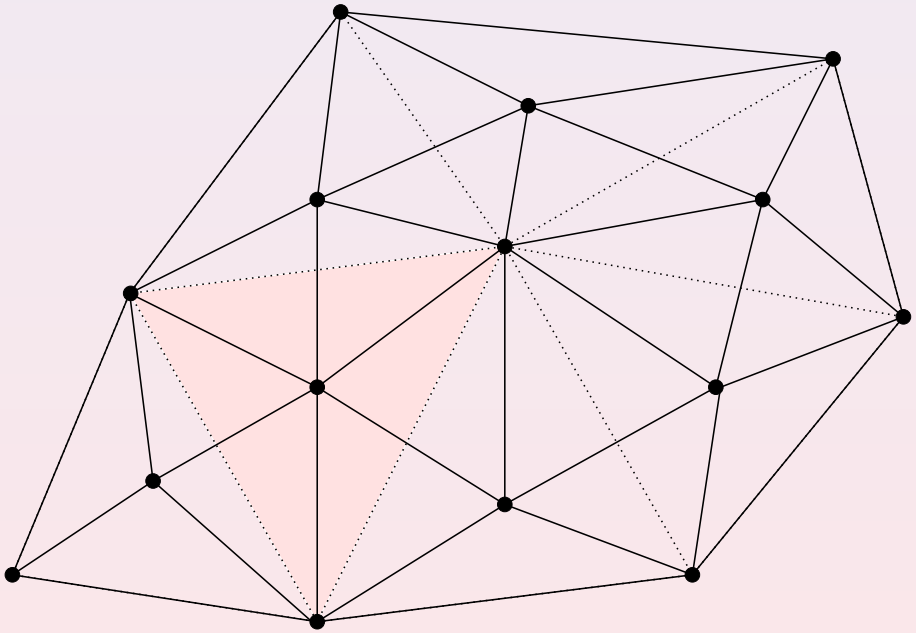
Possible refinements: $\sqrt{3}$-subdivsion



$2 \times \sqrt{3} =$ triadic $\Rightarrow$ complexity grows less fast

Subdivision

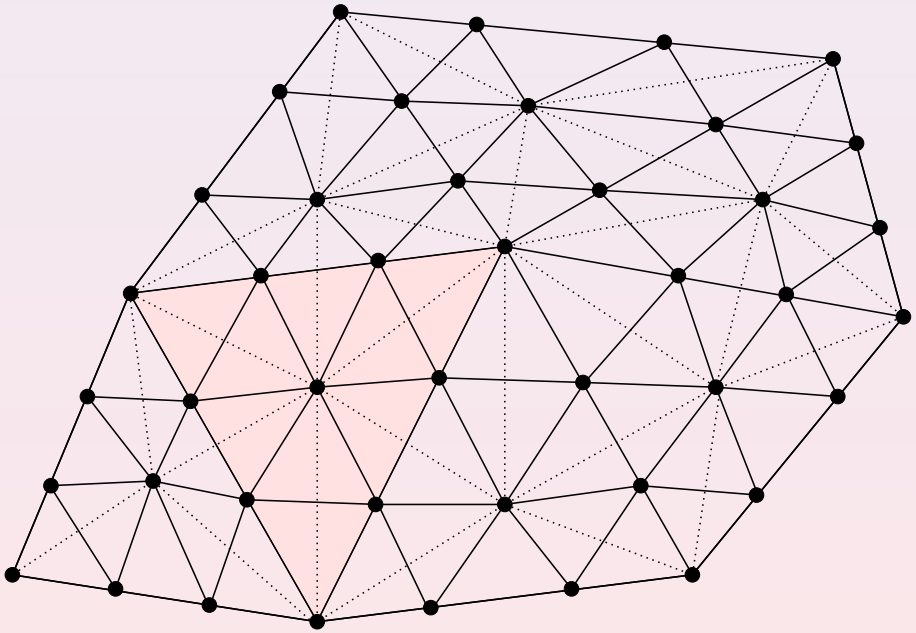
Possible refinements: $\sqrt{3}$-subdivsion



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Subdivision

Possible refinements: $\sqrt{3}$-subdivsion



$2 \times \sqrt{3} =$ triadic $\Rightarrow$ complexity grows less fast

Hierarchical basis

- triangulations $\Delta_0 \subset \Delta_1 \subset \Delta_2 \subset \cdots \subset \Delta_l \subset \cdots$
- PS 6-splits $\Delta_0^* \subset \Delta_1^* \subset \Delta_2^* \subset \cdots \subset \Delta_l^* \subset \cdots$
- PS splines $S_\ell = S_2^1(\Delta_\ell^*), \qquad S_\ell \subset S_{\ell+1}$
- B-spline bases: $S_\ell = \text{span}\{\phi_{\ell,j} : j = 1, \ldots, 3N_\ell\}$

$$s_\ell(x,y) = \phi_\ell c_\ell = \sum_{i=1}^{N_\ell} \sum_{j=1}^{3} B_{ij\ell}(x,y) c_{ij\ell}$$

$$\phi_{\ell+1} = [\phi_{\ell+1}^o \quad \phi_{\ell+1}^n],$$

$\phi_{\ell+1}^o$ correspond to old reused vertices of level ℓ
$\phi_{\ell+1}^n$ correspond to the new vertices of level $\ell+1$

Hierarchical basis

- $S_\ell \subset S_{\ell+1}$ hence $\phi_\ell = \phi_{\ell+1} A_\ell$

$$[\phi_\ell \ \phi^n_{\ell+1}] = [\phi^o_{\ell+1} \ \ \phi^n_{\ell+1}] \begin{bmatrix} O_\ell & 0 \\ N_\ell & I \end{bmatrix}.$$

- ϕ_ℓ are like scaling functions (coarse info)
- $\phi^n_{\ell+1}$ are like wavelet functions (detail info)
- $S_{\ell+1} = S_\ell \oplus W_\ell$

Hierarchical basis

Problems

- What kind of functions can be represented?
  (In which function spaces will this converge?)
- How to represent edges?
- How about stability?
  (small $\delta c_\ell \Rightarrow$ small $\delta s_\ell = \phi_\ell \cdot \delta c_\ell$?)
  Important e.g. for thresholding, compression!
- No (bi)orthogonality
- $\phi^n_{\ell+1}$ not really wavelets $\Leftarrow$ no vanishing moment

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Wavelet basis

Lifting scheme to construct biorthogonal wavelets.

$$[\phi_\ell \ \phi^n_{\ell+1}] = [\phi^o_{\ell+1} \ \ \phi^n_{\ell+1}] \begin{bmatrix} O_\ell & 0 \\ N_\ell & I \end{bmatrix}$$

Dual filters $\tilde{A}_\ell$, $\tilde{B}_\ell$

$$[\phi_\ell \ \ \psi_\ell] = \phi_{\ell+1}[A_\ell \ \ B_\ell] \quad \text{and} \quad \phi_{\ell+1} = [\phi_\ell \ \ \psi_\ell] \begin{bmatrix} \tilde{A}_\ell \\ \tilde{B}_\ell \end{bmatrix}$$

ensure orthogonality

$$\begin{bmatrix} \tilde{A}_\ell \\ \tilde{B}_\ell \end{bmatrix} [A_\ell \ B_\ell] = I$$

Wavelet basis

Lifting scheme to construct biorthogonal wavelets.

Do an extra update

$$[\phi_\ell \ \psi_\ell] = [\phi^o_{\ell+1} \ \ \phi^n_{\ell+1}] \begin{bmatrix} O_\ell & 0 \\ N_\ell & I \end{bmatrix} \begin{bmatrix} I & -C_\ell \\ 0 & I \end{bmatrix}$$

Dual filters $\tilde{A}_\ell$, $\tilde{B}_\ell$

$$[\phi_\ell \ \ \psi_\ell] = \phi_{\ell+1}[A_\ell \ \ B_\ell] \quad \text{and} \quad \phi_{\ell+1} = [\phi_\ell \ \ \psi_\ell] \begin{bmatrix} \tilde{A}_\ell \\ \tilde{B}_\ell \end{bmatrix}$$

ensure orthogonality

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Wavelet basis

Lifting scheme to construct biorthogonal wavelets.

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ensure orthogonality

$$\begin{bmatrix} \tilde{A}_\ell \\ \tilde{B}_\ell \end{bmatrix} [A_\ell \ B_\ell] = I$$

Wavelet basis

Primal			Dual		
$[\phi_\ell \ \psi_\ell]$	$=$	$\phi_{\ell+1} M_\ell$	$[\tilde{\phi}_\ell \ \tilde{\psi}_\ell]$	$=$	$\tilde{\phi}_{\ell+1} \tilde{M}_\ell$
$[\phi_\ell \ \psi_\ell]\tilde{M}_\ell$	$=$	$\phi_{\ell+1}$	$[\tilde{\phi}_\ell \ \tilde{\psi}_\ell] M_\ell$	$=$	$\tilde{\phi}_{\ell+1}$

$$\psi_\ell = \phi^n_{\ell+1} - \phi_\ell C_\ell$$

Problems

- Orthogonality $\Rightarrow C_\ell$ not sparse
- Fix C_ℓ sparse $\Rightarrow \psi_\ell$ local support
- i.e. $\tilde{\phi}_\ell$ has to reproduce constants
- Remaining orthogonality conditions approximated by least squares

Wavelet basis

Primal			Dual		
$[\phi_\ell \ \psi_\ell]$	$=$	$\phi_{\ell+1} M_\ell$	$[\tilde{\phi}_\ell \ \tilde{\psi}_\ell]$	$=$	$\tilde{\phi}_{\ell+1} \tilde{M}_\ell$
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- Want local support $\Rightarrow$ C_ℓ sparse
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- Fix C_ℓ sparse $\Rightarrow$ ψ_ℓ local support
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$$\psi_\ell = \phi^n_{\ell+1} - \phi_\ell C_\ell$$

Problems

- Want local support $\Rightarrow$ C_ℓ sparse
- Fix C_ℓ sparse $\Rightarrow$ ψ_ℓ local support, but basis unstable
- i.e. $\tilde{\phi}_\ell$ has to reproduce constants
- Remaining orthogonality conditions approximated by least squares

Wavelet basis

Primal			Dual		
$[\phi_\ell \ \psi_\ell]$	$=$	$\phi_{\ell+1} M_\ell$	$[\tilde{\phi}_\ell \ \tilde{\psi}_\ell]$	$=$	$\tilde{\phi}_{\ell+1} \tilde{M}_\ell$
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$$\psi_\ell = \phi^n_{\ell+1} - \phi_\ell C_\ell$$

Problems

- Want local support $\Rightarrow$ C_ℓ sparse
- Want stability $\Rightarrow$ need 1 vanishing moment for ψ_ℓ
- i.e. $\tilde{\phi}_\ell$ has to reproduce constants
- Remaining orthogonality conditions approximated by least squares

Wavelet basis

Primal			Dual		
$[\phi_\ell \ \psi_\ell]$	$=$	$\phi_{\ell+1} M_\ell$	$[\tilde{\phi}_\ell \ \tilde{\psi}_\ell]$	$=$	$\tilde{\phi}_{\ell+1} \tilde{M}_\ell$
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Problems

- Want local support $\Rightarrow$ C_ℓ sparse
- Want stability $\Rightarrow$ need 1 vanishing moment for ψ_ℓ
- An extra linear constraint
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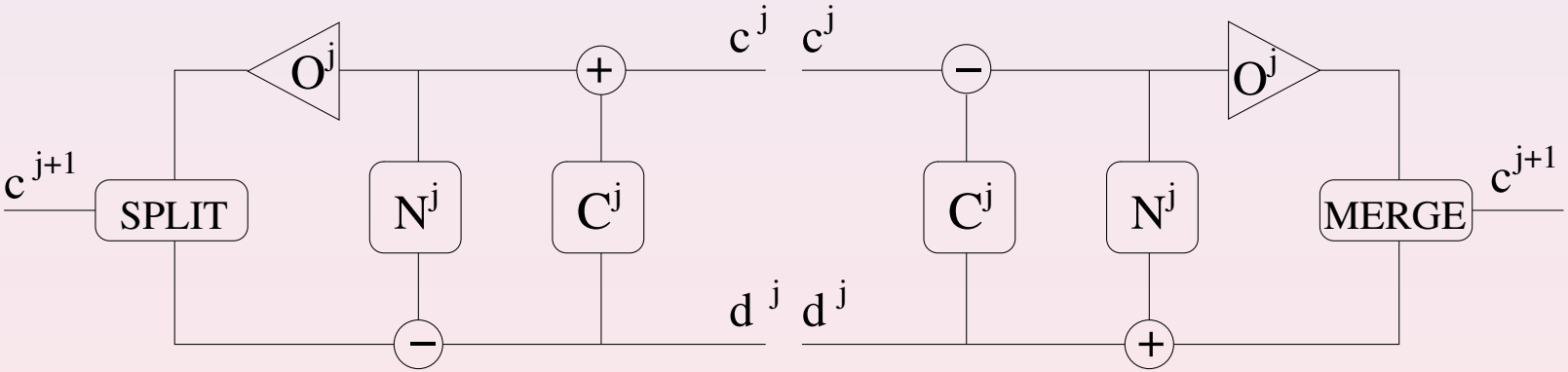
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Problems

- Want local support $\Rightarrow$ C_ℓ sparse
- Want stability $\Rightarrow$ need 1 vanishing moment for ψ_ℓ
- An extra linear constraint
- Remaining orthogonality conditions approximated by least squares

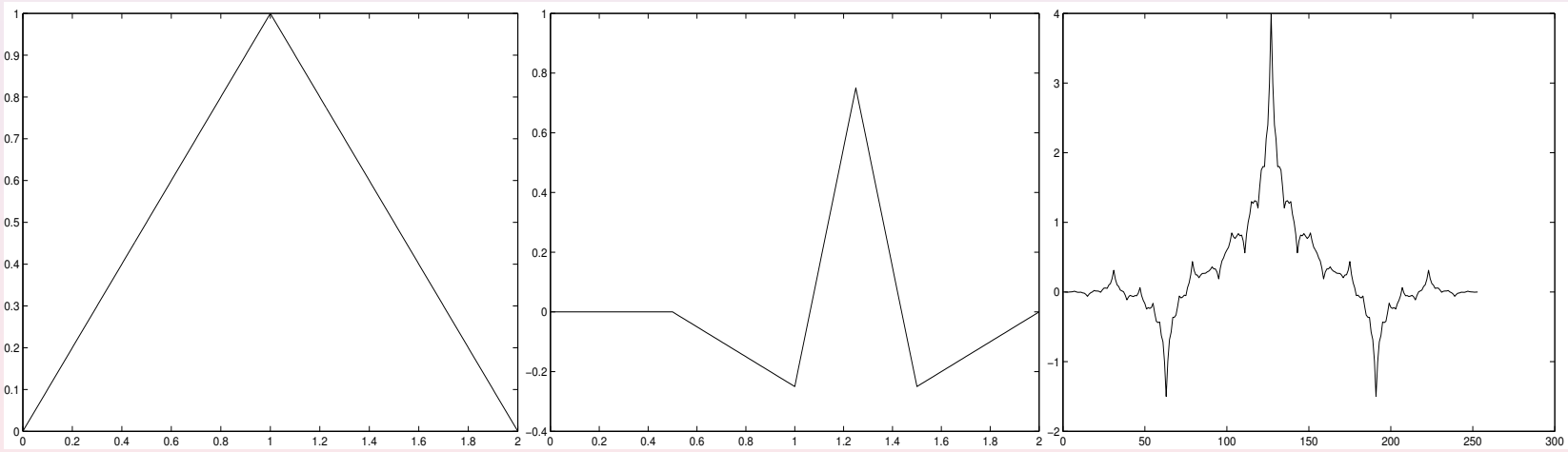
Lifting

Lifting with update C_j.



Some examples

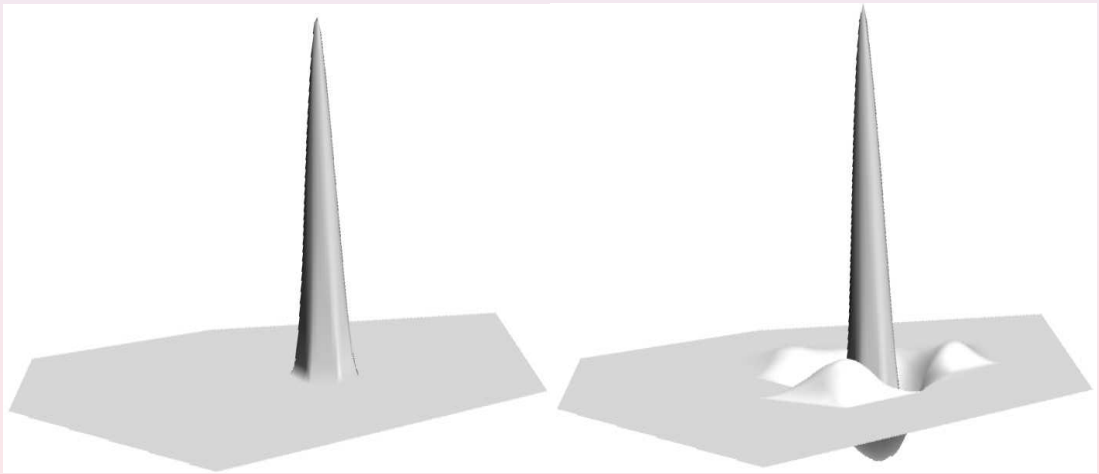
Univariate linear spline wavelets



ϕ ψ $\tilde{\phi}$

Some examples

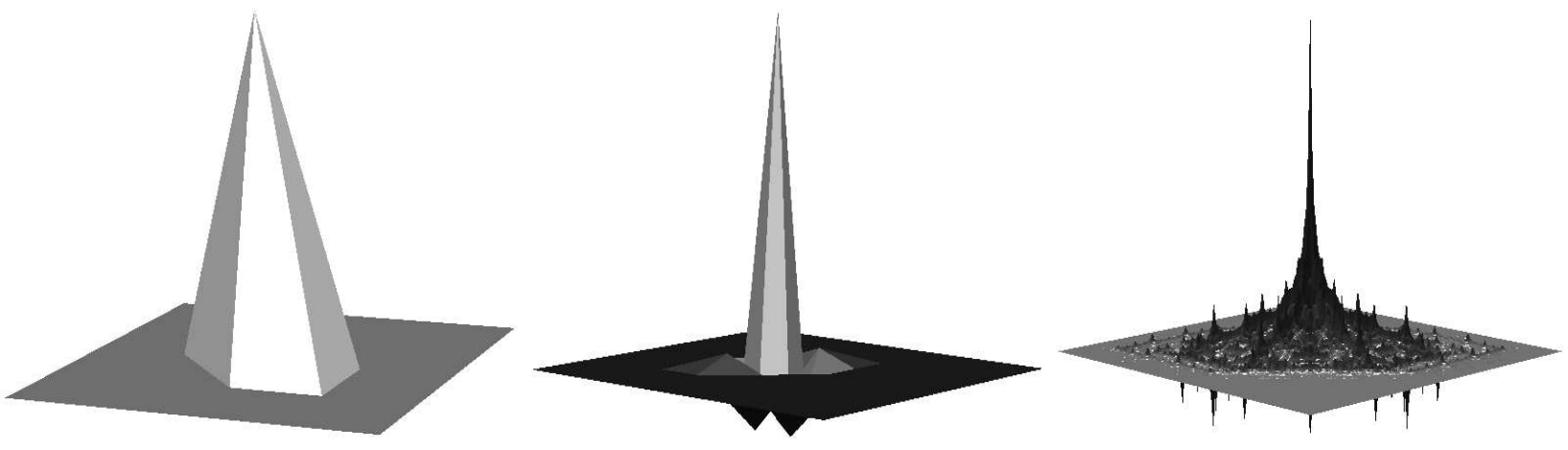
Bivariate linear spline wavelets



ϕ^n (hierarchic) ψ wavelet

Some examples

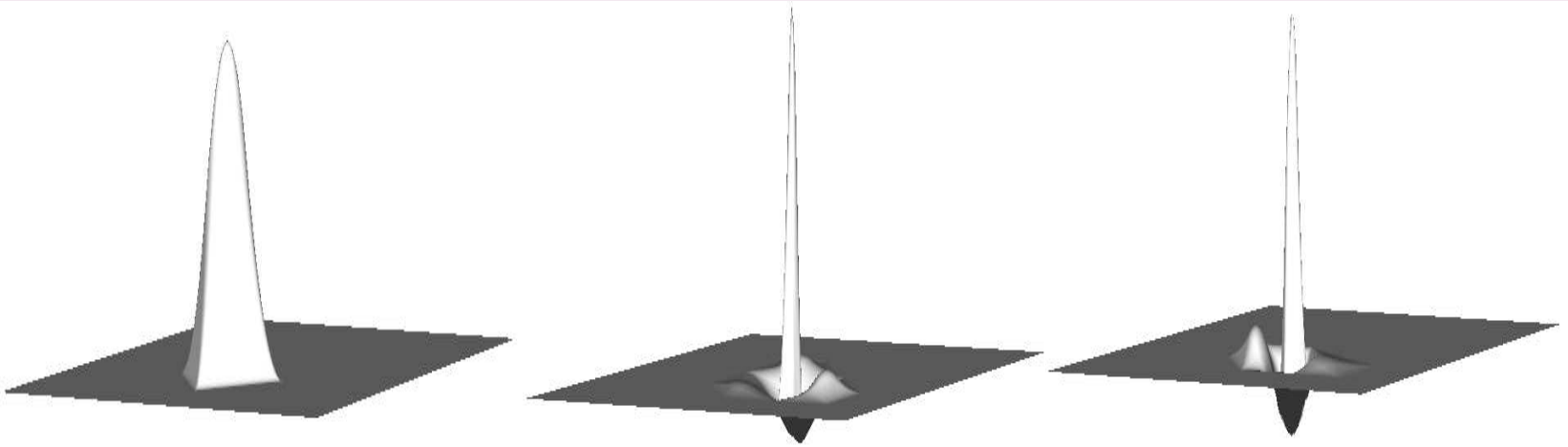
Bivariate linear spline wavelets



ϕ ψ $\tilde{\psi}$

Some examples

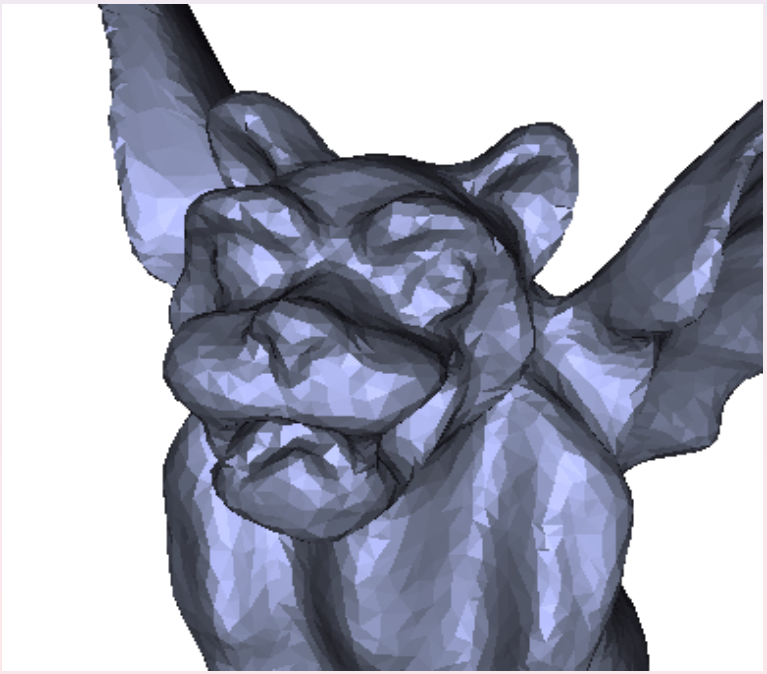
Bivariate quadratic spline wavelets



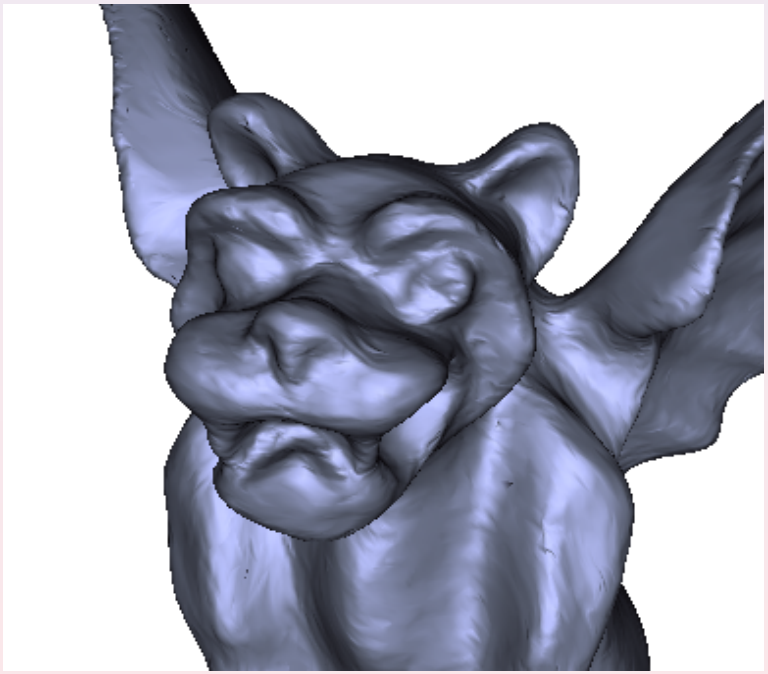
ϕ ψ internal ψ boundary

Application

Smoothing



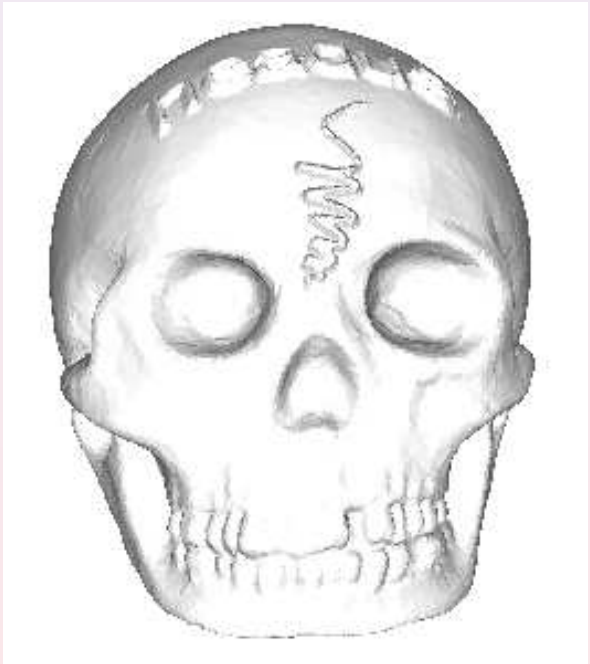
triangular mesh



PS splines

Application

Smoothing with less data



triangular mesh

(40000 triangles)

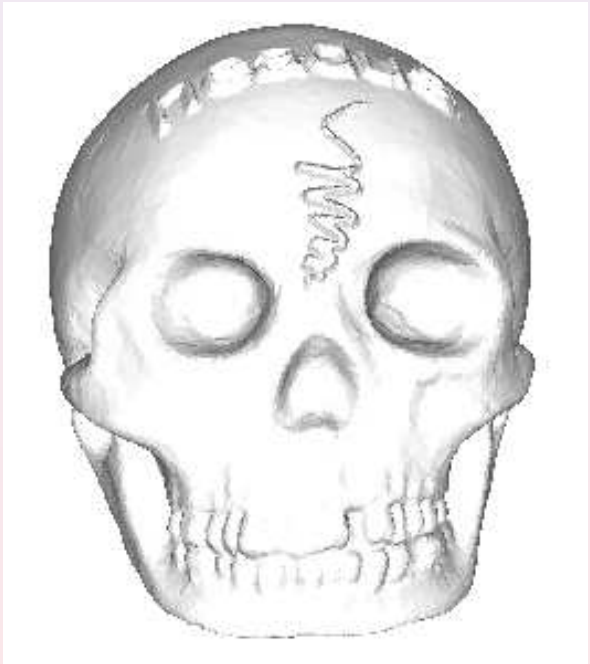


reduced mesh

(5000 triangles)

Application

Smoothing with less data



triangular mesh

(40000 triangles)



Powell–Sabin spline

(5000 control triangles)

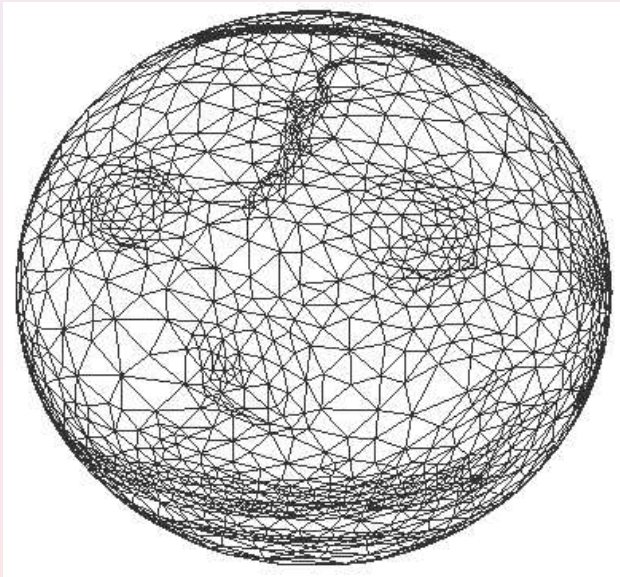
Application

Smoothing with less data

- genus zero mesh, can be mapped on a sphere
- needs to define PS spline wavelets on a sphere
- use local linearization (map to tangent plane and apply previous theory)
- control triangles: construct dual mesh on the sphere, original triangles are projections of control triangles of dual vertices.

Application

Smoothing with less data



parameterization on sphere

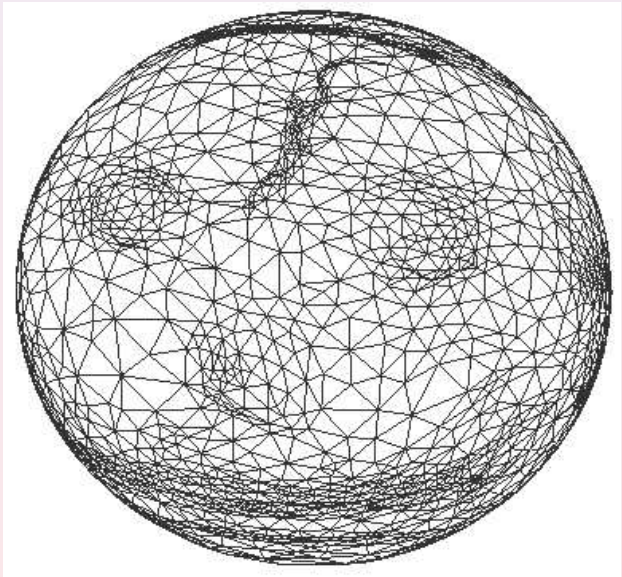


reduced mesh

(5000 triangles)

Application

Smoothing with less data



parameterization on sphere

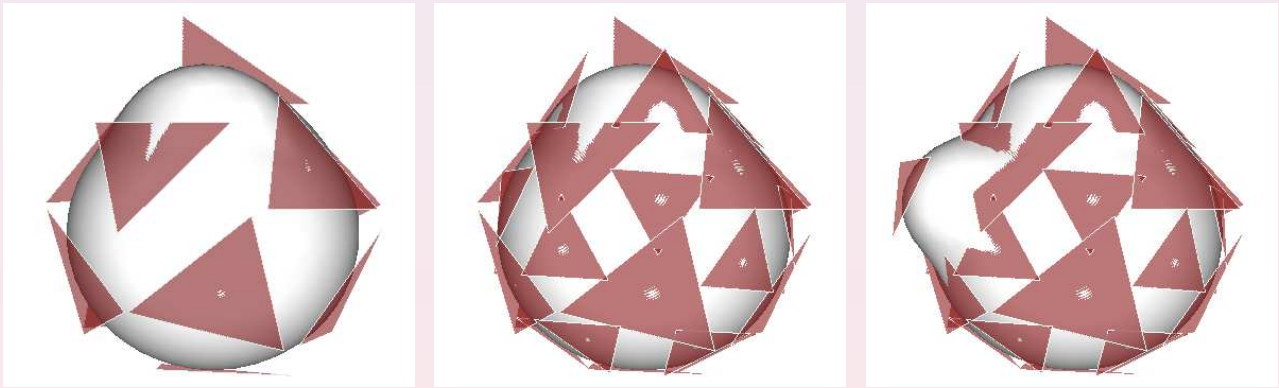


Powell–Sabin spline

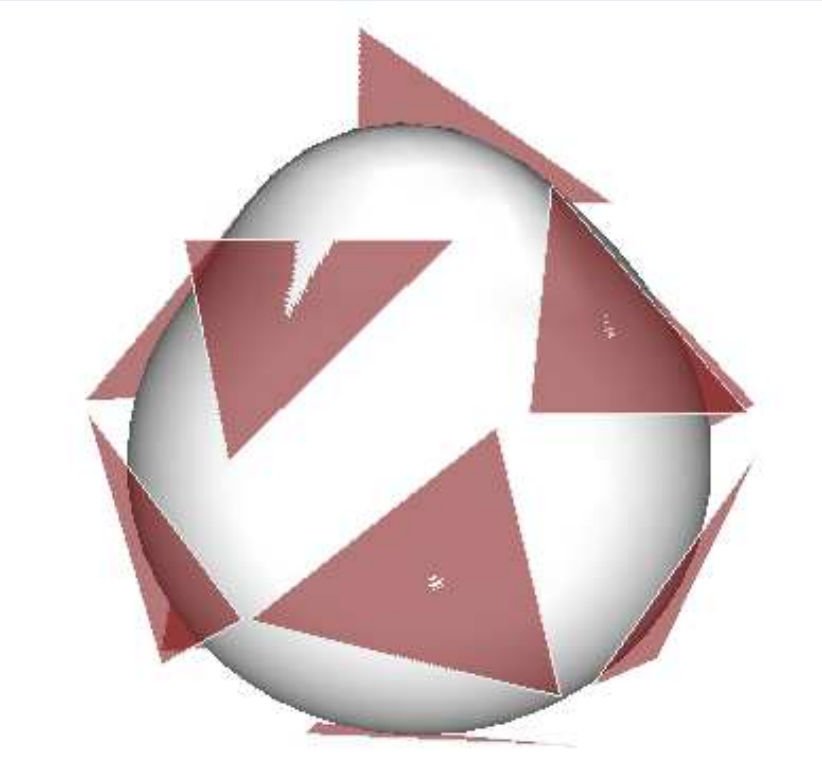
(5000 control triangles)

Application

Spherical PS spline wavelets give local control



The hierarchical basis



click image to start animation

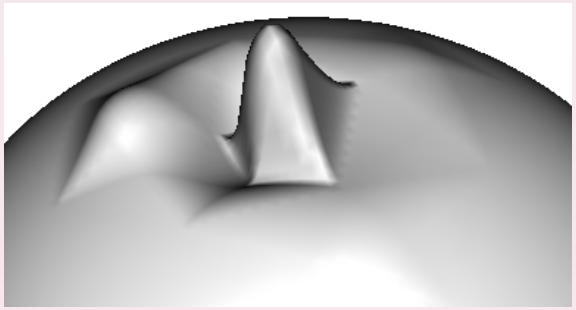
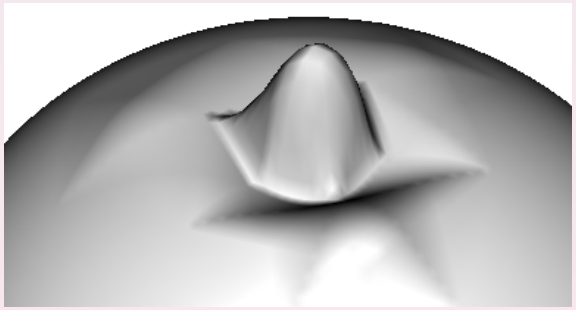
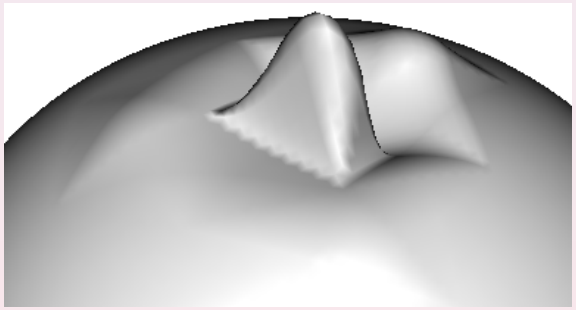
- $\{V_i \in \Delta_\ell\} \subset \{V_i \in \Delta_{\ell+1}\}$
- $\Delta_\ell^{PS} \subset \Delta_{\ell+1}^{PS}$
- $S_\ell := S_2^1(\Delta_\ell^{PS}), \quad S_\ell \subset S_{\ell+1}$

$$S_2 = S_0 \oplus W_0 \oplus W_1$$

- Largest triangles control S_0
- Medium triangles control W_0
- Smallest triangles control W_1

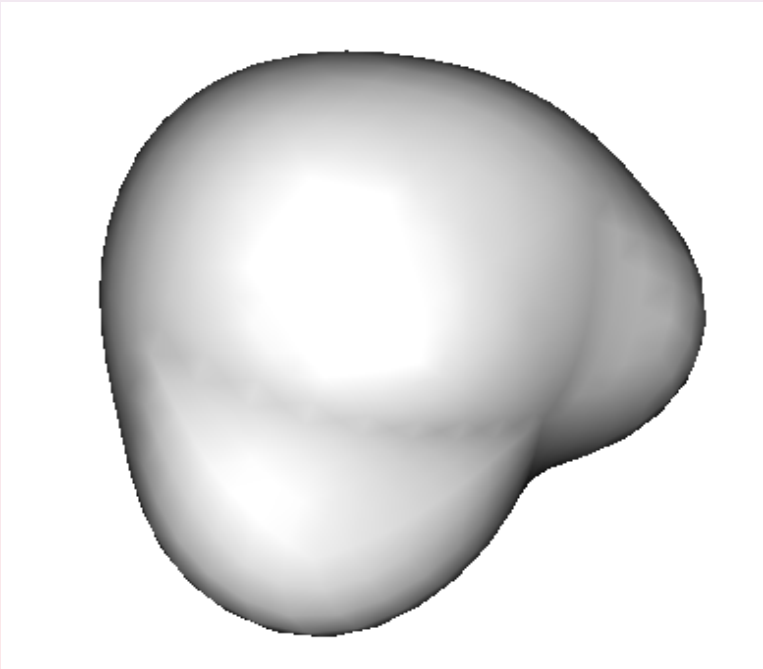
Spherical wavelets

3 wavelets associated with one vertex

Spherical wavelets

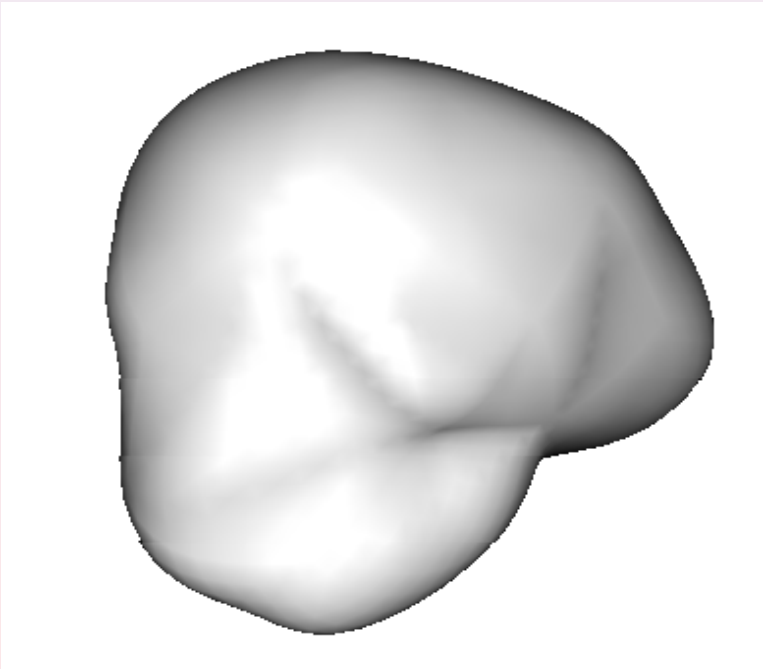
Different levels with $\sqrt{3}$ subdivision starting from an icosaeder



level 0

Spherical wavelets

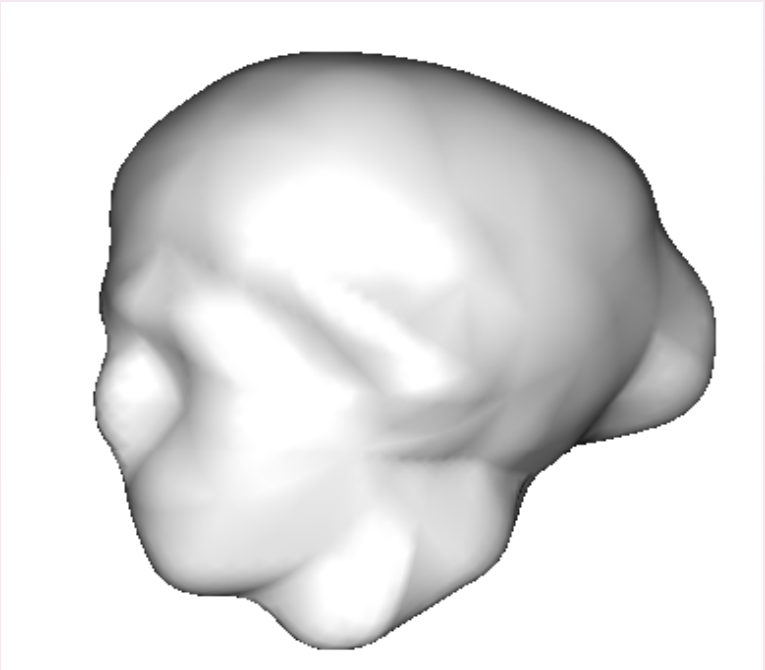
Different levels with $\sqrt{3}$ subdivision starting from an icosaeder



level 1

Spherical wavelets

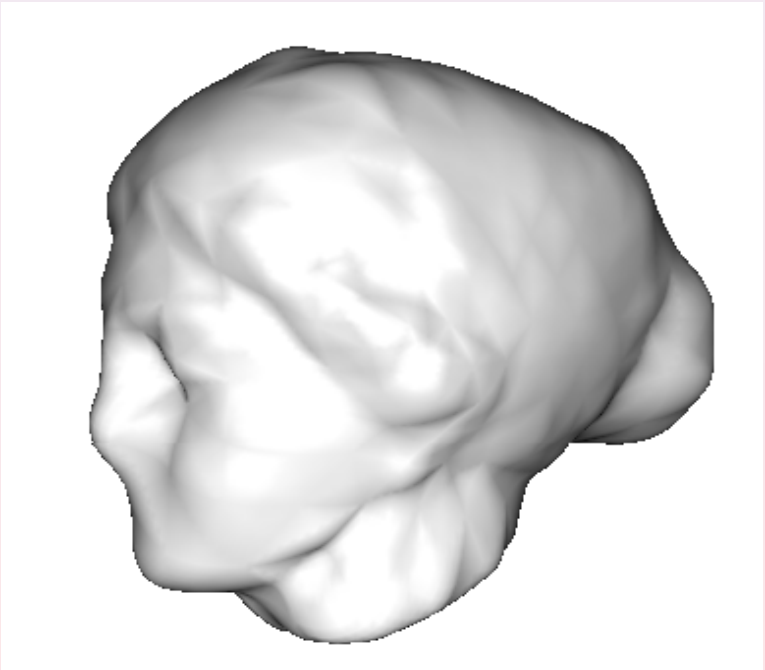
Different levels with $\sqrt{3}$ subdivision starting from an icosaeder



level 2

Spherical wavelets

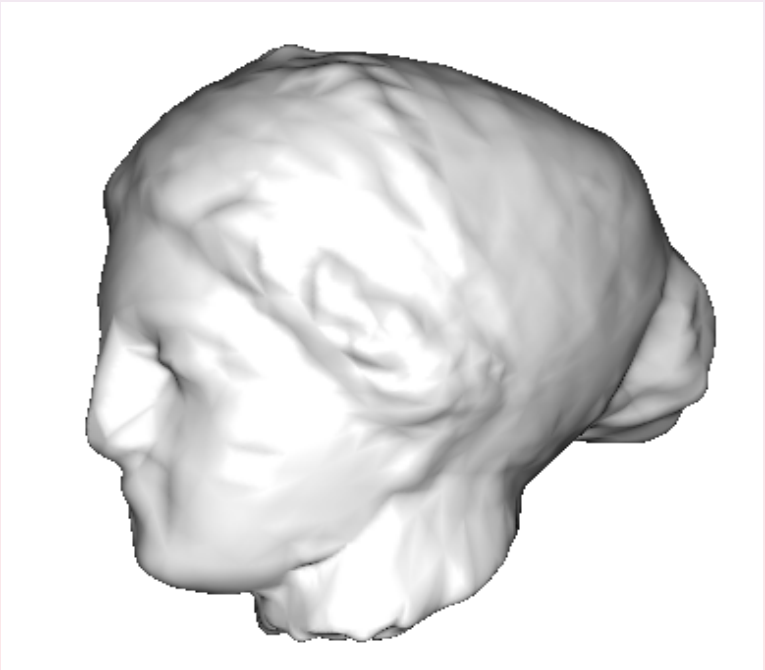
Different levels with $\sqrt{3}$ subdivision starting from an icosaeder



level 3

Spherical wavelets

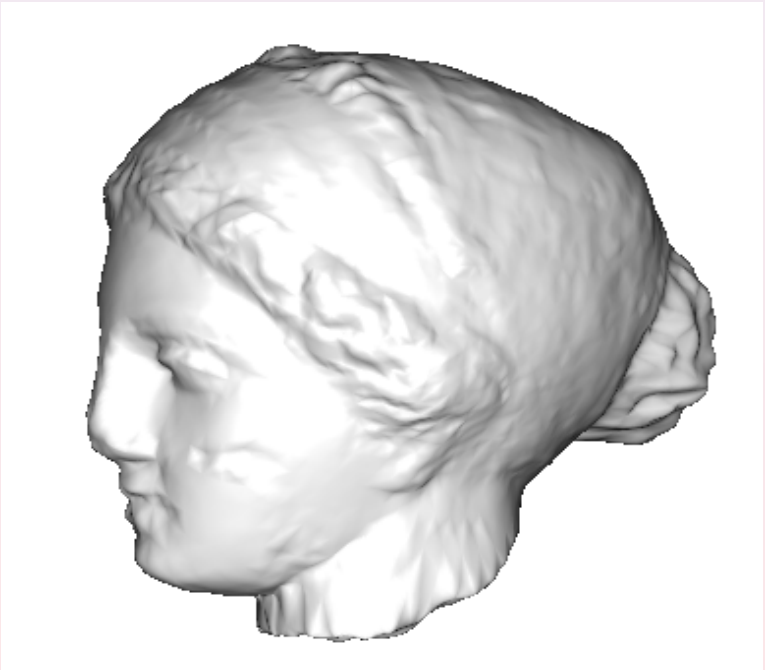
Different levels with $\sqrt{3}$ subdivision starting from an icosaeder



level 4

Spherical wavelets

Different levels with $\sqrt{3}$ subdivision starting from an icosaeder



level 5

Spherical wavelets

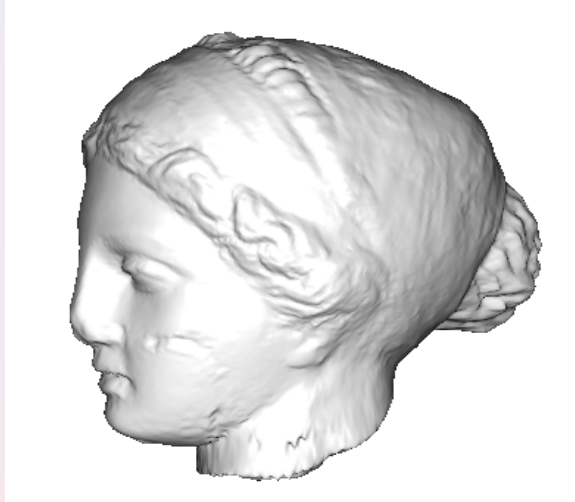
Different levels with $\sqrt{3}$ subdivision starting from an icosaeder

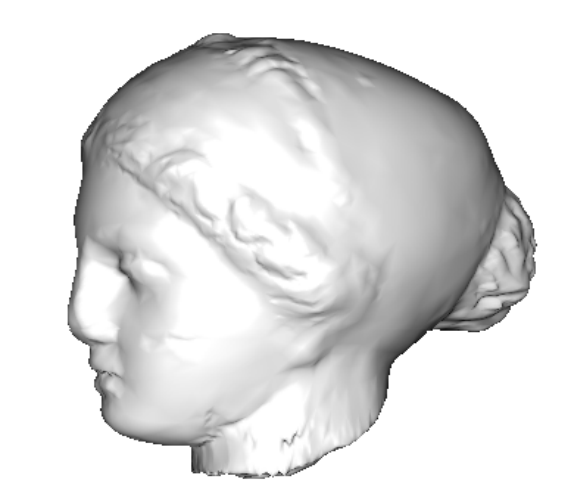


level 6

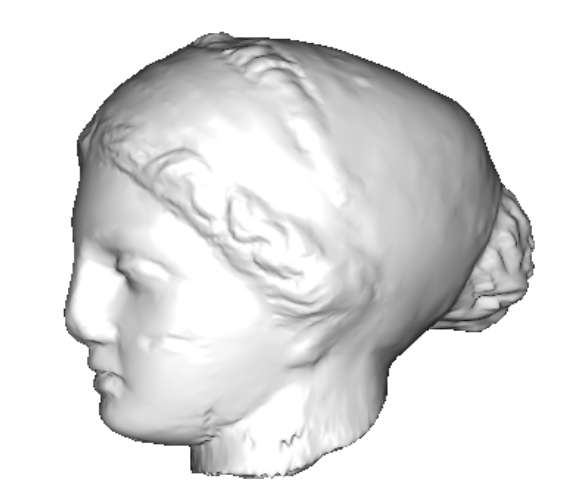
Application

Data reduction





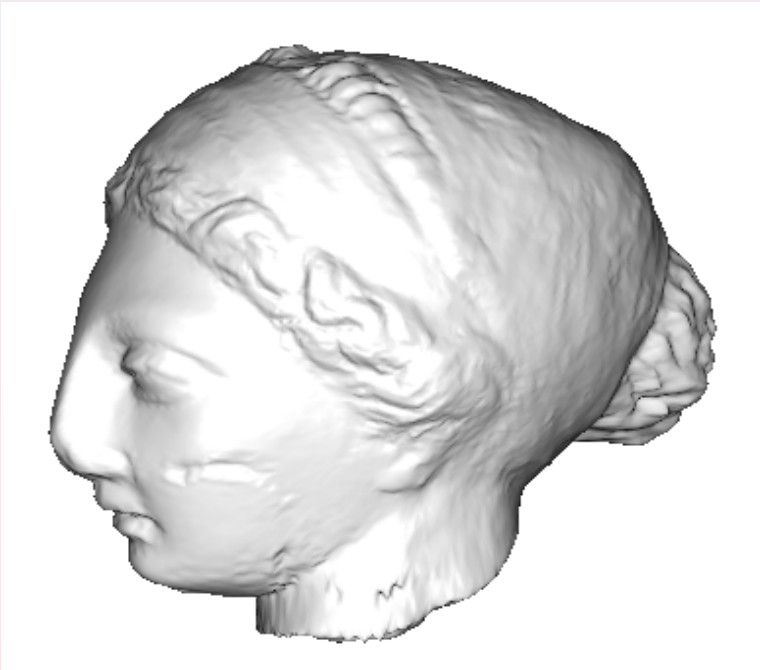
keep 14%

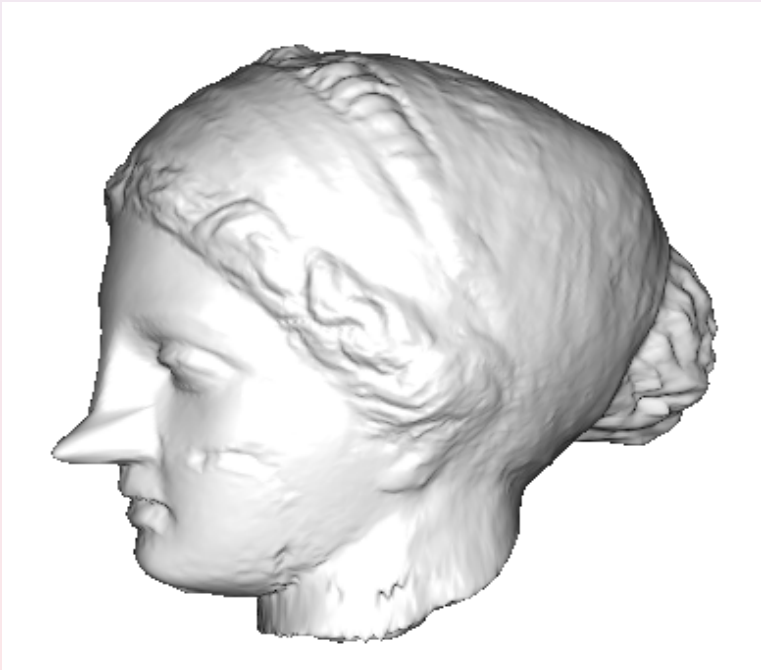


keep 26%

Application

Spherical PS spline wavelets give local control





Application

Spherical PS spline wavelets give local control parametric representation



